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FOURIER COEFFICIENTS OF PERIODIC FUNCTIONS OF GEVREY CLASSES AND ULTRADISTRIBUTIONS

By

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§ 0. Introduction.

Gevrey classes of ultradifferentiable functions on an open domain $\Omega\subset\boldsymbol{R}^n$ of type s $(s>1)$ and their dual spaces are extensively studied by Professor H. Komatsu ([1], [2]). In the case of one point support, J. P. Ramis ([7], [8]) has studied Gevrey classes of functions of type s $(-\infty<s<+\infty)$. It is known from the definition that for $s\leq 1$, these functions are analytic so that the usual technique of functional analysis breaks down. That is why Ramis uses formal power series to overcome some of these difficulties.

In this paper, we study Gevrey classes of ultradifferentiable functions of type s $(-\infty<s<+\infty)$ on the unit circle $\boldsymbol{T}$. We characterize these functions by growth conditions on their Fourier coefficients. It turns out that for $s<0$, ultradifferentiable functions are only constant functions but we do have analytic cases for $s\leq 1$. Since the space $\boldsymbol{T}$ is compact, we can still use the method of functional analysis to define and estimate the Fourier coefficients of dual elements i.e., ultradistributions of generalized sense. Following Ramis, we denote the space of Gevrey-Roumieu ultradifferentiable functions of type s by $\mathcal{C}_s(\boldsymbol{T})$ and that of Gevrey-Beurling ultradifferentiable ones of type s by $\mathcal{C}_{(s)}(\boldsymbol{T})$. Then, for example, $\mathcal{C}_1(\boldsymbol{T})=\mathcal{A}(\boldsymbol{T})$ is the space of all analytic functions on $\boldsymbol{T}$ so that the dual space $\mathcal{C}_1(\boldsymbol{T})^\wedge=\mathcal{B}(\boldsymbol{T})$ is the space of Sato-hyperfunctions. For $\mathcal{C}_{(1)}(\boldsymbol{T})$, this space can be identified with the space of all holomorphic functions on $\boldsymbol{C}\backslash\{0\}$ and hence the dual space $\mathcal{C}_{(1)}(\boldsymbol{T})^\wedge$ is the space of Morimoto's cohomological ultradistributions [6].

The main result (Theorem 3.3) characterizes the Gevrey-Roumieu ultradifferentiable functions by means of their Fourier coefficients. For $s=1$, such a result is already proved by Köthe [4] and possibly by others. Usual proof of this kind of results uses associated functions $\{M_n\}$ of Mandelbrojt (see for example Komatsu [1], I) but we use associated pairs which can be taken as smooth (i, e. C^∞) functions making calculations more transparent.

Another main result (Theorem 4.4) deals with the Fourier coefficients of elements of the dual spaces $\mathcal{C}_s(\boldsymbol{T})^\wedge$ and $\mathcal{C}_{(s)}(\boldsymbol{T})^\wedge$ for any $s: 0<s<\infty$ i.e., ultra-distributions. We use Köthe's theory of perfect spaces and of echelon spaces to complete the proof.

These main results are already used in our forthcomming paper [11].

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§1. Gevrey classes $\mathcal{C}_s(\boldsymbol{T})$ and $\mathcal{C}_{(s)}(\boldsymbol{T})$ $(-\infty<s<+\infty)$.

Let S^1 be the unit circle in the complex plane $\boldsymbol{C}$, i.e., $S^1=\{z\in\boldsymbol{C} \mid |z|=1\}$. We identify S^1 with $\boldsymbol{T}=\boldsymbol{R}/2\pi\boldsymbol{Z}$ by the usual transformation: $z=e^{it}$ $(i=\sqrt{-1})$ so that we think of a function f on S^1 as a function on $\boldsymbol{R}$ of period 2π. In particular $f\in C^\infty(\boldsymbol{T})$ means that $f\in C^\infty(\boldsymbol{R})$ and has period 2π. We denote $f^{(n)}(t)=d^n f(t)/dt^n$ for the n-th derivatives.

Definition 1.1. *Gevrey-Roumieu class $\mathcal{C}_s(\boldsymbol{T})$* is the totality of functions $f\in C^\infty(\boldsymbol{T})$ satisfying that for some $A>0$ and $C>0$,

$$\|f^{(n)}\|_\infty=\sup_t|f^{(n)}(t)|\le C\cdot A^n\cdot(n\,!)^s$$

for any integer $n\ge 0$. $f\in\mathcal{C}_s(\boldsymbol{T})$ is called a *ultradifferentiable function of Gevrey-Roumieu class of type s*.

Definition 1.2. *Gevrey-Beurling class $\mathcal{C}_{(s)}(\boldsymbol{T})$* is the totality of functions $f\in C^\infty(\boldsymbol{T})$ satisfying that for any $A>0$ there exists $C_A>0$ such that

$$\|f^{(n)}\|_\infty\le C_A\cdot A^n\cdot(n\,!)^s$$

for any integer $n\ge 0$. $f\in\mathcal{C}_{(s)}(\boldsymbol{T})$ is called a *ultradifferentiable function of Gevrey-Beurling class of type s*.

§2. Associated pair $(F(x), G(r))$.

Definition 2.1. Let $F(x)$, $G(r)$ be real valued functions on $x>0$, $r>0$ respectively. Then $F(x)$ and $G(r)$ are *associated* if they are connected by the following formulas:

$$\inf_{x>0} F(x)\cdot r^{-x}=G(r) \quad\text{and}\quad \sup_{r>0} G(r)\cdot r^x=F(x).$$

W call $(F(x), G(r))$ an *associated pair*.

Diffinition 2.2. Let $(F(x), G(r))$, $(\tilde{F}(x), \tilde{G}(r))$ be two associated pairs, then we call them *equivalent* if there exists $A>0$ such that

$$\tilde{F}(x)=A^x\cdot F(x) \quad\text{for all } x>0 \quad\text{and}\quad \tilde{G}(r)=G\left(\frac{r}{A}\right) \quad\text{for all } r>0.$$

Example 2.3. For each s $(0<s<\infty)$,

$$F(x)=x^{sx}(x>0) \quad \text{and} \quad G(r)=e^{-(s/e)r^{1/s}}(r>0)$$

are associated hence $(x^{sx}, e^{-(s/e)r^{1/s}})$ is an associated pair.

§ 3. Fourier coefficients of ultradifferentiable functions of Gevrey classes.

Definition 3.1. For any $f\in C^{\infty}(T)$, we define its *j-th Fourier coefficient* $f^{\wedge}(j)$ by

$$f^{\wedge}(j)=\frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ijt}dt$$

for any integer $j\in \boldsymbol{Z}$.

It is well known that for any $f\in C^{\infty}(\boldsymbol{T})$, the Fourier series

$$\sum_{j\in Z} f^{\wedge}(j)e^{ijt}$$

converges everywhere to $f(t)$. We quote also a well known estimate with its ingeneous proof (e.g. [3]):

Lemma 3.2. *For any $f\in C^{\infty}(\boldsymbol{T})$, we have*

$$|f^{\wedge}(j)|\leq \inf_{n\geq 0} \|f^{(n)}\|_{\infty}/|j|^{n}$$

for any $j\in \boldsymbol{Z}$, $j\neq 0$.

Proof. We know that for any $f\in C^{\infty}(\boldsymbol{T})$, we have

$$f^{(n)}(t)=\sum_{j} f^{\wedge}(j)(ij)^{n}e^{ijt} \qquad \text{for all} \quad n\geq 0.$$

Hence

$$\int_0^{2\pi} f^{(n)}(t)e^{-ijt}dt=2\pi f^{\wedge}(j)(ij)^{n} \qquad \text{for any} \quad j\in \boldsymbol{Z} \text{ and } n\geq 0,$$

From this we have

$$|f^{\wedge}(j)|\leq \frac{1}{2\pi|j|^{n}}\int_0^{2\pi}|f^{(n)}(t)|dt\leq \|f^{(n)}\|_{\infty}/|j|^{n}$$

for any $j\in \boldsymbol{Z}$, $j\neq 0$ and $n\geq 0$, hence the required estimate.

Theorem 3.3. *For any s $(0<s<\infty)$, we have*

(R): *$f\in \mathcal{C}_{s}(\boldsymbol{T})$ if and only if for some $B>0$ and $K>0$,*

$$|f^{\wedge}(j)|\leq K\cdot e^{-B|j|^{1/s}} \qquad \text{tor any} \quad j\in \boldsymbol{Z}.$$

(B): *$f\in \mathcal{C}_{(s)}(\boldsymbol{T})$ if aud only if for any $B>0$ there exists $K_B>0$ such that*

$$|f^{\wedge}(j)|\leq K_{B}\cdot e^{-B|j|^{1/s}} \qquad \text{for any} \quad j\in \boldsymbol{Z}.$$

Proof of (**R**). *Only if part.* Suppose $f\in \mathcal{C}_s(\boldsymbol{T})$. From Definition 1.1 and $n!\leq n^n$ we have for some $C>0$ and $A>0$, $\|f^{(n)}\|_\infty\leq C\cdot A^n\cdot n^{sn}$. Using Lemma 3.2, we have

$$|f^\wedge(j)|\leq\inf_{n\geq 0}\|f^{(n)}\|_\infty/|j|_u\leq C\cdot\inf_{n\geq 0}A^n\cdot n^{sn}|j|^{-n}$$

for any $j\in\boldsymbol{Z}$, $j\neq 0$. Now recall that $(x^{sx}, e^{-(s/e)r^{1/s}})$ is an associated pair i.e.,

$$\inf_{x>0}x^{sx}\cdot r^{-x}=e^{-(s/e)r^{1/s}}\qquad(r>0).$$

For a given $r>0$, there exist some $x_0>0$ and $n_0\in\boldsymbol{N}=\{0, 1, 2, \cdots\}$ such that

$$n_0<x_0\leq n_0+1\quad\text{and}\quad\inf_{x>0}x^{sx}\cdot(r/4^s)^{-x}=x_0^{sx_0}\cdot(r/4^s)^{-x_0},$$

The case of $n_0>0$: Using the following inequalities

$$n_0^{sn_0}\cdot r^{-(n_0+1)}\leq x_0^{sx_0}\cdot r^{-x_0}\qquad\text{for any}\quad r\geq 1$$

and

$$(n+1)^{n+1}\leq 4^n\cdot n^n\qquad\text{for any}\quad n\geq 0,$$

we have

$$(n_0+1)^{s(n_0+1)}\cdot r^{-(n_0+1)}\leq 4^{sn_0}\cdot n_0^{sn_0}\cdot r^{-(n_0+1)}\leq x_0^{sx_0}\cdot(r/4^s)^{-x_0}$$

so that

$$\inf_{n>0}n^{sn}\cdot r^{-n}\leq x_0^{sx_0}\cdot(r/4^s)^{-x_0}=e^{-(s/e)(r/4^s)^{1/s}}.$$

Letting $r=|j|/A$, we get

$$|f^\wedge(j)|\leq C\cdot\inf_{n\geq 0}A^n\cdot n^{sn}|j|^{-n}\leq C\cdot e^{-(s/e)|j/4^sA|^{1/s}}$$

for all $j\in\boldsymbol{Z}$; $|j|\geq A$. Defining

$$B=(s/4e)A^{-1/s}>0$$

and then

$$K=\mathrm{Max}(C, |f^\wedge(j)|\cdot e^{B|j|^{1/s}};\ j\in\boldsymbol{Z}\text{ and }|j|<A)>0,$$

we get the required inequality:

$$|f^\wedge(j)|\leq K\cdot e^{-B|j|^{1/s}}\qquad\text{for all}\quad j\in\boldsymbol{Z}.$$

The case of $n_0=0$: Denoting $0^0=1$ in the two inequalities at the beginning in the previous case, we can similarly get the required inequality.

Proof of (**R**). *If part.* Suppose that there are $B>0$ and $K>0$ such that

$$|f^\wedge(j)|\leq K\cdot e^{-B|j|^{1/s}}\qquad\text{for any}\quad j\in\boldsymbol{Z}.$$

We have to show that $f\in\mathcal{C}_s(\boldsymbol{T})$. Now,

$$|f^{(n)}(t)|=\left|\sum_{j\in\boldsymbol{Z}}f^\wedge(j)(ij)^n e^{ijt}\right|\leq 2K\cdot\sum_{j=1}^{j=\infty}|j|^n e^{-B|j|^{1/s}}$$

$$\leq 2K\cdot\int_0^\infty(x+1)^n e^{-Bx^{1/s}}dx.$$

The integral in the last term, denoted by I, is evaluated in the following form:

$$I \leq L \cdot 2^n \cdot \int_0^\infty x^n \cdot e^{-Bx^{1/s}} dx \qquad \text{for some integer} \quad L>0.$$

Changing variables: $x=y^s$ and then $y=(1/B)\cdot z$, the integral I is estimated as follows, i.e., for some integer $L>0$,

$$I \leq L \cdot 2^n \cdot \frac{s}{B^{s(n+1)}} \int_0^\infty z^{s(n+1)-1} e^{-z} dz = \frac{L \cdot 2^n \cdot s}{B^{s(n+1)}} \cdot \Gamma(s(n+1)).$$

Using the Stirling's formulas

$$\Gamma(z) \sim \sqrt{2\pi/z} \cdot z^z \cdot e^{-z} \qquad \text{for } z \text{ large}$$

and

$$(n+1)! \sim \sqrt{2\pi(n+1)} \cdot (n+1)^{n+1} \cdot e^{-(n+1)} \qquad \text{for } n \text{ large},$$

the integral I is more estimated as follows:

$$I \leq L \cdot 2^n \cdot (2\pi)^{(1-s)/2} \cdot s^{1/2} \cdot (n+1)^{-(1+s)/2} (s/B)^{s(n+1)} ((n+1)!)^s.$$

Using the fact $(n+1)! \leq 2^n \cdot n!$ and $|f^{(n)}(t)| \leq 2KI$ we have

$$|f^{(n)}(t)| \leq 2K \cdot L \cdot 2^n \cdot (2\pi)^{(1-s)/2} \cdot s^{1/2} \cdot (n+1)^{-(1+s)/2} \cdot ((s/B)^s)((s/B)^s)^n (2^s)^n (n!)^s$$

for large n. Since $(n+1)^{-(1+s)/2} \leq 1$, so by defining

$$A = 2 \cdot (2s/B)^s$$

and

$$C \geq 2KL \cdot (2\pi)^{(1-s)/2} \cdot s^{1/2} \cdot (s/B)^s$$

big enough to majorize $|f^{(n)}(t)|$ for small n, we get

$$\|f^{(n)}\|_\infty \leq C \cdot A^n (n!)^s \qquad \text{for all} \quad n \geq 0,$$

i.e., by definition, $f \in \boldsymbol{C}_s(\boldsymbol{T})$ as required.

Proof of (B). *Only if part.* Suppose $f \in \boldsymbol{C}_{(s)}(\boldsymbol{T})$. For a given $B>0$ define $A>0$ by solving $B=(s/4e)\cdot A^{-1/s}$. Then define K_B by C or sufficiently large positive number, then as in the case of (R), we have

$$|f^\wedge(j)| \leq K_B \cdot e^{-B|j|^{1/s}} \qquad \text{for all} \quad j \in \boldsymbol{Z}.$$

Proof of (B). *If part.* For a given $A>0$, define $B>0$ by solving $A=2(2s/B)^s$. Then, take $C_A \geq 2KL \cdot (2\pi)^{(1-s)/2} \cdot s^{1/2} (s/B)^s$ big enough as in the case of (R), we get

$$\|f^{(n)}\|_\infty \leq C_A \cdot A^n (n!)^s \qquad \text{for any} \quad n \geq 0.$$

Proposition 3.4. *$f \in \boldsymbol{C}_0(\boldsymbol{T})$ if and only if there exists $A>0$ such that*

$$|f^\wedge(j)| = 0 \qquad \textit{for all} \quad |j| > A,$$

that is, f is a trigonometric polynomial on $\boldsymbol{T}$:

$$f(t)=\sum_{|j|\leq A} f^{\wedge}(j)e^{ijt}.$$

Proof. From Definition 1.1, $f\in\mathcal{C}_0(\boldsymbol{T})$ implies that

$$\|f^{(n)}\|_\infty\leq C\cdot A^n \qquad \text{for all} \quad n\geq 0$$

with suitable $C>0$ and $A>0$. Since by Lemma 3.2,

$$|f^{\wedge}(j)|\leq\inf_{n\geq 0}\|f^{(n)}\|_\infty/|j|^n\leq C\cdot\inf_{n\geq 0}(A/|j|)^n$$

and

$$\inf_{n\geq 0}(A/|j|)^n=0 \text{ for } |j|>A, \quad \text{we get } |f^{\wedge}(j)|=0 \text{ for } |j|>A.$$

Conversely, if

$$f(t)=\sum_{|j|\leq A} f^{\wedge}(j)e^{ijt}$$

for some $A>0$, we have

$$|f^{(n)}(t)|\leq\sum_{|j|\leq A}|f^{\wedge}(j)|\cdot|j|^n\leq\max_{|j|\leq A}|f^{\wedge}(j)|(2A+1)A^n$$

that is, $f\in\mathcal{C}_0(\boldsymbol{T})$.

Proposition 3.5. *If $f\in\mathcal{C}_{(0)}(\boldsymbol{T})$, then $|f^{\wedge}(j)|=0$ for all $|j|>0$, that is, f is a constant function on $\boldsymbol{T}$.*

Proof. By Definition 1.2, if $f\in\mathcal{C}_{(0)}(\boldsymbol{T})$, for any $A>0$ there exists $C_A>0$ such that

$$\|f^{(n)}\|_\infty\leq C_A\cdot A^n \qquad \text{for any} \quad n\geq 0.$$

Take A such that $0<A<1$. Then, by Lemma 3.2,

$$|f^{\wedge}(j)|\leq C_A\cdot\inf_{n\geq 0}(A/|j|)^n \qquad \text{for any} \quad |j|\geq 1,$$

so that $|f^{\wedge}(j)|=0$ for any $|j|\geq 1$. That is, $f(t)=f^{\wedge}(0)$ is a constant function.

Proposition 3.6. *For any $s<0$,*

$$\mathcal{C}_s(\boldsymbol{T})=\mathcal{C}_{(s)}(\boldsymbol{T})=\boldsymbol{C}$$

the space of constant functions on $\boldsymbol{T}$.

Proof. From Definitions 1.1 and 1.2, we have for any $s<s'$

$$\mathcal{C}_{(s)}(\boldsymbol{T})\subset\mathcal{C}_s(\boldsymbol{T})\subset\mathcal{C}_{(s')}(\boldsymbol{T}).$$

From the preceeding proposition, $\mathcal{C}_{(0)}(\boldsymbol{T})=\boldsymbol{C}$, hence we get the results.

§4. Ultradistributions $\mathcal{C}_s(\boldsymbol{T})^\wedge$, $\mathcal{C}_{(s)}(\boldsymbol{T})^\wedge$ for any s, $0<s<\infty$.

From the theorem 3.3, the spaces $\mathcal{C}_s(\boldsymbol{T})$, $\mathcal{C}_{(s)}(\boldsymbol{T})$ are identified with sequence spaces over $\boldsymbol{Z}$. After Köthe [5] we define:

Definition 4.1. Let E be a sequence space over $\boldsymbol{Z}$, i.e., $E\subset \boldsymbol{C}^{\boldsymbol{Z}}$. The *dual* ($\alpha$-*dual* in the sense of Köthe) is defined as

$$E^\wedge=\{v\in \boldsymbol{C}^{\boldsymbol{Z}} \mid \sum_j |v_j|\,|u_j|<\infty \text{ for all } u\in E\}.$$

E is called *perfect* if $E^{\wedge\wedge}=E$. The *normal topology* on a sequence space E is a locally convex topology defined by the family of semi-norms of the form:

$$p_v(u)=\sum_j |v_j|\,|u_j|$$

for all $v\in E^\wedge$.

Definition 4.2. We identify $\mathcal{C}_s(\boldsymbol{T})$, $\mathcal{C}_{(s)}(\boldsymbol{T})$ with sequence spaces using Fourier coefficients of their elements. We consider normal topologies on them. Then, $v\in\mathcal{C}_s(\boldsymbol{T})^\wedge$ (resp. $v\in\mathcal{C}_{(s)}(\boldsymbol{T})$) if and only if it defines a linear continuous functional on $\mathcal{C}_s(\boldsymbol{T})$ (resp. on $\mathcal{C}_{(s)}(\boldsymbol{T})$). More precisely, if we define the *Fourier coefficient* $v^\wedge(j)$ as usual by

$$v^\wedge(j)=v(e^{ijt}),\qquad e^{ijt}\in\mathcal{C}_s(\boldsymbol{T}) \text{ (resp. } \in\mathcal{C}_{(s)}(\boldsymbol{T}))$$

then (e.g. [5], [9]):

Proposition 4.3. *$v\in\mathcal{C}_s(\boldsymbol{T})^\wedge$ if and only if, for any $f\in\mathcal{C}_s(\boldsymbol{T})$,*

$$\sum_j |v_j|\,|f^\wedge(j)|<\infty,$$

i.e., $\mathcal{C}_s(\boldsymbol{T})^\wedge$ is the dual of $\mathcal{C}_s(\boldsymbol{T})$ as a sequence space. Similarly for $\mathcal{C}_{(s)}(\boldsymbol{T})$ and $\mathcal{C}_{(s)}(\boldsymbol{T})^\wedge$.

We now prove another main result:

Theorem 4.4. *For any s, $0<s<\infty$,*
(R): *$v\in\mathcal{C}_s(\boldsymbol{T})^\wedge$ if and only if for any $B>0$, there exists $K_B>0$ such that*

$$|v_j|\le K_B\cdot e^{B|j|^{1/s}} \qquad \textit{for all}\quad j\in\boldsymbol{Z}.$$

(B): *$v\in\mathcal{C}_{(s)}(\boldsymbol{T})^\wedge$ if and only if there are $B>0$ and $K>0$ such that*

$$|v_j|\le K\cdot e^{B|j|^{1/s}} \qquad \textit{for all}\quad j\in\boldsymbol{Z}.$$

As a first step, we prove:

Lemma 4.5. *$v\in\mathcal{C}_s(\boldsymbol{T})^\wedge$ if and only if for any $B>0$,*

$$\sum_j |v_j| e^{-B|j|^{1/s}} < \infty .$$

Proof. If $v \in \mathcal{C}_s(T)^\wedge$, for any $f \in \mathcal{C}_s(T)$ we have

$$\sum_j |v_j| \, |f^\wedge(j)| < \infty .$$

From Theorem 3.3, for any $B>0$,

$$f(t) = \sum_j e^{-B|j|^{1/s}} e^{ijt} \in \mathcal{C}_s(T)$$

hence,

$$\sum_j |v_j| e^{-B|j|^{1/s}} < \infty .$$

Let, conversely, $f \in \mathcal{C}_s(T)$. Then there are some $B>0$ and $K>0$ such that

$$|f^\wedge(j)| \leq K \cdot e^{-B|j|^{1/s}} \qquad \text{for any} \quad j \in \boldsymbol{Z}.$$

Hence

$$\sum_j |v_j| \, |f^\wedge(j)| \leq K \cdot \sum_j |v_j| e^{-B|j|^{1/s}} < \infty .$$

We now prove Theorem 4.4.

Proof of (R). *If part.* Let $B>0$ be given. Define $B' = B/2$ and $K_{B'}>0$, so that

$$|v_j| \leq K_{B'} e^{B'|j|^{1/s}} \qquad \text{for any} \quad j \in \boldsymbol{Z}.$$

Then

$$\sum_j |v_j| e^{-B|j|^{1/s}} \leq K_{B'} \cdot \sum_j e^{-B' \cdot |j|^{1/s}} .$$

Since

$$e^{-B'|j|^{1/s}} \leq |j|^{-2} \qquad \text{for } |j| \text{ large},$$

we have for some $K>0$,

$$\sum_j |v_j| e^{-B|j|^{1/s}} \leq K_{B'} \cdot K \cdot \sum_j |j|^{-2} < \infty ,$$

that is $v \in \mathcal{C}_s(T)^\wedge$ by the preceeding lemma.

Proof of (R). *Only if part.* Let $\sum_j |v_j| e^{-B|j|^{1/s}} < \infty$ for any $B>0$. Take $j_0 > 0$ so that for any j, $|j| \geq j_0$,

$$|v_j| e^{-B|j|^{1/s}} \leq 1, \qquad \text{i.e.,} \quad |v_j| \leq e^{B|j|^{1/s}} .$$

Accordingly, there exists some $K_B>0$ such that

$$|v_j| \leq K_B \cdot e^{B|j|^{1/s}} \qquad \text{for any} \quad j \in \boldsymbol{Z}.$$

For the proof of (B), it is enough to show:

Lemma 4.6. *$v \in C_{(s)}(T)^\wedge$ if and only if, for some $B>0$*

$$\sum_j |v_j| e^{-B|j|^{1/s}} < \infty .$$

Proof. For any $B>0$, we consider so-called an *echelon space*:

$$E_B=\{a=(a_j)\,|\,\sum_j|a_j|e^{-B|j|^{1/s}}<\infty\}.$$

Since, $E_B\simeq l^1$ by a diagonal transformation, its dual space $E_B{}^\wedge\simeq l^\infty$ is given by

$$E_B{}^\wedge=\{b=(b_j)\,|\,\text{for some } K>0,\ |b_j|\leq Ke^{-B|j|^{1/s}}\}.$$

By Theorem 3.3, we know that

$$f\in \mathcal{C}_{(s)}(\boldsymbol{T}) \text{ if and only if } (f^\wedge(j))\in\bigcap_{B>0}E_B{}^\wedge.$$

We now use Köthe's theory ([5], § 30.8 (1)) relating echelon and coechelon spaces to obtain the dual space:

$$v\in\mathcal{C}_{(s)}(\boldsymbol{T})^\wedge \text{ if and only if } (v_j)\in\bigcup_{B>0}E_B$$

i.e., for some $B>0$,

$$\sum_j|v_j|e^{-B|j|^{1/s}}<\infty.$$

Remarks. (1) Theorem 4.4 shows in particular that for any s, $0<s<\infty$, the spaces $\mathcal{C}_s(\boldsymbol{T})$, $\mathcal{C}_{(s)}(\boldsymbol{T})$ are perfect in the sense of Köthe. $\mathcal{C}_{(0)}(\boldsymbol{T})=\boldsymbol{C}$ is not perfect but $\mathcal{C}_0(\boldsymbol{T})$=the space of trigonometric polynomials $\simeq\boldsymbol{C}^{(\boldsymbol{Z})}$=the space of all finite sequences is perfect with the dual space $\mathcal{C}_0(\boldsymbol{T})^\wedge$=the space of all formal trigonometric series $\simeq\boldsymbol{C}^{\boldsymbol{Z}}$.

(2) $\mathcal{C}_1(\boldsymbol{T})^\wedge=\mathcal{B}(\boldsymbol{T})$ is the space of Sato-hyperfunctions on $\boldsymbol{T}$. In fact, $v\in\mathcal{C}_1(\boldsymbol{T})^\wedge$ if and only if for any $B>0$ there exists $K_B>0$ such that

$$|v_j|\leq K_Be^{B|j|}\qquad\text{for any }\ j\in\boldsymbol{Z}.$$

Consider the power series $v=\sum_j v_jz^j$ with $z\in\boldsymbol{C}$ then

$$\limsup_{|j|\to\infty}\sqrt[|j|]{|v_j|}\leq\lim_{|j|\to\infty}\sqrt[|j|]{K_B}\cdot e^B\qquad\text{for any }\ B>0,$$

that is, $\limsup\sqrt[|j|]{|v_j|}\leq 1$. We conclude that the series $v^+=\sum_{j\geq 0}v_jz^j$, $v^-=\sum_{j<0}v_jz^j$ are convergent inside and outside of the unit circle respectively and v represents a *boudary value* i.e., a Sato-hyperfunction.

(3) $\mathcal{C}_{(1)}(\boldsymbol{T})^\wedge$ may be considered the space of Morimoto's cohomological ultra-distributions [6]. In fact, the series $v^+=\sum_{j\geq 0}v_jz^j$ converges for z, $|z|\leq e^{-B}<1$ and the series $v^-=\sum_{j<0}v_jz^j$ converges for z, $|z|\geq e^B>1$ for some $B>0$ with $v=v^++v^-$ for any $v\in\mathcal{C}_{(1)}(\boldsymbol{T})^\wedge$.

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