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SOME INFINITE SERIES ASSOCIATED WITH THE RIEMANN ZETA FUNCTION

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1. Introduction.

For the Riemann zeta function $\zeta(s)$ defined, when $\mathrm{Re}(s)>1$, by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^{s}}=\frac{1}{1-2^{-s}}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{s}}, \tag{1.1}$$

Landau's formula (*cf.* [1, p. 274, Equation (3)]; see also [5, p. 33, Equation (2.14.1)])

$$\zeta(s)=1+\frac{1}{s-1}-\sum_{n=1}^{\infty}\frac{(s)_{n}}{(n+1)!}\{\zeta(s+n)-1\} \tag{1.2}$$

is capable of providing the analytic continuation of $\zeta(s)$ over the whole s-plane. Here, for convenience,

$$(s)_{0}=1 \quad \text{and} \quad (s)_{n}=s(s+1)(s+2)\cdots(s+n-1), \qquad n=1, 2, 3, \cdots. \tag{1.3}$$

Another formula, which can be used in a similar way, is attributed to Ramaswami (*cf.* [3, p. 166]; see also [5, p. 33, Equation (2.14.2)]):

$$(1-2^{1-s})\zeta(s)=\sum_{n=1}^{\infty}\frac{(s)_{n}}{n!}\frac{\zeta(s+n)}{2^{s+n}}. \tag{1.4}$$

Motivated by these well-known results [(1.2) and (1.4)] in the theory of the Riemann zeta function $\zeta(s)$, which obviously is meromorphic everywhere in the s-plane except for a simple pole at $s=1$ (with residue 1), Singh and Verma [4] have recently derived the following infinite series involving $\zeta(s)$:

$$\zeta(s)=\frac{1}{2}+\frac{1}{s-1}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n(s)_{n+1}}{(n+2)!}\zeta(s+n+1), \qquad \mathrm{Re}(s)<1, \tag{1.5}$$

and

$$\zeta(s)=1+\frac{1}{2^{s+1}}\frac{s+3}{s-1}+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n(s)_{n+1}}{(n+2)!}\{\zeta(s+n+1)-1\}. \tag{1.6}$$

The proofs of (1.5) and (1.6) by Singh and Verma [4, Sections 2 and 3] depend rather heavily upon the integral representation [5, p. 14, Equation (2.1.4)]

$$\zeta(s)=\frac{1}{2}+\frac{1}{s-1}-s\int_{1}^{\infty}\left(x-[x]-\frac{1}{2}\right)\frac{dx}{x^{s+1}}, \qquad \mathrm{Re}(s)>-1. \tag{1.7}$$

In our attempt to give relatively simple proofs of (1.5) and (1.6), without using the integral representation (1.7), we are led naturally to an interesting unification (and generalization) of (1.5) and (1.6) involving the generalized (Hurwitz's) zeta function defined usually by (*cf.* [5, p. 36])

$$(1.8) \qquad \zeta(s, a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}, \qquad \mathrm{Re}(s)>1; \quad a\neq 0, -1, -2, \cdots,$$

so that, obviously,

$$(1.9) \qquad \zeta(s, 1)=\zeta(s), \quad \zeta(s, 2)=\zeta(s)-1, \quad \zeta\left(s, \frac{1}{2}\right)=(2^s-1)\zeta(s).$$

The elementary techniques employed here are shown to apply also to the derivation of numerous other results for $\zeta(s, a)$ including, for example, some useful analogues of (1.2) and (1.4).

2. Derivation of the main result.

The derivation of our unification (and generalization) of (1.5) and (1.6) is based simply upon the familiar binomial expansion:

$$(2.1) \qquad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}z^n=(1-z)^{-\lambda}, \qquad |z|<1.$$

Indeed it follows readily from (2.1) and the definition (1.8) that (*cf.*, *e.g.*, [6, p. 90, Equation (1)])

$$(2.2) \qquad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\zeta(\lambda+n, a)t^n=\zeta(\lambda, a-t), \qquad |t|<|a|,$$

which holds true, by the analytic continuation of $\zeta(s, a)$, for all values of $\lambda\neq 1$.

Now replace the summation index in (2.2) by $n+2$, set $\lambda=s-1$, and divide both sides of the resulting equation by t^2. We thus observe from (2.2) that

$$(2.3) \qquad \sum_{n=0}^{\infty}\frac{(s-1)_{n+2}}{(n+2)!}\zeta(s+n+1, a)t^n$$
$$=\{\zeta(s-1, a-t)-\zeta(s-1, a)\}t^{-2}-(s-1)\zeta(s, a)t^{-1}, \qquad 0<|t|<|a|.$$

Differentiating both sides of (2.3) with respect to t, and noticing from the definitions (1.3) and (1.8) that

$$(2.4) \qquad (s-1)_{n+2}=(s-1)(s)_{n+1}, \qquad \frac{\partial}{\partial t}\{\zeta(s-1, a-t)\}=(s-1)\zeta(s, a-t),$$

we have

$$(2.5) \qquad \sum_{n=1}^{\infty}\frac{n(s)_{n+1}}{(n+2)!}\zeta(s+n+1, a)t^{n-1}$$
$$=\{\zeta(s, a-t)+\zeta(s, a)\}t^{-2}-\frac{2}{s-1}\{\zeta(s-1, a-t)-\zeta(s-1, a)\}t^{-3}, \quad 0<|t|<|a|.$$

For $t=-1$, (2.5) readily yields the desired unification (and generalization) of (1.5) and (1.6) in the form:

$$\zeta(s,\ a)=a^{-s}\left(\frac{1}{2}+\frac{a}{s-1}\right)+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n(s)_{n+1}}{(n+2)\,!}\zeta(s+n+1,\ a), \tag{2.6}$$

provided that the series converges.

3. Applications.

In the special case of our main result (2.6) when $a=1$, the series converges if $\operatorname{Re}(s)<1$, and we immediately obtain (1.5). Furthermore, in view of the first two identities in (1.9), our formula (2.6) with $a=2$ is precisely the known result (1.6).

It may be of interest to remark here that alternative proofs of the well-known results (1.2) and (1.4), based upon the integral representation [5, p. 18, Equation (2.4.1)]

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{x^{s-1}}{e^{x}-1}dx, \qquad \operatorname{Re}(s)>1, \tag{3.1}$$

were given by Menon [2]. As a matter of fact, both (1.2) and (1.4) can be deduced fairly easily from (2.2).

Replacing the summation index n in (2.2) by $n+1$, and setting $\lambda=s-1$, we have

$$\sum_{n=0}^{\infty}\frac{(s)_n}{(n+1)\,!}\zeta(s+n,\ a)t^{n+1}=\frac{1}{s-1}\{\zeta(s-1,\ a-t)-\zeta(s-1,\ a)\}, \quad |t|<|a|, \tag{3.2}$$

where we have made use of the first identity in (2.4).

By virtue of the definition (1.8), (3.2) with $t=1$ assumes the form:

$$\zeta(s,\ a)=\frac{(a-1)^{1-s}}{s-1}-\sum_{n=1}^{\infty}\frac{(s)_n}{(n+1)\,!}\zeta(s+n,\ a), \tag{3.3}$$

which, in view of the second identity in (1.9), yields Landau's formula (1.2) for $a=2$. On the other hand, (3.2) with $t=1/2$ (and s replaced by $s+1$) or (2.2) with $t=1/2$ (and $\lambda=s$) similarly yields

$$\zeta(s,\ 2a-1)-2^{1-s}\zeta(s,\ a)=\sum_{n=1}^{\infty}\frac{(s)_n}{n\,!}\frac{\zeta(s+n,\ a)}{2^{s+n}}, \tag{3.4}$$

which, in view of the first identity in (1.9), leads us immediately to Ramaswami's formula (1.4) upon setting $a=1$.

For $t=-1$, (3.2) yields

$$\zeta(s,\ a)=\frac{a^{1-s}}{s-1}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(s)_n}{(n+1)\,!}\zeta(s+n,\ a), \tag{3.5}$$

and (3.2) with $t=-1/2$ (and s replaced by $s+1$) or (2.2) with $t=-1/2$ (and $\lambda=s$) gives

$$\zeta(s, 2a)-2^{1-s}\zeta(s, a)=-\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(s)_n}{n\,!}\frac{\zeta(s+n, a)}{2^{s+n}}. \tag{3.6}$$

Formulas (3.5) with $a=2$ and (3.6) with $a=1$ provide interesting analogues of (1.2) and (1.4), respectively; in fact, this indicated analogue of (1.4) [that is, (3.6) with $a=1$] was also given by Ramaswami [3, p. 166]. It is not difficult to deduce (2.6) as a natural consequence of (3.5).

Numerous other consequences of the general results (2.2), (2.6), and (3.2) can be deduced by suitably specializing the parameter t in a manner detailed above.

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