YOKOHAMA MATHEMATICAL
JOURNAL VOL. 35, 1987

EXISTENCE AND STABILITY OF MILD SOLUTIONS OF SEMILINEAR DIFFERENTIAL EQUATIONS IN LOCALLY CONVEX SPACES

By

S. B. AGASE*

(Received August 14, 1986)

Abstract: A semilinear differential equation of the type

$$\dot{u}=Au+f(t,u), \quad u(a)=z \tag{*}$$

is considered in locally convex space X, for the existence and stability of its mild solutions. A is assumed to be the generator of an equicontinuous C_0-semigroup of linear operators and the function $f(t,u)$ satisfies certain condition in terms of the measure of noncompactness. Existence and stability results are obtained via fixed point theorem. Examples are given to illustrate an abstract theory developed here.

1. Introduction.

The main aim in this study is to develop abstract theory of existence and stability of mild solutions of semilinear differential equations of the form

$$\dot{u}=Au+f(t, u) \tag{1.1}$$

$$u(0)=z \tag{1.2}$$

where A is closed densely defined operator on a locally convex topological vector space (LCTVS) X, $z\in X$, $f: \boldsymbol{R}^{+}\times X\to X$ and $u: \boldsymbol{R}^{+}\to X$. The motivation behind such a study in abstract spaces is mainly because of its successful applications in the study of partial differential equations (for example see [5], [6], [9]). Our main tool is the fixed point theorem via the measure of noncompactness in LCTVS. In this approach we combine the measure of noncompactness of a bounded set $B\subset X$ and the images $S(t)B$, $t\geqq 0$ under the C_0-semigroup $\{S(t): t\geqq 0\}$ generated by the operator A in (1.1). As per our knowledge, this approach seems to be new one. In Agase and Raghavendra [2] we have studied (1.1), (1.2) in case X is a Banach Space. In this work we generalise some results in

Keywords and phrases: Mild solution, Measure of noncompactness, C_0-semigroup, Q-condensing. AMS (MOS) Subject classification (1980): Primary 34 G 20; Secondary 58 D 25; 35 R 15.

(*) On leave from: Department of Mathematics, Science College, NANDED 431602 (INDIA).

[2] and add some existence and stability results obtained on the lines of Agase [1] and Yuasa [11]. The generalization does not seem to be trivial because of complicated structure of LCTVS. For example semigroup $\{S(t),\ t\geqq 0\}$ of translations is a C_0-semigroup of contractions generated by $A\equiv d/dx$ in case of a Banach space but it is not so in LCTVS, (see [8]).

In Section 2, definitions and preliminary results are given. Existence results are established in Section 3. Conditions sufficient to satisfy our hypothesis on A are also obtained. Stability theorems are given in Section 4. In Section 5, two examples of LCTVS are given where in our results may have applications. The purpose of these examples is theoritical and as such it may not have any bearing on physical problems.

2. Preliminaries.

Let X be a Hausdorff locally convex topological vector space (LCTVS) and $\mathcal{P}$ denote a family of seminorms inducing the given topology on X. At times we denote by $\mathcal{U}$, (in place of $\mathcal{P}$), the fundamental system of neighbourhoods of $0\in X$. Let $S=\{S(t):\ t\geqq 0\}$ denote a C_0-semigroup of linear operators on X, that is, for each $t\geqq 0$, $S(t):\ X\rightarrow X$ is a bounded linear operator with,

$$S(t)S(h)=S(t+h)\,;\quad S(0)=I \tag{2.1}$$

$$\lim_{h\to t} S(h)x=S(t)x\,,\quad t\geqq 0\,,\quad x\in X \tag{2.2}$$

S is called a C_0-semigroup of contractions, (or $\mathcal{P}$-contraction C_0-semigroup), if for every $p\in\mathcal{P}$, $x\in X$, and $t\geqq 0$

$$p(S(t)x)\leqq p(x)\,. \tag{2.3}$$

S is said to be an equicontinuous semigroup of class C_0 if for any $p\in\mathcal{P}$, there exists a $q\in\mathcal{P}$ such that

$$p(S(t)x)\leqq q(x)\qquad \text{for all}\quad t\geqq 0\,,\ x\in X. \tag{2.4}$$

The generator A of $\{S(t):\ t\geqq 0\}$ is a linear operator with domain $D(A)$,

$$D(A)=\left\{x\in X:\ \lim_{h\to 0+}\frac{1}{h}[S(h)-I]x \text{ exists in } X\right\},$$

and

$$Ax=\lim_{h\to 0+}\frac{1}{h}[S(h)x-x]\qquad \text{for all}\quad x\in D(A)\,.$$

It is well known that $D(A)$ is dense in X.

Let $\boldsymbol{R}^+ : [0,\infty)$, $I=[a,\ a+T]$, $a\in\boldsymbol{R}^+$, $T>0$. For a function $u:I\rightarrow X$, $\dot{u}\equiv\frac{du}{dt}$ is the derivative in the usual sense and $\int_a^t u(s)ds$ is the integral in the Riemann sense. Let D be an open subset of X, $f:I\times D\rightarrow X$ and $z\in D$. For any A,

$B\subset X$, $\mathrm{cl}\,B$(or $\bar{B}$) and $\mathrm{co}\,B$ denote closure of B, convex hull of B respectively, and $A+B$: $\{x+y: x\in A, y\in B\}$. We consider the semilinear differential equation (SDE)

$$\dot{u}=Au+f(t, u), \qquad u(a)=z \tag{2.5}$$

where A is the generator of an equicontinuous semigroup of class C_0.

2.1. Definition. A function $u: I\to X$ is said to be a strong solution of SDE (2.5) if $u(t)\in D\cap D(A)$ for all $t\in I$, u is differentiable on I and u satisfies SDE (2.5) on I. It is easy to see that a solution of SDE (2.5) satisfies the following integral equation

$$u(t)=S(t-a)z+\int_a^t S(t-s)f(s, u(s))ds \tag{2.6}$$

but the converse need not be true,

2.2. Definition. A continuous function $u: I\to D$ is called a mild solution of SDE (2.5) if u satisfies the integral equation (2.6).

We use the notion of measure of noncompactness in LCTVS defined by Yuasa [11] which is the modified form of the definition introduced by S. Reich [10].

2.3. Definition ([10]). For a bounded subset B of X, the measure of noncompactness, denoted by $M(B)$, is defined as

$$M(B): \{U\in\mathcal{U}: B\subset K+U \text{ for some precompact set } K\subset X\}.$$

2.4. Definition ([11]). For a bounded $B\subset X$, the measure of non-precompactness, $Q(B)$, is defined as ($\mathcal{U}$ being fixed):

$$Q(B): \{U\in\mathcal{U}: \text{For any } \varepsilon>0, \text{ there exists a precompact set } K\subset X, \text{ such that } B\subset K+(1+\varepsilon)U\},$$

It is obvious that $M(B)\subset Q(B)$. These measures satisfy the properties listed in the following:

2.5. Theorem. *Let X be a complete LCTVS and A, $B\subset X$ be bounded. Then the following hold.*

a) *If $A\subset B$ then $Q(A)\supset Q(B)$*
b) $Q(A\cup B)=Q(A)\cap Q(B)$
c) *$Q(A)=\mathcal{U}$ if and only if A is precompact*
d) $Q(A)=Q(\mathrm{cl}\,A)$
e) $Q(B)=Q(\mathrm{co}\,B)=Q(\mathrm{cl}\,\mathrm{co}\,B)$
f) *If $a\geqq b\geqq 0$ then $Q(bA)\supset Q(aA)$*

g) $Q(A)=Q(\bigcup_{0\leq t\leq 1} tA)$

h) $Q(A+B)=Q(A)$ *for every precompact set* $B\subset X$.

Proof. (a)-(f) are proved in Yuasa [11] and (g) in Agase [1]. We prove (h).

Let A be bounded and B be precompact subset of X. If $U\in Q(A+B)$ then for each $t>0$ there exists a precompact subset $K_t\subset X$ such that $A+B\subset K_t+(1+t)U$. Since K_t-B is precompact it follows that $U\in Q(A)$ and hence $Q(A+B)\subset Q(A)$. Reverse inclusion can also be realised in the same way and (h) follows.

We denote the class of all functions F from X into Y (Y—LCTVS), which map bounded subsets of X into bounded subsets of Y, as $B(X, Y)$. Further if F is continuous also then we write $F\in BC(X, Y)$.

2.6. Difinition. An operator $F: X\to X$ is said to be Q-condensing if $F\in B(X, X)$ and $Q(B)\subsetneqq Q(F(B))$ for every bounded $B\subset X$. A function $f: \boldsymbol{R}^+\times X\to X$ is said to be Q-condensing if $f\in B(\boldsymbol{R}^+\times X, X)$ and for any closed interval $I\subset \boldsymbol{R}^+$ and bounded $B\subset X$, $Q(B)\subsetneqq Q(f(I\times B))$.

2.7. Theorem. *Let K be a nonempty closed convex subset of a complete Hausdorff LCTVS. Suppose $F: K\to K$ is continuous and has bounded range. If F is Q-condensing then F has a fixed point in K.*

For the details of the proof see Istratescu ([4] page 185-187).

Let $Y=C[I, X]$ the space of all continuous X-valued functions on I. We consider the locally convex topology on Y defined by the family of seminorms $\{q_\alpha\}$, where

$$q_\alpha(u)=\sup_{t\in I} p_\alpha(u(t)), \qquad \text{for all} \quad u\in Y \text{ and } p_\alpha\in\mathcal{P}.$$

2.8. Lemma. *Let $f\in C[I\times X, X]$ and G be defined on Y by*

$$G(y)(t)=f(t, y(t)) \qquad \textit{for all} \quad y\in Y \textit{ and } t\in I.$$

Then

(i) $G: Y\to Y$

(ii) $G\in C(Y, Y)$

(iii) *G and rf, $(0\leqq r\leqq 1)$, are Q-condensing whenever f is so.*

Proof. For (i) and (ii) we refer to Millionscikov [7]. (iii) is proved in [1] when f is M-condensing. For completeness we give the proof in the present situation.

Let $B\subset Y$ be bounded. So that $B(I)$ is bounded subset of X and then $f(I, B(I))$ is bounded in X. Since f is Q-condensing we choose $U_0\in\mathcal{U}$ such that

$$U_0 \in Q(f(I, B(I))) \text{ but } U_0 \notin Q(B(I)).$$

Let p_0 denote the Minkowski functional of U_0 ($p_0 \in \mathcal{P}$) and q_0 the corresponding seminorm on Y. Set $U_0' = \{y \in Y : q_0(y) \leqq 1\}$. The following claims establish that G is Q-condensing.

Claim 1. $U_0' \in Q(G(B))$.

If the claim is not true then there exists $\delta > 0$ such that for any precompact $K \subset Y$, $G(B) \not\subset K + (1+\delta)U_0'$. But for this $\delta > 0$, by the choice of U_0' there is a precomepact $K_0 \subset X$ with $f(I, B(I)) \subset K_0 + (1+\delta)U_0$ and $B(I) \not\subset K_0 + (1+\delta)U_0$. Let $g \in C[I, R]$ be such that $g(t) = 1 - t \in I$ and write $K_0' = \{xg : x \in K_0\}$. Then K_0' is precompact subset of Y. Since $G(B) \not\subset K_0 + (1+\delta)U_0'$, there is a $y \in B$ with $G(y) \notin K_0 + (1+\delta)U_0'$. Thus for some $t \in I$,

$$G(y)(t) = f(t, y(t)) \neq xg(t) + (1+\delta)u_0'(t)$$

for all $x \in K_0$ and $u_0' \in U_0'$. Since the set $\hat{U}_0 : \{u_0 g : u_0 \in U_0\} \subset U_0'$, it follows that $G(y)(t) \neq x + (1+\delta)u_0$ for all $x \in K_0$ and $u_0 \in U_0$. But this contradicts the choice of U_0 and K_0.

Claim 2. $U_0' \notin Q(B)$.

From the choice of U_0, it is clear that there exists $d > 0$ and a precompact set $K_0 \subset X$ with $G(I, B(I)) \subset K_0 + (1+d)U_0$ but $B(I) \not\subset K + (1+d)U_0$ for any precompact $K \subset X$. Now if the claim is not true then for this $d > 0$, there is a precompact set $K' \subset Y$ such that $B \subset K' + (1+d)U_0'$ which further shows that $B(I) \subset K'(I) + (1+d)U_0'(I)$. Since $U_0'(I) \subset U_0$ and $K'(I)$ is precompact we arrive at a contradiction to the choice of U_0.

For $0 < r \leqq 1$, rf is Q-condensing follows easily and hence the lemma.

2.9. Lemma. *Assume that* $\{S(t) : t \geqq 0\}$ *is an equicontinuous semigroup of class* C_0 *on* X *such that, for any bounded* $B \subset X$

$$Q\Big(\bigcup_{0 \leq t \leq 1} S(t)B\Big) = Q(B) \tag{2.7}$$

Let $f \in C[I \times X, X]$ *and be* Q*-condensing. Define a function* F *on* $I \times Y$ *by*

$$F(t, y)(s) = \begin{cases} S(t-s)f(s, y(s)) & a \leqq s \leqq t \\ f(s, y(s)), & t \leqq s \leqq T \end{cases} \tag{2.8}$$

Then $F \in C[I \times Y, Y]$ *and* F *is* Q*-condensing on* $I \times Y$.

Proof. Let $B \subset Y$ be bounded. Then $F(I, B)$ is bounded and $F(I, B) \subset \bigcup_{0 \leq \lambda \leq 1} S(\lambda)G(B)$ where G is as in Lemma 2.8. By using (2.7) and Q-condensing property of G we have $Q(F(I, B)) \supset Q(G(B)) \supsetneqq Q(B)$ and hence the lemma.

2.10. Lemma. *Let $\{S(t)\}$, f and F be as in Lemma* 2.9. *Then for all $t\in I$ and $y\in Y_t=C[[a, t], X]$,*

$$\int_a^t F(t, y)(s)ds\in(t-a)\,\mathrm{cl\,co}\,(\{F(t, y)(s):\ s\in(a, t]\}).$$

Proof. Obvious from the definition of Riemann integral.

2.11. Lemma. *Let $\{S(t)\}$, f and F be as in Lemma* 2.9. *Define the operator T on Y by*

$$Ty(t)=S(t-a)z+\int_a^t S(t-s)f(s, x(s))ds$$

for all $t\in[a, a+T]$, $y\in Y$ and z fixed in X. Then

(i) *$T\in BC[Y, Y]$*

(ii) *T is Q-condensing on Y.*

Proof. (i) follows easily.

(ii) Let $B\subset Y$ be bounded. Then $B(I)$ is bounded in X. For each $y\in B$ and $t\in I$,

$$Ty(t)\in S(t-a)z+(t-a)\,\mathrm{cl\,co}\,(\{F(t, y)(s):\ s\in(a, t]\}).$$

Hence

$$T(B)(I)\supset\bigcup_{t\in I}S(t-a)z+\bigcup_{t\in I}(t-a)\,\mathrm{cl\,co}\,(\{F(t, B)(I)\}).$$

Since $\bigcup_{t\in I}S(t-a)z$ is compact, it follows that

$$Q(T(B)(I)))\supset Q(\bigcup_{0\le\lambda\le1}\lambda\,\mathrm{cl\,co}\,(\{F(I, B)(I)\})=Q(F(I, B)(I)).$$

Following the same procedure as in Lemma 2.9 and using Q-condensing property of F we obtain that

$$Q(T(B))\supset Q(F(I, B))\supsetneqq Q(B).$$

3. Existence Theorems.

3.1. Theorem. *Suppose that*

(i) *A is the infinitesimal generator of an equicontinuous semigroup $\{S(t): t\geqq0\}$ of class C_0 satisfying condition* (2.7).

(ii) *$I=[a, a+T]$, $0<T\leqq1$, $a\in\boldsymbol{R}^+$, $f: I\times X\to X$ is continuous and there exists $0<\alpha\leqq T$ and $c>0$ such that cf is Q-condensing on $[a, a+\alpha]\times X$.*

(iii) *There is $U_0\in\mathcal{U}$ such that $S(\lambda)f(t, x)\in M_0U_0$ for all $(t, x)\in I\times X$, $\lambda\in[0, 1]$ and some $M_0>0$.*

Then there exists β, $0<\beta\leqq T$ such that for each $z\in X$, SDE (2.5) *has a mild solution on $[a, a+\beta]$.*

Proof. Let $\beta=\min(\alpha, c)$ and $J=[a, a+\beta]$. We note that since cf is Q-condensing on $[a, a+\alpha]\times X$, Lemma 2.5(f) shows that βf is Q-condensing. It is obvious that $\beta f\in B[J\times X, X]$. Assumption (2.7) now shows that $\bigcup_{0\le\lambda\le 1} S(\lambda)\beta f(J\times B)$ is bounded for every bounded subset B of X. Further by assumption (iii), $p_0(\bigcup_{0\le\lambda\le 1} S(\lambda)f(t, x))\leqq M_0$ where p_0 is the Minkowski functional corresponding to U_0.

If $u(t)$ is a mild solution of SDE (2.5) on J then for $t, t'\in J$,

$$u(t)-u(t')\in[S(t-a)-S(t'-a)]z+|t-t'|\operatorname{cl}\operatorname{co}(\{\bigcup_{\lambda} S(\lambda)f(J, u(J))\}).$$

Here we have used the fact that for any convex $D\subset X$ and $\alpha, \beta>0$, $\alpha D+\beta D=(\alpha+\beta)D$. Then $u(t)$ satisfies,

$$p_0(u(t)-u(t'))\leqq p_0([S(t-a)-S(t'-a)]z)+|t-t'|M_0 \tag{3.1}$$

From this observation we try to investigate a solution in the subset H of Y given by H: $\{u\in C[J, X]: u(a)=z$ and u satisfies inequality (3.1)$\}$. Then H is nonempty, closed and convex. Define the operator $T: H\to H$ as in Lemma 2.11. Then T is continuous, T has bounded range and T is Q-condensing. Theorem 2.7 then assures the existence of a fixed point of T in H which is a required mild solution on J.

3.2. Theorem. *If the hypothesis of Theorem* 3.1 *is true with* $c=1$ *and* $\alpha=T$ *then for each* $z\in X$, *SDE* (2.5) *has a mild solution existing on* I.

3.3. Theorem. *Let* $I_1=[a, a+\beta]$, $0<\beta<\infty$. *Assume* $c=1$ *and* $\alpha=T$ *in Theorem* 3.1. *Then SDE* (2.5) *has a mild solution on* I_0.

Proof of theorems 3.2 and 3.3 are similar to that of Theorems 2 and 3 of [1] and hence omitted.

In fact we have a more general theorem.

3.4. Theorem. *Assume the hypothesis of Theorem* 3.1 *with* $c=1$, $I=\boldsymbol{R}^+$ *and* (iii) *replaced by* (iii)*: *Suppose that there exists* $U_0\in\mathcal{U}$ *and a continuous function* $G(t, r)$ *from* $\boldsymbol{R}^+\times\boldsymbol{R}^+$ *into* $\boldsymbol{R}^+$ *such that*

a) *for each* t, $G(t, r)$ *is nondecreasing in* r.

b) *for* $(t, x)\in\boldsymbol{R}^+\times X$ *and* $\lambda>0$,

$$p_0(S(\lambda)f(t, x))\leqq G(t, p_0(x))$$

c) *for any* $z\in X$, *there exists a function* g *defined on* $\boldsymbol{R}^+$ *such that*

$$g(t)\geqq p_0(S(t-a)z)+\int_a^t G(s, g(s))ds \tag{3.2}$$

$$g(a)=p_0(z). \tag{3.3}$$

Then for each $z\in X$, *the SDE* (2.5) *has a mild solution on any compact subinterval* $I=[a, a+\beta]$, $0<\beta<\infty$.

Proof. Let g be as given. Define a set $H: \{u\in C[\boldsymbol{R}^+, X]: p_0(u(t))\leqq g(t)\}$. Then H is closed and convex. Define T from H into $C[\boldsymbol{R}^+, X]$ by

$$Tu(t)=S(t-a)z+\int_a^t S(t-s)f(s, x(s))ds \tag{3.4}$$

for all $u\in H$ and $t\geqq a$. Then since $p_0(Tu(t))\leqq g(t)$, $T: H\to H$. Rest of the part now is routine (see Agase [1]).

3.5. Theorem. *Assume the hypothesis of Theorem* 3.4 *with* $p_0(S(t-a)z)$ *in* (iii)* (b) *replaced by any function* $u\in C[[a, a+\beta], X]$. *Then the integral equation*

$$w(t)=u(t)+\int_a^t S(t-s)f(s, w(s))ds \tag{3.5}$$

has a solution $w\in C[[a, a+\beta], X]$.

Proof. Follows from the proof of Theorem 3.4.

3.6. Remark. Theorems 3.4 and 3.5 generalise and improve similar theorems in Pazy [9] (page 184) except uniqueness.

Now we restrict $f: I\times D\to X$ where D is open subset of X and combine ideas in Yuasa [11] to establish the following theorem.

3.7. Theorem. *Assume that;*

C_1) *Hypothesis* (i) *of Theorem* 3.1 *holds.*

C_2) *There exists a closed convex bounded set* F *with* $z\in F\subset D$ *such that the set* $B_0=\mathrm{cl\,co}(\{\bigcup_{0\leq\lambda\leq 1} S(\lambda)f(I, F)\}\cup\{0\})$ *is bounded and* $z+\alpha_0 B_0\subset F$ *for some* $\alpha_0>0$.

C_3) *For any bounded set* $B_1\subset\bar{B}_1\subset D$, *there is an interval* $I_1=[a, a+\beta]\subset I$ *and a constant* $c>0$ *such that* cf *is Q-condensing on* $I_1\times B_1$.

Then for each $z\in D$ *there exists* $0<\alpha<a$ *such that SDE* (2.5) *has a mild solution on* $[a, a+\alpha]$.

3.8. Remark. We note that when X is a Banach space then condition C_2 is redundant. (C_2) follows from Lemma 3.4 of [2].

Proof of Theorem 3.7. Let $z\in D$, $F\subset D$ and B_0 be as in (C_2). For this F we choose $\beta>0$ and $c>0$ such that cf is Q-condensing on $[a, a+\beta]\times\hat{F}$. Now set $\alpha=\min(\alpha_0, \beta, C)$ and $J=[a, a+\alpha]$. Obviously cf is Q-condensing on $J\times F$. Define the set H by,

$$H=\{u\in C[J, F]: u(a)=z \text{ and for } t, t'\in J,$$
$$u(t)-u(t')\in[S(t-a)-S(t'-a)]z+|t-t'|B_0\}.$$

Then H is closed convex and bounded in $C[J, F]$. The operator $T: H \to H$, defined as in lemma 2.11, is continuous, and Q-condensing (this follows as in theorem 3.1). Then a fixed point of T, assured by Theorem 2.7, is a required mild solution of SDE (2.5).

3.9. Remarks. (*i*) By using arguments similar to the proof of Lemma 2.8 (iii), one can easily verify that the function $F(t, x)$ defined by (2.8) is Q-condensing, for each fixed $t \in I$, on $t \times Y_t$ where $Y_t = C[[a, t], X]$. This, together with equicontinuity of H, helps in the arguments similar to Yuasa [11] to get a fixed point of T.

(*ii*) Theorem 3.7 generalizes Theorem 3.3 in Agase and Raghavendra [2].

(*iii*) When $A=0$ then $S(t)=I$ for all $t>0$, Theorem 3.7 yields Theorem 2 of Yuasa [11] and Theorem 3.3 gives theorem 3 of Agase [1]. Further results on noncontinuability and uniqueness of mild solution of SDE (2.5) can be obtained on the similar lines. The details are omitted.

(*iv*) If $f: I \times D \to X$ is locally Lipschitzian ([11]) then the condition (C_8) is satisfied.

We conclude this section by showing that a C_0-semigroup of contractions satisfies our assumption (2.7).

3.10. Theorem. *If $\{S(t): t \geqq 0\}$ is a C_0-semigroup of $\mathcal{P}$-contractions on a complete LCTVS X then for any bounded $B \subset X$*

$$Q\Big(\bigcup_{0 \le t \le T} S(t)B\Big) = Q(B), \qquad T>0.$$

Proof. Let $B \subset X$ be bounded and write $D = \bigcup_{0 \le t \le T} S(t)B$ for some $T>0$. Since $B \subset D$ it follows that $Q(B) \supset Q(D)$.

Now let $U \in Q(B)$ and $\varepsilon > 0$. There is a precompact set $K = K(\delta)$ such that $B \subset K + (1+\varepsilon/2)U$. We conclude the theorem via following claims.

Claim 1. There exists $\delta > 0$ such that

$$U \in Q\Big(\bigcup_{0 \le t \le \delta} S(t)B\Big).$$

Proof. Clearly $D \subset \bigcup_{0 \le t \le T} S(t)K + (1+\varepsilon/2)U$. Since K is precompact there exist points $x_1, x_2, \cdots, x_n$ such that $K \subset \{x_i\}_{i=1}^n + (\varepsilon/4)U$, that is, $K \subset \bigcup_{i=1}^n B(x_i, p, \varepsilon/4)$ where p is the Minkowski functional of U and

$$B\Big(x, p, \frac{\varepsilon}{4}\Big): \Big\{y \in X: p(x-y) < \frac{\varepsilon}{4}\Big\}.$$

Now let $x \in B(x_i, p, \varepsilon/4)$ and $t > 0$. Then

$$p(S(t)x-x_i)\leqq p(S(t)x-S(t)x_i)+p(S(t)x_i-x_i)$$
$$\leqq\frac{\varepsilon}{4}+p(S(t)x_i-x_i). \tag{3.6}$$

By (2.2), there is a $\delta_i=\delta_i(x_i, p, \varepsilon)$ such that for all $t\in[0, \delta_i]$, $p(S(t)x_i-x_i)<\varepsilon/4$ and hence (3.6) yields that $S(t)x\in B(x_i, p, \varepsilon/2)$ for all $t\in[0, \delta_i]$. Set $\delta=\min_{1\leq i\leq n}\delta_i$. So that for all $t\in[0, \delta]$ $S(t)B(x_i, p, \delta/4)\subset B(x_i, p, \varepsilon/2)$ which implies that, for each $i=1, 2, \cdots, n$,

$$\bigcup_{0\leq t\leq\delta}S(t)B\left(x_i, p, \frac{\varepsilon}{4}\right)\subset B\left(x_i, p, \frac{\varepsilon}{2}\right).$$

Thus we obtain

$$\bigcup_{0\leq t\leq\delta}S(t)K\subset\left|\bigcup_{0\leq t\leq\delta}S(t)\left[\bigcup_{i=1}^{n}B\left(x_i, p, \frac{\varepsilon}{4}\right)\right]\right.$$
$$=\bigcup_{i=1}^{n}\left[\bigcup_{0\leq t\leq\delta}S(t)B\left(x_i, p, \frac{\varepsilon}{4}\right)\right]$$
$$\subset\bigcup_{i=1}^{n}B\left(x_i, p, \frac{\varepsilon}{2}\right).$$

So that

$$\bigcup_{0\leq t\leq\delta}S(t)B\subset\bigcup_{0\leq t\leq\delta}K+\left(1+\frac{\varepsilon}{2}\right)$$
$$\subset\{x_i\}_{i=1}^{n}+\frac{\varepsilon}{2}U+\left(1+\frac{t}{2}\right)$$
$$=\{x_i\}_{i=1}^{n}+(1+\varepsilon)U.$$

Therefore $U\in Q(\bigcup_{0\leq t\leq\delta}S(t)B)$ where $\delta=\delta(x_1, \cdots, x_n, p, \varepsilon)$.

Claim 2. $Q(\bigcup_{0\leq t\leq\delta}S(t)B)=Q(\bigcup_{0\leq t\leq n\delta}S(t)B)$ for any positive integer n.

Proof. Take $n=2$. Then by Theorem 2.5(b)

$$Q(\bigcup_{0\leq t\leq 2\delta}S(t)B)=Q(\bigcup_{0\leq t\leq\delta}S(t)B)\cap Q(\bigcup_{\delta\leq t\leq 2\delta}S(t)B) \tag{3.7}$$

Now

$$\bigcup_{\delta\leq t\leq 2\delta}S(t)B=S(\delta([\bigcup_{\delta\leq t\leq 2\delta}S(t-\delta)B]$$
$$=S(\delta([\bigcup_{0\leq t\leq\delta}S(t)B].$$

Since $S(\delta)$ is a contraction, it follows that

$$Q(\bigcup_{0\leq t\leq\delta}S(t)B)\subset Q(S(\delta)[\bigcup_{0\leq t\leq\delta}S(t)B])$$
$$=Q(\bigcup_{\delta\leq t\leq 2\delta}S(t)B).$$

(3.7) now shows that the claim is true for $n=2$. The claim obviously holds for any finite n.

We choose n such that $[0, T]\subset[0, n\delta]$. Since $D\subset\bigcup_{0\leq t\leq n\delta}S(t)B$, $Q(D)\supset Q(\bigcup_{0\leq t\leq\delta}S(t)B)$ and so that $U\in Q(D)$. Hence the theorem.

4. Stability Theorems.

4.1. Theorem. *Let the hypothesis of Theorem* 3.4 *be true with* (iii)* (b) *holding for every* $U\in\mathcal{V}$ *and* (iii)* (c) *replaced by the following: for each* $p_\alpha\in\mathcal{P}$ *there exists a function* q_α *such that*

$$q_\alpha(t)=p_\alpha(S(t-a)z)+\int_a^t G(S, g_\alpha(s))ds \tag{4.1}$$

$$g_\alpha(a)=p_\alpha(z).$$

If g_α *is bounded for each* α, *then for every* $z\in X$, *SDE* (2.5) *has a bounded mild solution on* $[a, a+T]$, $0<T<\infty$.

4.2. Theorem. *Assume that*

(i) *A is the infinitesimal generator of an equicontinuous semigroup of class C_0 satisfying* (2.7).

(ii) $f:\boldsymbol{R}^+\times X\to X$ *be contiuuous and Q-condensing,* $f(t, 0)\equiv 0$.

(iii) *There exists a continuous function $G(t, r)$ from $\boldsymbol{R}^+\times\boldsymbol{R}^+\to\boldsymbol{R}^+$ such that*

(a) $G(t, 0)\equiv 0$, *nondecreasing in r for each $t\in\boldsymbol{R}^+$,*

(b) *for* $(t, x)\in\boldsymbol{R}^+\times X$ *and* $\lambda>0$,

$$p(S(\lambda(f(t, x))\leqq G(t, p(x)) \quad \forall p\in\mathcal{P} \tag{4.2}$$

(d) *for any $z\in X$ and $p_\alpha\in\mathcal{P}$ there exist functions $g_\alpha: {}^+\boldsymbol{R}\to\boldsymbol{R}^+$ satisfying* (4.1). *Then the stability* (*or asymptotic stability*) *of the null solution of scalar integral equations* (4.1) *for every α implies the stability* (*or asymptotic stability*) *of the trivial solution of*

$$\frac{du}{dt}=Au+f(t, u).$$

Proofs of Theorems 4.1 and 4.2 are almost similar to Theorems 4 and 5 of Agase [1] and hence omitted.

5. Applications.

5.1. Let X be the space of rapidly decreasing functions on $\boldsymbol{R}$ with the family of seminorms $\mathcal{P}=\{\bar{p}_{kn}: k, n=0, 1, \cdots\}$ defined by

$$\bar{p}_{kn}(f)=\sup_{t\geq 0} p_{kn}(e^{-t}R(t)f)$$

where $[R(t)f](x)=f(t+x)$ $\forall t\geqq 0$, $x\in \boldsymbol{R}$ and $p_{kn}(g)=\max\limits_{0\leq j\leq k}\sup\limits_{x\in R}|x^j g^{(n)}(x)|$. $\mathscr{P}$ induces a locally convex topology on X ([3]).

Consider the differential equation

$$\frac{\partial}{\partial t}u(t, x)=\frac{\partial^2}{\partial x^2}u(t, x)+f(t, u(t, x)) \quad \text{in} \quad \boldsymbol{R}^+\times X \tag{5.1}$$

$$u(0, x)=u_0(x), \qquad u_0\in X \tag{5.2}$$

The operator $A\equiv\partial^2/\partial x^2$ generates a $\mathscr{P}$-contraction C_0-semigroup $\{P(s): s\geqq 0\}$ defined by $P(0)=I$.

$$[P(s)f](x)=\frac{1}{\sqrt{2\pi s}}\int_{-\infty}^{\infty}e^{-(x-y)^2/2s}f(y)dy$$

for all $x\in\boldsymbol{R}$ (see [3]). Define $y: \boldsymbol{R}^+\to X$ by $y(t)(x)=u(t, x)$ $\forall x\in\boldsymbol{R}$. Then the problem (5.1) (5.2) can be written as

$$\frac{dy}{dt}=Ay+f(t, y) \quad \text{in} \quad \boldsymbol{R}^+\times X \tag{5.3}$$

$$y(0)=u_0, \qquad u_0\in X \tag{5.4}$$

which is of the form (2.5). If $f(t, y)$ is assumed to be locally Lipschitzian with respect to $\mathscr{P}$ (see [11]) then f satisfies condition C_3 of Theorem 3.7. Further if (C_2) is also satisfied then Theorem 3.7 assures existence of mild solution of (5.1) (5.2) on some suitable interval.

5.2. Let H be the space of real valued C^∞ functions on $\boldsymbol{R}^m$ whose partial derivatives of all orders belong to $L^2(\boldsymbol{R}^m)$. It is known (Miyadera [8] or Choe [3]) that it is a pre-Hilbert space with inner product

$$\langle f, g\rangle_n=\sum_{|\alpha|\leq n}\int_{R^m}D^\alpha f(x)dx$$

for all $f, g\in H$. Then for each n, $\langle f, f\rangle_n=\|f\|_n^2$ $\forall f\in H$, defines a norm on H. For each multiindex $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ a seminorm p_α is defined on H by

$$p_\alpha(f)=\|D^\alpha f\|_0=\left(\int_{R^m}(D^\alpha f(x))^2dx\right)^{1/2}$$

for all $f\in H$. The totality Γ of these seminorms p_α corresponding to all multiindices α induces a metrizable LCTVS on H.

Consider the Differential Equation

$$\frac{\partial}{\partial t}u(t, x)=Au(t, x)+f(t, u(t, x)) \quad \text{in} \quad \boldsymbol{R}^+\times\boldsymbol{R}^m \tag{5.5}$$

$$u(0, x)=u_0(x), \qquad u_0\in H,\ x\in\boldsymbol{R}^m \tag{5.6}$$

where $A=\sum\limits_{i=1}^m\left(\frac{\partial}{\partial x_i}\right)^2$.

It is known that ([3]) A generates the Γ-contraction C_0-semigroup $[P(s): s\geqq 0]$ on H defined by

$$P(0)=I$$

$$[P(s)f](x)=\frac{1}{(2\pi s)^{m/2}}\int_{R^m} e^{-(|x-y|^2/2s)}f(y)dy\,.$$

Again a map $y:(0, \infty)\rightarrow H$ such that $y(t)(x)=u(t, x)$ $\forall x\in \boldsymbol{R}^m$, reduces (5.5) (5.6) in the form (5.3) (5.4) and can be studied similarly.

As mentioned in the introduction, at present we do not know any physical problem governed by such equations with a Lipschitz or non-Lipschitz f (having Q-condensing property). We mention these examples only to illustrate the abstract theory developed here. In a forthcoming paper we study regularity of mild solutions of SDE (2.5) in LCTVS.

Acknowledgements. The author would like to thank Dr. V. Raghavendra for interesting discussions with him. Thanks are also due to the Management of Science College Nanded for providing leave during his stay at this institute.

References

[1] Agase S.B.: Existence and stability of ordinary differential equations in locally convex spaces. *Nonlinear Analysis—T.M.A.* **5** (1981), 713–719.

[2] Agase S.B. and Raghavendra V.: Existence of mild solutions of semilinear differential equations in Banach spaces [Communicated].

[3] Choe Y.H.: C_0-semigroups on a locally convex space. *J. Math. Anal. Appl.* **106** (1985), 293–320.

[4] Istratescu V.I.: *Introducere in Teoria Punctelor Fixe* Ed. Academiei R.S. Romania, Bucuresti 1973.

[5] Lakshmikantham V. and Leela S.: *Nonlinear Differential Equations in Abstract Spaces*, Pergamon Press (1981).

[6] Martin R.H. Jr.: *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley-Interscience NY (1976).

[7] Millionscikov V.M.: A contribution to the theory of differential equations $dx/dt=f(x,t)$ in locally convex spaces, *Soviet Math. Dokl.* **288** (1960).

[8] Miyadera I.: Semigroups of operators in Frecher space and applications to partial differential equations—*Tohoku Math. J.*, **11** (1959), 162–183.

[9] Pazy A.: *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer Verlag. N.Y. (1983).

[10] Reich S.: Fixed points in locally convex spaces. *Math. Z.* **125** (1972), 17–31.

[11] Yuasa T.: Differential equations in a locally convex space via the measure of nonprecompactness. *J. Math. Anal. Appl.* **84** (1981), 534–554.

Department of Mathematies
Indian Institute of Technology
Kanpur—208016
India