YOKOHAMA MATHEMATICAL
JOURNAL VOL. 35, 1987

SPHERICAL SUBMANIFOLDS WITH POINTWISE 3- OR 4-PLANAR NORMAL SECTIONS

By

SHI-JIE LI

(Received July 31, 1986)

1. Introduction.

Let M be an n-dimensional submanifold in a Euclidean $(n+p)$-space E^{n+p}. For any point q in M and any unit vector t at q tangent to M, the vector t and the normal space $T_q^{\perp}M$ of M at q determine a $(p+1)$-dimensional vector subspace $E(q, t)$ of E^{n+p}. The intersection of M and $E(q, t)$ gives rise to a curve γ (in a neighborhood of q) which is called the normal section of M at q in the direction of t. In general the normal section γ is a twisted space curve in $E(q, t)$. In particular, $\gamma'\wedge\gamma''\wedge\gamma'''\neq 0$ at q in general. A submanifold M is said to have pointwise k-planar $(2\leq k\leq p)$ normal sections if each normal section γ at q satisfies $\gamma'\wedge\gamma''\wedge\cdots\wedge\gamma^{(k+1)}=0$ at q for each q in M. Let h be the second fundamental form and ∇h the covariant derivative of h. The following results were obtained by B. Y. Chen (2, 3).

Theorem A. *An n-dimensional submanifold M of E^{n+p} has pointwise 2-planar normal sections if and only if $(\nabla_t h)(t, t)\wedge h(t, t)=0$, for any $t\in TM$.*

Theorem B. *An n-dimensional spherical submanifold M of E^{n+p} has pointwise 2-planar normal sections if and only if M has parallel second fundamental form, i.e., $\nabla h=0$.*

Using Theorem A and other results of [3], B. Y. Chen and the author [4] have classified surfaces with pointwise 2-planar normal sectoins.

In this paper, we shall study submanifolds with pointwise 3- or 4-planar normal sections and generalize Chen's results.

The author would like to express his hearty thanks to Professor B. Y. Chen for many valuable suggestions.

2. Basic formulas.

In this section, we shall derive some formulas involving the second fundamental form h. Let M be an n-dimensional submanifold in E^{n+p}. We choose a local field of orthonormal frames $(e_1, \cdots, e_n; e_{n+1}, \cdots, e_{n+p})$ in E^{n+p} such that, restricted to M, the vectors $e_1, \cdots, e_n$ are tangent to M and $e_{n+1}, \cdots e_{n+p}$ are normal to M. We denote by $(\theta^1, \cdots, \theta^{n+p})$ the field of dual frames. The structure equations of E^{n+p} are given by

$$d\theta^A = -\sum_B \theta^A_B \wedge \theta^B, \qquad \theta^A_B + \theta^B_A = 0, \tag{2.1}$$

$$d\theta^A_B = -\sum_C \theta^A_C \wedge \theta^C_B, \qquad A, B, C, \cdots = 1, 2, \cdots, n+p. \tag{2.2}$$

Restricting these forms on M, we have $\theta^r = 0$, $r, s, u \cdots = n+1, \cdots, n+p$. Since

$$0 = d\theta^r = -\sum_i \theta^r_i \wedge \theta^i, \qquad i, j, k, l, m, q, \cdots = 1, 2, \cdots, n, \tag{2.3}$$

Cartan's lemma implies

$$\theta^r_i = \sum_j h^r_{ij} \theta^j, \qquad h^r_{ij} = h^r_{ji}. \tag{2.4}$$

From these formulas we obtain

$$d\theta^i = -\sum_j \theta^i_j \wedge \theta^j, \qquad \theta^i_j + \theta^j_i = 0, \tag{2.5}$$

$$d\theta^i_j = -\sum_k \theta^i_k \wedge \theta^k_j + \Omega^i_j, \qquad \Omega^i_j = \frac{1}{2} \sum_{k,l} R^i_{jkl} \theta^k \wedge \theta^l, \tag{2.6}$$

$$R^i_{jkl} = \sum_r (h^r_{ik} h^r_{jl} - h^r_{il} h^r_{jk}).$$

$$d\theta^r_s = -\sum_u \theta^r_u \wedge \theta^u_s + \Omega^r_s, \qquad \Omega^r_s = \frac{1}{2} \sum_{i,j} R^r_{sij} \theta^i \wedge \theta^j, \tag{2.7}$$

$$R^r_{sij} = \sum_k (h^r_{ki} h^s_{k} - h^r_{kj} h^r_{ki}).$$

The Riemannian connection of M is defined by (θ^i_j). The form (θ^r_s) defines a connection D in the normal bundle of M, then we have $h = \sum_{i,j,r} h^r_{ij} \theta^i \theta^j e_r$. We call $H = (1/n) \operatorname{tr} h$ the mean curvature vector of M. We take exterior differentiation of (2.4) and define h^r_{ijk} by

$$\sum_k h^r_{ijk} \theta^k = dh^r_{ij} - \sum_m h^r_{mj} \theta^m_i - \sum_m h^r_{im} \theta^m_j + \sum_s h^s_{ij} \theta^r_s. \tag{2.8}$$

Then we have the following equation of Codazzi.

$$h^r_{ijk} = h^r_{ikj}. \tag{2.9}$$

We take exterior differentiation of (2.8) and define h^r_{ijkl} by

$$\sum_l h^r_{ijkl} \theta^l = dh^r_{ijk} - \sum_m h^r_{mjk} \theta^m_i - \sum_m h^r_{imk} \theta^m_j - \sum_m h^r_{ijm} \theta^m_k + \sum_s h^s_{ijk} \theta^r_s. \tag{2.10}$$

Then

$$h^r_{ijkl} - h^r_{ijlk} = \sum_m h^r_{im} R^m_{jkl} + \sum_m h^r_{mj} R^m_{ikl} - \sum_s h^s_{ij} R^r_{skl}. \tag{2.11}$$

We take exterior differentiation of (2.10) and define h^r_{ijklm} by

$$\sum_m h^r_{ijklm}\theta^m = dh^r_{ijkl} - \sum_m h^r_{mjkl}\theta^m_i - \sum_m h^r_{imkl}\theta^m_j - \sum_m h^r_{ijml}\theta^m_k - \sum_m h^r_{ijkm}\theta^m_l + \sum_s h^r_{ijkl}\theta^r_s. \tag{2.12}$$

Then

$$h^r_{ijklm} - h^r_{ijkml} = \sum_q h^r_{qjk}R^q_{ilm} + \sum_q h^r_{iqk}R^q_{jlm} + \sum_q h^r_{ijq}R^q_{klm} - \sum_s h^s_{ijk}R^r_{slm}. \tag{2.13}$$

If we denote by ∇ and $\tilde{\nabla}$ the covariant detivatives of M and E^{n+p}, respectively. Then, for any two vector fields x, y tangent to M and any vector field ξ normal to M, we have

$$\tilde{\nabla}_x y = \nabla_x y + h(x, y), \tag{2.14}$$

$$\tilde{\nabla}_x \xi = -A_\xi x + D_x \xi, \tag{2.15}$$

where A_ξ denotes the Weingarten map with respect to ξ. Then

$$\langle A_\xi x, y\rangle = \langle h(x, y), \xi\rangle. \tag{2.16}$$

For the first three covariant derivatives $\bar{\nabla}h$, $\bar{\nabla}\bar{\nabla}h$ and $\bar{\nabla}\bar{\nabla}\bar{\nabla}h$ of h, we have the following, respectively, (see, for instance, (1)).

$$(\bar{\nabla}_x h)(y, z) = D_x(h(y, z)) - h(\nabla_x y, z) - h(y, \nabla_x z), \tag{2.17}$$

$$(\bar{\nabla}_w \bar{\nabla}_x h)(y, z) = D_w((\bar{\nabla}_x h)(y, z)) - (\bar{\nabla}_x h)(\nabla_w y, z) - (\bar{\nabla}_x h)(y, \nabla_w z) - (\bar{\nabla}_{\nabla_w x} h)(y, z), \tag{2.18}$$

$$(\bar{\nabla}_u \bar{\nabla}_w \bar{\nabla}_x h)(y, z) = D_u((\bar{\nabla}_w \bar{\nabla}_x h)(y, z)) - (\bar{\nabla}_w \bar{\nabla}_x h)(\nabla_u y, z) - (\bar{\nabla}_w \bar{\nabla}_x h)(y, \nabla_u z) - (\bar{\nabla}_{\nabla_u w} \bar{\nabla}_x h)(y, z) - (\bar{\nabla}_w \bar{\nabla}_{\nabla_u x} h)(y, z), \tag{2.19}$$

where $x, y, z, w, u \in TM$. Comparing (2.8) with (2.17), (2.10) with (2.18), and (2.12) with (2.19), respectively, we have

$$\sum_r h^r_{ijk} e_r = (\bar{\nabla}_{e_k} h)(e_i, e_j), \tag{2.20}$$

$$\sum_r h^r_{ijkl} e_r = (\bar{\nabla}_{e_l} \bar{\nabla}_{e_k} h)(e_i, e_j), \tag{2.21}$$

$$\sum_r h^r_{ijklm} e_r = (\bar{\nabla}_{e_m} \bar{\nabla}_{e_l} \bar{\nabla}_{e_k} h)(e_i, e_j). \tag{2.22}$$

3. Derivatives of normal sections.

Throughout .his setion and the followings, we assume that t is a unit vector tangent to M and $\gamma(s)$ is the normal section of M at q in the direction of t with s as its length and $\gamma(o)=q$. We denote by $T=\gamma'(s)$ the unit vector tangent to the normal section $\gamma(s)$. We choose a local field of orthonormal frame $(e_1, \cdots e_n;$

$e_{n+1}, e_{n+p})$ as in Section 2 and assume that, restricted to the normal section γ, $T=e_1$. Then we have

$$\gamma'(s)=T=e_1. \tag{3.1}$$

By differentiating (3.1) and using (2.4), we have

$$\gamma''(s)=\tilde{\nabla}_T T=\sum_i \theta_1^i(e_1)e_i+\sum_r h_{11}^r e_r. \tag{3.2}$$

$$\begin{aligned}\gamma'''(s)=\tilde{\nabla}_T\tilde{\nabla}_T T&=\sum_i e_1(\theta_1^i(e_1))e_i+\sum_{i,j,r}\theta_1^i(e_1)[\theta_i^j(e_1)e_j+\theta_i^r(e_1)e_r]\\&\quad+\sum_r e_1(h_{11}^r)e_r+\sum_{i,r,s}h_{11}^r[\theta_r^i(e_1)e_i+\theta_r^s(e_1)e_s]\\&=\sum_i(e_1(\theta_1^i(e_1))+\sum_i\theta_1^j(e_1)\theta_j^i(e_1)-\sum_r h_{11}^r h_{1i}^r]e_i\\&\quad+\sum_r[h_{111}^r+3\sum_i h_{1i}^r\theta_1^i(e_1)]e_r.\end{aligned} \tag{3.3}$$

By differentiating (3.3) and using (2.4), (2.8)-(2.10), we have

$$\begin{aligned}\gamma^{\mathrm{iv}}(s)=\tilde{\nabla}_T\tilde{\nabla}_T\tilde{\nabla}_T T&=\sum_i[e_1^2(\theta_1^i(e_1))+\sum_i e_1(\theta_1^j(e_1)\theta_j^i(e_1))\\&\quad-\sum_r e_1(h_{11}^r h_{1i}^r)]e_i+\sum_i[e_1(\theta_1^i(e_1))+\sum_j\theta_1^j(e_1)\theta_j^i(e_1)\\&\quad-\sum_r h_{11}^r h_{1i}^r][\sum_k\theta_i^k(e_1)e_k+\sum_s\theta_i^s(e_1)e_s]\\&\quad+\sum_r[e_1(h_{111}^r)+3\sum_i e_1(h_{1i}^r\theta_1^i(e_1)]e_r+\sum_r[h_{111}^r\\&\quad+3\sum_i h_{1i}^r\theta_1^i(e_1)][\sum_k\theta_r^k(e_1)e_k+\sum_s\theta_r^s(e_1)e_s]\\&=\sum_i[e_1^2(\theta_1^i(e_1))+2\sum_j e_1(\theta_1^j(e_1))\theta_j^i(e_1)\\&\quad+\sum_j\theta_1^j(e_1)e_1(\theta_j^i(e_1))+\sum_{j,k}\theta_1^k(e_1)\theta_k^j(e_1)\theta_j^i(e_1)-2\sum_r h_{111}^r h_{1i}^r\\&\quad-5\sum_{j,k}h_{1i}^r h_{1j}^r\theta_1^j(e_1)-\sum_r h_{11}^r h_{11i}^r-\sum_{j,r}h_{11}^r h_{ij}^r\theta_1^j(e_1)]e_i\\&\quad+\sum_r[4\sum_i e_1(\theta_1^i(e_1))h_{1i}^r+4\sum_{i,j}h_{1i}\theta_j^i(e_1)\theta_1^j(e_1)+h_{1111}^r\\&\quad-\sum_{i,s}h_{1i}^r h_{11}^s h_{1i}^s+6\sum_i h_{11i}^r\theta_1^i(e_1)+3\sum_{i,j}h_{ij}^r\theta_1^j(e_1)\theta_1^i(e_1)]e_r.\end{aligned} \tag{3.4}$$

In this paper, we shall also need the normal component $(\gamma^{v}(s))^{\perp}$ of $\gamma^{v}(s)$. So similarly, by differentiating (3.4), we have

$$\begin{aligned}(\gamma^{\mathrm{v}}(s))^{\perp}=(\tilde{\nabla}_T\tilde{\nabla}_T\tilde{\nabla}_T\tilde{\nabla}_T T)^{\perp}&=\sum_{i,u}[e_1^2(\theta_1^i(e_1))+2\sum_j e_1(\theta_1^j(e_1))\theta_j^i(e_1)\\&\quad+\sum_j\theta_1^j(e_1)e_1(\theta_j^i(e_1))+\sum_{j,k}\theta_1^k(e_1)\theta_k^j(e_1)\theta_j^i(e_1)-2\sum_r h_{111}^r h_{1i}^r\\&\quad-5\sum_{j,r}h_{1i}^r h_{1j}^r\theta_1^j(e_1)-\sum_r h_{11}^r h_{11i}^r-\sum_{j,r}h_{11}^r h_{ij}^r\theta_1^j(e_1)]\theta_i^u(e_1)e_u\\&\quad+\sum_r[4\sum_i e_1^2(\theta_1^i(e_1))h_{1i}^r+4\sum_i e_1(\theta_1^i(e_1))e_1(h_{1i}^r)\end{aligned} \tag{3.5}$$

$$+4\sum_{i,j} e_1(h^r_{1i}\theta^i_j(e_1))\theta^j_1(e_1)+4\sum_{i,j} h^r_{1i}\theta^i_j(e_1)e_1(\theta^j_1(e_1))$$

$$+6\sum_i e_1(h^r_{11i})\theta^i_1(e_1)+6\sum_i h^r_{11i}e_1(\theta^i_1(e_1))$$

$$+3\sum_{i,j} e_1(h^r_{ij}\theta^j_1(e_1)\theta^i_1(e_1))-\sum_{i,s} e_1(h^r_{1i}h^s_{11}h^s_{1i})+e_1(h^r_{1111})]e_r$$

$$+\sum_{r,u}[4\sum_i e_1(\theta^i_1(e_1)h^r_{1i}+4\sum_{i,j} h^r_{1i}\theta^i_j(e_1)\theta(^j_1e_1)+6\sum_i h^r_{11j}\theta^i_1(e_1)$$

$$+3\sum_{i,j} h^r_{ij}\theta^j_1(e_1)\theta^i_1(e_1)-\sum_{i,s} h^s_{11}h^s_{1i}h^r_{1i}+h^r_{1111}]\theta^u_r(e_1)e_u.$$

Since $\gamma(s)$ is the normal section of M at q in the direction t, at $\gamma(o)=q$, $\gamma''(o)$, $\gamma'''(o)$, $\gamma^{\text{iv}}(o)$ and $\gamma^{\text{v}}(o)$ all lie in the $(p+1)$-space $E(q, t)$. We recall that $E(q, t)$ is spanned by t and $T^\perp_q M$, thus (3.2) and (3.3) give

(3.6) $$\theta^i_1(t)=0, \qquad i=1, \cdots, n.$$

(3.7) $$\gamma''(o)=\sum_r h^r_{11}e_r,$$

(3.8) $$e_1(\theta^i_1(e_1))_q=\sum_r h^r_{11}h^r_{1i}, \qquad i=2, \cdots, n.$$

(3.9) $$\gamma'''(o)=-\sum_r (h^r_{11})^2 t+\sum_r h^r_{111}e_r.$$

And (3.4) gives, with the help of (3.6) and (3.8),

(3.10) $$e^2_1(\theta^i_1(e_1))_q=\sum_r [2h^r_{111}h^r_{1i}+h^r_{11i}h^r_{11}+2\sum_j h^r_{11}h^r_{ij}\theta^j_i(e_l)], \qquad i=2, \cdots, n.$$

(3.11) $$\gamma^{\text{iv}}(o)=\sum_r [-3h^r_{11}h^r_{11}+2\sum_s h^r_{11}h^s_{11}\theta^r_s(e_1)]t+\sum_r [h^r_{1111}$$
$$+3\sum_{i,s} h^s_{11}h^s_{1i}h^r_{1i}-2\sum_s h^s_{11}h^s_{11}h^r_{11}]e_r.$$

Lastly, (3.5) gives, with the help of (3.6), (3.8) and (3.10),

(3.12) $$(\gamma^{\text{v}}(o))^\perp=\sum_r [7\sum_{i,s} h^s_{111}h^s_{1i}h^r_{1i}+3\sum_{i,s} h^s_{11i}h^s_{11}h^r_{1i}+9\sum_{i,s} h^r_{11i}h^s_{11}h^s_{1i}$$
$$+h^r_{11111}-15\sum_s h^s_{111}h^s_{11}h^r_{11}-10\sum_s h^s_{11}h^s_{11}h^r_{111}]e_r.$$

Then from (3.7), we have

(3.13) $$\gamma''(o)=h(t, t).$$

Applying (2.20) to (3.9), we have

(3.14) $$\gamma'''(o)=-\langle h(t, t), h(t, t)\rangle t+(\bar{\nabla}_t h)(t, t).$$

From (2.16), we have

(3.15) $$A_{h(t,t)}t=\sum_i \langle A_{h(t,t)}t, e_i\rangle e_i$$
$$=\sum_i \langle h(t, t), h(t, e_i)\rangle e_i=\sum_{i,s} h^s_{11}h^s_{1i}e_i.$$

Then applying (2.20), (2.21) and (3.15) to (2.11), we obtain

$$(3.16)\qquad (\gamma^{iv}(o))^{\perp}=(\nabla_t\nabla_t h)t,\ t)+3h(t,\ A_{h(t,t)}t)-2\langle h(t,\ t),\ h(t,\ t)\rangle h(t,\ t).$$

We recall that a submanifold M of E^{n+p} is said to be isotropic (in the sense of O'Neill [7]), if for each point q in M and each unit vector t tangent to M at q, the length of $h(t, t)$, $\|h(t, t)\|$, depends only on q, not on t at q. In particular, when $\|h(t, t)\|$ is also independent of the point q in M, then M is said to be constant isotropic. It is known (see (7)) that the submanifold M is isotropic at q if and only if h satisfies

$$(3.17)\qquad \langle h(x,\ x),\ h(x,\ y)\rangle=0,$$

for any orthogonal vectors x, y of the tangent space T_qM, where $\langle\,,\rangle$ denotes the inner product of E^{n+p}. If M is isotropic, from (3.17), we have

$$(3.17')\qquad \sum_s h^s_{11}h^s_{1i}=0,\qquad i=2,\cdots,n.$$

Differentiating (3.17′) and using (2.8), (2.9), (3.6) and (3.17′), we have

$$\begin{aligned}(3.18)\qquad 0&=e_1(\sum_s h_{11}h_{1i})=\sum_s e_1(h^s_{11})h^s_{1i}+\sum_s h^s_{11}e_1(h^s_{1i})\\ &=\sum_s h^s_{111}h^s_{1i}+2\sum_{j,s}h^s_{1j}\theta^j_1(e_1)h^s_{1i}+\sum_{s,u}h^u_{11}\theta^s_u(e_1)h^s_{1i}+\sum_s h^s_{11}h^s_{11i}\\ &\quad+\sum_{j,s}h^s_{11}h^s_{ij}\theta^j_1(e_1)+\sum_{j,s}h^s_{11}h^s_{1j}\theta^j_i(e_1)-\sum_{s,u}h^s_{11}h^u_{1i}\theta^s_u(e_1)\\ &=\sum_s h^s_{111}h^s_{1i}+\sum_s h^s_{11}h^s_{11i},\qquad i=2,\cdots,n.\end{aligned}$$

From (2.16), we have

$$\begin{aligned}(3.19)\qquad A_{(\nabla_t h)(t,t)}t&=\sum_i\langle A_{(\nabla_t h)(t,t)}t,\ e_i\rangle e_i\\ &=\sum_i\langle(\nabla_t h)(t,\ t),\ h(t,\ e_i)\rangle e_i=\sum_{i,s}h^s_{111}h^s_{1i}e_i,\end{aligned}$$

then applying (3.17′)-(3.19) to (3.12) and using (2.20)-(2.22), we have

$$\begin{aligned}(3.20)\qquad (\gamma^{v}(o))^{\perp}&=\sum_r[4\sum_{i,s}h^s_{111}h^s_{1i}h^r_{1i}+h^r_{1111}-\sum_s h^s_{11}h^s_{11}h^r_{111}-15\sum_s h^s_{111}h^s_{11}h^r_{11}]e_r\\ &=(\nabla_t\nabla_t\nabla_t h)(t,\ t)+4h(t,\ A_{(\nabla_t h)(t,t)}t)\\ &\quad-\langle h(t,\ t),\ h(t,\ t)\rangle(\nabla_t h)(t,\ t)-15\langle(\nabla_t h)(t,\ t),\ h(t,\ t)\rangle h(t,\ t).\end{aligned}$$

4. Submanifolds with pointwise 3- or 4-planar normal sections.

In this section we shall study submanifolds with 3- or 4-planar normal sections. First we give the following.

Lemma 1. *An n-dimensional submanifold of E^{n+p} has pointwise 3-planar* *normal sections if and only if*

$$[(\nabla_t\nabla_t h)(t,\ t)+3h(t,\ A_{h(t,t)}t)]\wedge(\nabla_t h)(t,\ t)\wedge h(t,\ t)=0. \tag{4.1}$$

for any $t\in TM$.

Proof. From (3.13), (3.14) and (3.16), we have

$$\begin{aligned}&\gamma'(o)\wedge\gamma''(o)\wedge\gamma'''(o)\wedge\gamma^{\mathrm{iv}}(o)\\&=t\wedge h(t,\ t)\wedge(\nabla_t h)(t,\ t)\wedge[(\nabla_t\nabla_t h)(t,\ t)+3h(t,\ A_{h(t,t)}t)].\end{aligned} \tag{4.2}$$

Since t lies in T_qM, by the definition of submanifolds with pointwise 3-planar normal sections, we get immediately from (4.2) the conclusion of the lemma.

If M is isotropic, substituting (3.17′) into (3.15), we have

$$A_{h(t,t)}t=\langle h(t,\ t),\ h(t,\ t)\rangle t. \tag{4.3}$$

Then from Lemma 1, using (4.3), we could obtain the following.

Collary [5]. *An n-dimensional isotropic submanifold M of* E^{n+p} *has pointwise 3-planar normal sections if and only if*

$$(\nabla_t\nabla_t h)(t,\ t)\wedge(\nabla_t h)(t,\ t)\wedge h(t,\ t)=0, \tag{4.4}$$

for any $t\in TM$.

Lemma 2. *An n-dimensional isotropic submanifold M of* E^{n+p} *has pointwise 4-planar normal sections if and only if*

$$[(\nabla_t\nabla_t\nabla_t h)(t,\ t)+4h(t,\ A_{(\nabla_t h)(t,t)}t)]\wedge(\nabla_t\nabla_t h)(t,\ t)\wedge(\nabla_t h)(t,\ t)\wedge h(t,\ t)=0, \tag{4.5}$$

for any $t\in TM$.

Proof. From (3.13), (3.14), (3.16) and (3.20), we have, with the help of (4.4),

$$\begin{aligned}&\gamma'(o)\wedge\gamma''(o)\wedge\gamma'''(o)\wedge\gamma^{\mathrm{iv}}(o)\wedge\gamma^{\mathrm{v}}(o)\\&=t\wedge h(t,\ t)\wedge(\nabla_t h)(t,\ t)\wedge(\nabla_t\nabla_t h)(t,\ t)\wedge[(\nabla_t\nabla_t\nabla_t h)(t,\ t)+4h(t,\ A_{(\nabla_t h)(t,t)}t)].\end{aligned} \tag{4.6}$$

Since t lies in T_qM, by the definition of submanifolds with pointwise 4-planar normal sections, the lemma follows (4.6) directly.

If M is constant isotropic, we have

$$\sum_s h^s_{11}h^s_{11}=\text{const.} \tag{4.7}$$

Differentiating (4.7) and using (2.8) (3.17′), we have

$$\begin{aligned}0&=e_i\Big(\sum_s h^s_{11}h^s_{11}\Big)=3\sum_s e_i(h^s_{11})h^s_{11}\\&=2\sum_s h^s_{11i}h^s_{11}+4\sum_{j,s}h^s_{1j}\theta^j_1(e_i)h^s_{11}-2\sum_{u,s}h^u_{11}\theta^s_u(e_i)h^s_{11}\end{aligned} \tag{4.8}$$

$$=2\sum_s h^s_{11}h^s_{11i}, \qquad i=1, \cdots, n.$$

since $\sum_{u,s} h^u_{11}\theta^s_u(e_i)h^s_{11}=0$. Combining (3.18) and (4.8), we have

$$\sum_s h^s_{111}h^s_{1i}=0, \qquad i=2, \cdots, n. \tag{4.9}$$

Thus (3.19) becomes

$$A_{(\nabla_t h)(t,t)}t=(\sum_s h^s_{111}h^s_{11})t \tag{4.10}$$

$$=\langle(\nabla_t h)(t, t), h(t, t)\rangle t.$$

Substituting (4.10) into (4.5), we get the following.

Corollary. *An n-dimensional constant isotropic submanifold M of E^{n+p} has pointwise 4-planar normal sections if and only if*

$$(\nabla_t\nabla_t\nabla_t h)(t, t)\wedge(\nabla_t\nabla_t h)(t, t)\wedge(\nabla_t h)(t, t)\wedge h(t, t)=0, \tag{4.11}$$

for any $t\in TM$.

5. Spherical submanifolds with poinwise 3- or 4-planar normal sections.

In this section, we shall generalize Chen's results [3] concerning spherical submanifolds. We assume that M is an n-dimensional spherical submanifold of E^{n+p}. Without loss of generity, we may assume that M lies in a unit hypersphere S^{n+p-1} of E^{n+p}. We choose a local field of orthonormal frame $(e_1, \cdots, e_n; e_{n+1}, \cdots, e_{n+p})$ as in Section 2 and moreover we may choose e_{n+p} as the unit outerward normal of S^{n+p-1} in E^{n+p}. Then we have

$$h^{n+p}_{ij}=\delta_{ij}, \tag{5.1}$$

where δ_{ij} is Kronecker delta, and

$$D_x e_{n+p}=\sum_s \theta^s_{n+p}(x)e_s=0. \tag{5.2}$$

Differentiating (5.1) and using (2.8), (5.1) and (5.2), we have

$$0=e_k(h^{n+p}_{ij})=h^{n+p}_{ijk}+\sum_m h^{n+p}_{mj}\theta^m_i(e_k)+\sum_m h^{n+p}_{im}\theta^m_j(e_k)-\sum_s h^s_{ij}\theta^{n+p}_s(e_k)=h^{n+p}_{ijk}. \tag{5.3}$$

Similarly, we may obtain

$$h^{n+p}_{ijkl}=0, \tag{5.4}$$

and

$$h^{n+p}_{ijklm}=0. \tag{5.5}$$

Theorem 1. *Let M be an n-dimensional spherical submanifold of E^{n+p}. If M is constant isotropic, then M has pointwise 4-planar normal sections if and only if*

(5.6) $$(\nabla_t\nabla_t\nabla_t h)(t, t)\wedge(\nabla_t\nabla_t h)(t, t)\wedge(\nabla_t h)(t, t)=0,$$

for any $t\in TM$.

Proof. Let M be an n-dimensional spherical submanifold of E^{n+p} and constant isotropic If M has pointwise 4-planar normal sections, according to the corollary of Lemma 2, we have (4.11), which implies

(5.7) $$\alpha(\nabla_t\nabla_t\nabla_t h)(t, t)+\beta(\nabla_t\nabla_t h)(t, t)+\gamma(\nabla_t h)(t, t)+\zeta h(t, t)=0,$$

where functions α, β, γ and $\tilde{\delta}$ are not all zero. In particular, we have

(5.8) $$\alpha h^{n+p}_{11111}+\beta h^{n+p}_{1111}+\gamma h^{n+p}_{111}+\delta h^{n+p}_{11}=0.$$

Substituting (5.1) and (5.3)-(5.5) into (5.8), we obtain $\delta=0$. Thus there are functions α, β and γ not all zero such that

(5.9) $$\alpha(\nabla_t\nabla_t\nabla_t h)(t, t)+\beta(\nabla_t\nabla_t h)(t, t)+\gamma(\nabla_t h)(t, t)=0,$$

for any $t\in TM$. Consequently we get (5.6). The converse of this is trivial,

Similar to Theorem 1, we may get the following.

Theorem 2. *Let M be an n-dimensional spherical submanifold of E^{n+p}. If M is isotropic, then M has pointwise* 3*-planar normal sections if and only if*

(5.10) $$(\nabla_t\nabla_t h)(t, t)\wedge(\nabla_t h)(t, t)=0, \quad \textit{for any} \quad t\in TM.$$

Proof. Let M be an n-dimensional spherical submanifold of E^{n+p} and isotropic. If M has pointwise 3-planar normal sections, by the corollary of Lemma 1, there are functions α, β and γ not all zero such that

(5.11) $$\alpha(\nabla_t\nabla_t h)(t, t)+\beta(\nabla_t h)(t, t)+\gamma h(t, t)=0.$$

In particular, we have

(5.12) $$\alpha h^{n+p}_{1111}+\beta h^{n+p}_{111}+\gamma h^{n+p}_{11}=0.$$

Substituting (5.1), (5.3) and (5.4) into (5.12), we obtain $\gamma=0$. Then we may get (5.10). The converse of this is trivial.

Theorem 3. *Let M be an n-dimensional spherical submanifold. If M is constant isotropic, then M has pointwise* 2*-planar normal sections if and only if the second fundamental form is parallel, i.e.,* $\nabla h\equiv 0$.

Proof. Let M be an n-dimensional spherical submanifold of E^{n+p} and constant isotropic. Then we have (4.8). In particular, we have

(5.13) $$\sum_s h^s_{11}h^s_{111}=0.$$

Differentiating (5.13) and using (2.8), (2.10), (4.7) and (4.8), we obtain

$$(5.14)\qquad \begin{aligned} 0&=e_1(\sum_s h^s_{11}h^s_{111})=\sum_s e_1(h^s_{11})h^s_{111}+\sum_s h^s_{11}e_1(h^s_{111}) \\ &=\sum_s h^s_{111}h^s_{111}+2\sum_{i,s} h^s_{1i}\theta^i_1(e_1)h^s_{111}-\sum_{u,s} h^u_{11}\theta^s_u(e_1)h^s_{111} \\ &\quad+\sum_s h^s_{11}h^s_{1111}+3\sum_{i,s} h^s_{11}h^s_{11i}\theta^i_1(e_1)-\sum_{s,u} h^s_{11}h^u_{11}\theta^s_u(e_1) \\ &=\sum_s h^s_{111}h^s_{111}+\sum_s h^s_{11}h^s_{1111}. \end{aligned}$$

If M has pointwise 3-planar normal sections, by Theorem 2, there are two func tions α and β not all zero such that

$$(5.15)\qquad \alpha h^r_{1111}+\beta h^r_{111}=0, \qquad r=n+1, \cdots, n+h.$$

Combining (5.14) and (5.15), we have, with the help of (5.13),

$$(5.16)\qquad \alpha\sum_s h^s_{111}h^s_{111}=0.$$

If $\alpha\equiv 0$, from (5.15) we have $h^r_{1l1}=0$, $r=n+1, \cdots, n+p$, since α and β are not all zero. If $\alpha\not\equiv 0$, from (5.16), we also have $h^r_{111}=0$, $r=n+1, \cdots, n+p$. Thus we obtain

$$(5.17)\qquad (\nabla_t h)(t, t)=0, \qquad \text{for any} \quad t\in TM.$$

According to Theorem 2 of (2), it implies $\nabla h\equiv 0$. The converse of this is trivial.

A pointwise k-planar normal section γ is said to be proper pointwise k-planar if, locally, γ is not pointwise $(k-1)$-planar. By Theorem A, we know that in Theorem 3, M must be a submanifold with pointwise 2-planar normal sections. Thus we have the following.

Corollary 1. *There is no constant isotropic spherical submanifold with proper pointwise 3-planar normal sections in Euclidean space.*

Corollary 2. *There is no constant isotropic surface in Euclidean space with proper pointwise 3-planar normal sections, if its mean curvature vector is parallel.*

Proof. Let M be a constant isotropic surface in a E^{2+p} with parallel mean curvature vector. From a result of Chen [6], M is one of the followings:

(i) an open portion of a 2-plane or a 2-sphere of a E^3 in E^{2+p};

(ii) a minimal surface in a S^{p+1} of E^{2+p} with $\|H\|^2\geq 3K$, where K is the Gauss curvature of M. The equality holds if and only if M is a Veronese surface.

In Case (i), M has $\nabla h\equiv 0$ obviously. In Case (ii), if M has pointwise 3-planar normal sections, by Theorem 3, M has $\nabla h\equiv 0$. Thus from Theorem A, M must have pointwise 2-planar normal sections and the corollary is obtained.

References

[1] B.Y. Chen: *Total Mean Curvature and Submanifolds of Finite Type*, World Sci., Singapore, 1984.

[2] B.Y. Chen: Submanifolds with planar normal sections, *Soochow J. Math.*, 7 (1931), 19-24.

[3] B.Y. Chen: Differential geometry of submanifolds with planar normal sections, *Ann. Mat. Pura Appl.*, 130 (1982), 59-66.

[4] B.Y. Chen and S.J. Li: Classification of surfaces with pointwise planar normal sections and its application to Fomenko's conjecture, *J. Geom.*, 26 (1986), 21-34.

[5] S.J. Li: Isotropic submanifolds with pointwise 3-planr normal sections, to appear in *Boll. Uni. Mat. Ital.*

[6] B.Y. Chen: On the surfaces with parallel mean curvature vector, *Indiana Univ. Math. J.*, 22 (1973), 655-666.

[7] B. O'Neill: Isotropic and Kaehler immersions, *Canadian J. Math.*, 17 (1965), 907-915.

Department of Mathematics
South China Normal University
Guangzhou, China