YOKOHAMA MATHEMATICAL
JOURNAL VOL. 35, 1987

ON NIELSEN'S THEOREM FOR 3-MANIFOLDS

By

WOLFGANG HEIL and JEFFREY L. TOLLEFSON

(Received April 14, 1986)

§1. Introduction.

A classical result of Nielsen [N] states that a map g of a (closed, orientable) surface F to itself such that the nth iterate g^n is homotopic to the identity is homotopic to a homeomorphism whose nth iterate is equal to the identity. In this paper we prove an analogous theorem for irreducible, sufficiently large 3-manifolds.

Nielsen's Theorem is a solution of a special case of the following *realization problem*: Let M be a compact n-dimensional manifold, let ϕ: Homeo$(M)\to\mathcal{H}(M)$ be the homomorphism that assigns to each homeomorphism of M its isotopy class in the homeotopy group of M. Let H be a finite subgroup of $\mathcal{H}(M)$. Does there exist a subgroup G of Homeo(M) such that $\phi|G$ is an isomorphism of G onto H?

For the 2-dimensional case various partial results have been obtained (see the discussion in [Z]), and a full solution has been given by Kerckhoff [K]. For 3-dimensional Seifert fiber spaces the problem was solved in [H. T.] for the case when H is the cyclic group of order 2, and subsequently in [Z. Z.] for an arbitrary finite group H. Using the Splitting Theorem of Jaco-Shalen and Johannson and Thurston's work on hyperbolic manifolds, we obtained in [H. T. II] the solution for the case that M is an orientable Haken manifold and $H=\boldsymbol{Z}_2$. B. Zimmermann following the same approach, has generalized them ethods of [Z. Z.] to obtain the solution in this case for an aribtrary finite group H. The approach of Zieschang-Zimmermann differs drastically from ours, since the former use the theory of crystollographic groups on the universal cover of M while we use methods from the topology of 3-manifolds. Still another proof for the case of Seifert fiber spaces has been given by W. Neumann and F. Raymond in an unpublished manuscript.

In the present paper we generalize the methods of [H. T.] and [H. T. II] to obtain a proof for the case when H is a finite cyclic group and M is an orientable Haken manifold. Our proof proceeds according to the following outline. First we obtain a relative version of the Nielsen Theorem for the case that M is a sufficiently large Seifert fiber space.

Theorem 1. *If g is a map of a compact, orientable, irreducible Seifert fiber space M with nonempty boundary such that $g^n \simeq 1$ rel ∂M then there is a homeomorphism h of M such that $h \simeq g$ rel ∂M and $h^n = 1$.*

The idea of the proof of Theorem 1 is to deform g such that afterwards for an essential annulus A in M, $\hat{A} = \cup g^i(A)$ is a system of disjoint annuli and g^n is homotopic to the identity by a homotopy that is constant on $\partial M \cup \hat{A}$. Then splitting M along the annuli of $\hat{A}$ we consider g restricted to the components of M split along $\hat{A}$ and use an induction argument to show that g can be deformed rel ∂M to the desired homeomorphism h.

Next we consider the case that M is a (possibly closed) Seifert fiber space.

Theorem 2. *Let M be a compact, orientable, irreducible Seifert fiber space which contains an incompressible fibered torus and let g be a map of M to itself such that g^n is homotopic to the identity. Then g is homotopic to a homeomorphism h with $h^n = 1$ if and only if* $\mathrm{Obs}(Z_n, \pi_1(M), \Psi) = 0$.

The obstruction that appears in the statement of Theorem 2 is interpreted geometrically in Lemma 6. We show that if it vanishes then we can find an essential torus T in M and a deformation of g such that afterwards $g^n \simeq 1$ rel T. Then we proceed as in the proof of Theorem 1.

The next step is to prove a relative version of the Nielsen Theorem for certain simple manifolds (Thm 3), by using the absolute version given by Thurston's and Mostow's work. Finally, to treat the general case when M is a closed Haken manifold we use the Splitting Theorem to split M along incompressible tori into simple 3-manifolds and Seifert fiber spaces and deform g such that afterwards $g^n \simeq 1$ by a homotopy that is constant on the splitting tori, and obtain the following result.

Theorem 4. *Let M be a closed Haken manifold that is not a Seifert fiber space. Suppose that g is a map of M to itself such that g^n is homotopic to the identity. Then g is homotopic to a homeomorphism h with $h^n = 1$.*

We will work in the PL-category throughout this paper. A surface F in M is usually assumed to be *properly* embedded, i.e. $F \cap \partial M = \partial F$. The terms *irreducible, parallel, essential, sufficiently large,* are standard (see [H] or [J]) and are also defined in [H.T., § 1]. Similarly the terms *Seifert fiber space, fibered solid torus, orbit surface, fiber-preserving,* are well-known (see e.g. [S.T.]) and are recalled in [H.T., § 1]. A homotopy $G: M \times I \to M$ is also denoted by G_t, where $G_0 = G | M \times 0$ and $G_1 = G | M \times 1$. The notation $G: f \simeq g$ rel F is used to mean that the homotopy G from f to g is constant on F, i.e. $G(x, t) = f(x) = g(x)$ for $x \in F$ and all $t \in I$. If $g: M \to M$ and $A \subset M$ then we let $\hat{A} = \bigcup_i g^i(A)$.

§ 2. Periodic maps on surfaces and liftings of isotopies.

Let F be a compact surface. An *essential arc* in F is a proper arc that is not homotopic rel ∂F to an arc on ∂F. An *essential curve* in F is a simple closed noncontractible curve.

Lemma 1. *Let $f: F \to F$ be a homeomorphism of period $n \geq 2$. Assume F is different form S^2, D^2, P^2. If $\partial F \neq \emptyset$ then there is an essential arc c in F such that $\{\cup f^i(c)\}$ is a union of pairwise disjoint arcs. If $\partial F = \emptyset$ then there is an essential curve c in F such that either* (i) *$\{\cup f^i(c)\}$ is a union of pairwise disjoint simple closed curves or* (ii) *the orbit surface F/f is a 2-sphere with three branch points, c intersects each $f^i(c)$ transversally and each component of $F - \{\cup f^i(c)\}$ is an open disk.*

Proof. Let $p: F \to F/f$ be projection onto the orbit surface and let $B \subset F/f$ be the set of branch points. If F/f differs from S^2, D^2, P^2 we can lift an essential arc (if $\partial F/f \neq \emptyset$) or an essential curve (if $\partial F = \emptyset$) that misses the zero dimensional components of B, to obtain the desired c.

(a) If $F/f = D^2$, observe that 1-dimensional components of B can only occur in ∂D^2 and since $F \neq S^2$, D^2, there are either no such components or at least two. In the former case B consists of at least two points and we find an essential arc c by lifting an arc on F/f that separates points of B. In the latter case we get c by lifting an arc joining two 1-dimensional components of B.

(b) If $F/f = P^2$ there are at least two branchpoints (since otherwise $F = S^2$ or P^2). We obtain c by lifting a simple closed curve on F/f that bounds a disk containing two branchpoints.

(c) If $F/f = S^2$ there are at least three branchpoints. If there are more than three, we obtain c as in case (b). Thus assume that B consists of three points. The surface $p^{-1}(F/f - U(B))$, where U is a regular neighborhood of B, is not planar and thus contains an essential curve d that is not homotopic to a product of boundary curves of F. Now $p(d)$ is homotopic in $F/f - U(B)$ to a curve α that is a product of two simple closed curves a and b that are generators of $\pi_1(F/f - U(B))$. If $p^{-1}(\alpha)$ is a simple closed curve, then we let this be c. Otherwise $p^{-1}(\alpha) \simeq (wc_1w^{-1})c_2$ where w is an arc starting at the basepoint of $\pi_1(F)$, c_1 is a simple closed curve and $c_2 = wc_3$ (for some arc c_3) is a closed curve. Since $p^{-1}(\alpha) \simeq d$, at least one of wc_1w^{-1} or c_2 is not homotopic to a product of boundary curves of F. If this is true for wc_1w^{-1} we let $c = c_1$. Otherwise we repeat this process using c_2 in place of $p^{-1}(\alpha)$. Eventually we must find a simple closed subloop c of $p^{-1}(\alpha)$ with the desired property. In either case c is essential in F (since F is obtained from $p^{-1}(F/f - U(B))$ by filling in the boundary curves with disks) and $F/f - p(c)$ consists of three disks, each containing a branchpoint.

Lemma 2. *Let F be a torus. If $f: F \to F$ has period n and if α is a simple closed curve on F with $f(\alpha)$ homotopic to α, then there is a simple closed curve β homotopic to α such that $\hat{\beta} = \cup f^i(\beta)$ is a system of simple closed curves.*

Proof. Define a Riemannian metric on F with respect to which f is an isometry. Since in every homotopy class of simple closed curves there is a geodesic, we can choose a geodesic simple closed curve β that is (freely) homotopic to α. Since $f^i(\beta) \simeq f^j(\beta)$ then if $f^i(\beta) \cap f^j(\beta) \neq \emptyset$ and $f^i(\beta) \neq f^j(\beta)$ there is a disk D on F with ∂D a union of an arc on $f^i(\beta)$ and an arc on $f^j(\beta)$. But then by replacing one of these arcs by an arc of shorter length in D we could construct a simple closed curve of shorter length in the homotopy class of α, a contradiction. Thus $f^i(\beta) = f^j(\beta)$ or $f^i(\beta) \cap f_j(\beta) = \emptyset$.

We now use Nielsen's Theorem and the method of [H. T. II, Theorem 2] to derive the following relative version of Nielsen's Theorem.

Proposition 3. *If $g: F \to F$ is a map such that $g^n \simeq 1$ $\operatorname{rel} \partial F$ there is a homeomorphism $h \simeq g$ $\operatorname{rel} \partial F$ such that $h^n = 1$.*

Proof. Since g induces an isomorphism on the fundamental group we can assume that g is a homeomorphism.

Case (1) F is an annulus, $F = S^1 \times I$.

Without loss of generality, we may assume that g is standard on ∂F. Thus, suppose $g | \partial F(z, \varepsilon) = (\alpha(z), \lambda(\varepsilon))$, where $\alpha(z) = \bar{z}$ or zz_0 for some fixed $z_0 \in S^1$, $\varepsilon = 0$ or 1, and $\lambda(t) = t$ or $1 - t$. Let l denote the arc $\{1\} \times I$. Observe that the homotopy class $\operatorname{rel} \partial F$ of any map $g: F \to F$ is determined by $g | \partial F \cup l$. Since $g(l) \simeq \{(\alpha(1) e^{2\pi i q t}, \lambda(t)) | 0 \leq t \leq 1\}$ $\operatorname{rel} \partial F$ for some integer q, it follows that $g \simeq h$ $\operatorname{rel} \partial F$, where $h(z, \mathrm{t}) = (\alpha(z) e^{2\pi i q t}, \lambda(\mathrm{t}))$ with α, q and λ determined by g. Clearly $h^n \simeq 1$ $\operatorname{rel} \partial F$ implies $h^n = 1$.

Case (2) F is not an annulus.

By a homotopy of g, constant on ∂F, we may assume that there is a neighborhood $U = \partial F \times I$ such that $g(U) = U$ and the homotopy $G: g^n \simeq 1$ carries U to itself at each stage. Let $F' = F \backslash \mathring{U}$. By Nielsen's Theorem [N] there is a homeomorphism h' of F', $h' \simeq g | F'$ and $(h')^n = 1$. By Baer's Theorem [Z], h' is isotopic to $g | F'$. Extend this isotopy to an isotopy of F constant on ∂F to get a map $h; F \to F$ such that $g \simeq h$ $\operatorname{rel} \partial F$, $h' = h | F'$ and $h | \partial F$ are maps of period n, and $h^n \simeq 1$ $\operatorname{rel} \partial F$ by a homotopy G with $G(U \times I) = U$. Let $h'' = h | U$. By case (1) it suffices to show that there is a homotopy $G_1: (h'')^n \simeq 1$ $\operatorname{rel} \partial U$.

To see this, assume that ∂F is connected (otherwise the construction applies to each component) and note that $h^n(x_0) = x_0$, for a basepoint $x_0 \in \partial F' \subset \partial U$, and that the trace τ of the cyclic homotopy $G | F' \times I: h'^n \simeq 1$ represents an element of the center of $\pi_1(F')$, which is trivial. Hence we can assume by a 2-dimensional version of lemma 7 that $\tau=x_0$. Thus we now have $h''|\partial U$ is periodic and $G''=G|U\times I$ is a homotopy $(h'')^n\simeq 1 \operatorname{rel}(\{x_0\}\cup\partial F)$ with $G_t''(\partial F')\subset\partial F'$. The homotopy $G''=G_0$ is now deformed by a homotopy $H: U\times I\times I\to U$ to a desired homotopy G_1 by $H|U\times I\times\{0\}=G_0$, $H(x, 0, t)=(h'')^n(x)$, and H is constant on

$$\partial F\times I\times I\cup U\times\{1\}\times I\cup\partial F'\times I\times\{1\}\cup\{x_0\}\times I\times I.$$

Thus H is defined on the boundary of the 3-cell $(\partial F'-\{x_0\})\times I\times I$ and can be extended over this 3-cell. Now H is defined on $U\times I\times\{0\}\cup\partial(U\times I)\times I$ and extends over $U\times I\times I$ to give the desired $G_1=H|U\times I\times\{1\}$.

The next proposition is a lifting theorem for Seifert fiber spaces. For such a space M let $p: M\to M^*$ be the projection onto its orbit surface, let E be the set of exceptional fibers and $E^*=p(E)$.

Proposition 4. *Let M be a sufficiently large Seifert fiber space. Let $h^*: M^*\to M^*$ be a map isotopic to the identity* $\operatorname{rel}(\partial M^*\cup E^*)$. *Then there is a map $h: M\to M$ that is isotopic to the identity* $\operatorname{rel}(\partial M\cup E)$ *such that $ph=h^*p$.*

Proof. Case (a) $\partial M\neq\emptyset$: The proof is by induction on the length of a hierachy for M defined by fibered essential annuli reducing M to solid tori. If $M^*=D^2$ and $E^*=\emptyset$ the assertion is trivial. If $M^*=D^2$ and E^* consists of one point, let l be an arc from a point on ∂M^* to E^* and proceed in a way similar to what follows. In any other case there is an essential arc l in $M^*\backslash E^*$ such that l and $h^*(l)$ are in general position. Then l is isotopic to $h^*(l)$ $\operatorname{rel}(\partial M^*\cup E^*)$ and $A=p^{-1}(l)$ is an essential annulus isotopic $\operatorname{rel}(\partial M\cup E)$ to $p^{-1}(h^*(l))$ by a fiber preserving isotopy that is constant on the fibers of $A\cap p^{-1}(h^*(l))$. Thus there is a fiber preserving isotopy $k: M\to M$, constant on $\partial M\cup E$, with $k^*h^*(l)=l$. Now there is a fiber preserving isotopy $j: M\to M$, constant on $\partial M\cup E$, with $j^*k^*h^*|U(l)$ the identity on a regular neighborhood U of l. But then (by the argument below in case (b)) $j^*k^*h^*$ is isotopic to the identity by an isotopy constant on $\partial M^*\cup E^*\cup U(l)$. By induction there is a fiber preserving $f: M\backslash\mathring{U}(A)\to M\backslash\mathring{U}(A)$ with $f^*|(M^*\backslash\mathring{U}(l))=j^*k^*h^*|(M^*\backslash\mathring{U}(l))$ and f isotopic to the identity $\operatorname{rel}(\partial(M\backslash\mathring{U}(A))\cup E)$. Extend f over U by the identity to get $\hat{f}: M\to M$. Then $k^{-1}j^{-1}\hat{f}$ induces h^*.

Case (b) $\partial M=\emptyset$: First suppose M contains an incompressible fibered torus T. We wish to apply the argument of case (a) to $l=p(T)$ and we obtain fiber preserving isotopies k and j, constant on E, with $j^*k^*h^*|l$ the identity on l and $j^*k^*h^*$ isotopic to the identity by a homotopy $G: M^*\times I\to M^*$, $\operatorname{rel} E^*$. To proceed with the proof in case (a), we wish to replace G by a homotopy that is constant on $E^*\cup l$. If we pick the basepoint x_0 on l, the trace $\tau=G(x_0\times I)$ commutes with l in $\pi_1(M^*\backslash E^*)$. Thus, if $M^*\backslash E^*$ is not a torus, it is homotopic to a power of l and G can be deformed near l so that afterwards $G(l\times I)\subset l$

and then by a further deformation of G we can assume that G is constant on l. If $M^*\backslash E^*$ is a torus we let t be a simple closed curve that meets l transversely in one point. Then $j^*k^*h^*(t)$ is isotopic to t by an isotopy fixed on l and clearly this isotopy lifts. Thus we can assume that $j^*k^*h^*|t$ is the identity on t. But now $j^*k^*h^*$ is isotopic to the identity by an isotopy constant on $t\cup l$.

If M does not contain an incompressible fibered torus, we have $M^*=S^2$ and E^* consists of 3 points. By an isotopy of M we can assume that $h^*\simeq 1$ rel $U(E^*)$. By case (a) there is a lift $h': M\backslash p^{-1}(U)\to M\backslash p^{-1}(U)$ of $h'^*=h^*|(M^*\backslash U)$ that is (fiber) isotopic to the identity rel ∂U. Extend h' over U by the identity to obtain a lift h of h^*.

Corollary 5. *Let M be a sufficiently large Seifert fiber space and let $f: M\to M$ be a homeomorphism such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ N & \xrightarrow{f^*} & N \end{array}$$

commutes. If f^* is isotopic to h^* rel$(\partial M^*\cup E^*)$ *then there is a lift h of h^* such that h is isotopic* rel$(\partial M\cup E)$ *to f.*

Proof. Apply proposition 4 to get a lift g of $(f^*)^{-1}h^*$ and let $h=fg$.

§3. Making homotopies constant on surfaces.

In this section, given a homeomorphism $g: M\to M$ of a Haken 3-manifold and a homotopy $G: g^n\simeq 1$ rel ∂M, and an essential annulus or torus F in M such that $\hat{F}=\cup g^i(F)$ is a union of disjoint annuli or tori, we would like to deform g and G so that afterwards $G: g^n\simeq 1$ rel$(\partial M\cup \hat{F})$.

Let $g: M\to M$ be normalized by a homotopy such that a basepoint x_0 of M is fixed by g. Then g induces an automorphism g_* of $\pi=\pi_1(M, x_0)$. Let $\Psi: Z_n\to \mathrm{Out}\,\pi=\mathrm{Aut}\,\pi/\mathrm{Inn}\,\pi$ be $\Psi(k)=[g_*^k]$. Recall that a necessary condition for g to be homotopic to a map h of period n is that the abstract kernel (Z_n, π, Ψ) has an extension (see [H. T.] §2 or [C. R.]).

Lemma 6. *Suppose* center $(\pi)\simeq \boldsymbol{Z}$, *and let $\tau(t)=G_t(x_0)$ be the trace of the homotopy $G: g^n\simeq 1$. Then* $\mathrm{Obs}(Z_n, \pi, \Psi)=0$ *if and only if G can be chosen such that $g_*(\tau)=\tau$.*

Proof. If $\mathrm{Obs}(Z_n, \pi, \Psi)=0$ there is an extension E of the abstract kernel

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi & \longrightarrow & E & \longrightarrow & Z_n & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\Psi} & & \\ 1 & \longrightarrow & \operatorname{Inn}\pi & \longrightarrow & \operatorname{Aut}\pi & \longrightarrow & \operatorname{Out}\pi & \longrightarrow & 1 \end{array}$$

and we can choose $e \in E$ such that $\mu(e)=g_*$ (where $\mu(\alpha)$ is the inner automorphism $\alpha^{-1}(\ )\alpha$). Then $\mu(e^n)=g_*^n=\mu(\tau)$. Thus there exists $c \in \text{center}\ (\pi)$ such that $e^n=\tau c$. It follows that $g_*(\tau c)=\mu(e)(\tau c)=\mu(e)(e^n)=e^n=\tau c$. Since M admits an action of S^1 with center (π) generated by a principal orbit, there is a cyclic homotopy $H: 1 \simeq 1$ with trace c (namely $H(x, t)=\alpha(t)x$, where $\alpha(t)$ is a path in $(S^1, 1)$ with $f_*^{x_0}([\alpha])=c$ for the evaluation map $f^{x_0}: S^1 \to M$). Composing G with this cyclic homotopy yields a homotopy $g^n \simeq 1$ with trace τc.

Conversely, suppose $g_*(\tau)=\tau$, and $g_*^n=\mu(\tau)$. We define $\phi(\bar{s})=g_*^s$ $(0 \le s < n)$, where $Z_n=\{\bar{r} \mid 0 \le r < n\}$, and $f(\bar{r}, \bar{s})=\begin{cases} 0 & \text{if } r+s<n \\ \tau & \text{if } r+s \ge n \end{cases}$.

It is easily checked that the identities of lemma 8.1 in [McL] are satisfied and hence that (Z_n, π, Ψ) has an extension.

Lemma 7. *Let $G: M \times I \to M$ be a homotopy of pairs (M, F), where F is a surface properly embedded in M. Suppose $G_0(x_0)=G_1(x_0)=x_0 \in F$ and τ' is a loop in F homotopic to $G_t(x_0)$* rel x_0 *in F. Then there is a homotopy $H: M \times I \to M$ of pairs (M, F) such that $H_0=G_0$, $H_1=G_1$, H agrees with G outside a neighborhood of F and H has trace τ'.*

Proof. This is lemma 3.1 of [H. T.].

Lemma 8. *Let $\hat{F}$ be a system of incompressible surfaces in M and suppose that $f: M \to M$ is a homeomorphism with $f(\hat{F})=\hat{F}$ and $f \simeq 1$. Suppose* a) *or* b) *holds.*

a) *$F' \cap \partial M=\partial F' \neq \emptyset$ for each component F' of $\hat{F}$ and $f \simeq 1$* rel ∂M.

b) *Each component of $\hat{F}$ is closed and there is a homotopy $H: f \simeq 1$ such that for some points $x_1, \cdots, x_m$ (one in each component of $\hat{F}$) the traces $\tau_i(t)=H(x_i, t)$ are constant.*

Then there is a homotopy $G: f \simeq 1$ (constant on ∂M in case (a)*) such that $G(\hat{F} \times I) \subset \hat{F}$. Furthermore, if $f \mid \hat{F}=id \mid \hat{F}$, then G can be taken constant on $\hat{F}$.*

Proof. Case (a) is just lemma (4.4) of [H. T.]. We give the proof in case (b) which is similar:

Let F be the component of $\hat{F}$ containing x_1 and let $h_0=H \mid F \times I: F \times I \to M$. We first define $h_1: F \times I \to F$ as follows. Let $c_1, \cdots, c_k$ be based simple closed curves on F that cut F into a disk and that are mutually disjoint, except that they meet at x_1. H defines a based homotopy from $f(c_i)$ to c_i. Since F is incompressible, there is a based homotopy from $f(c_i)$ to c_i *on* F (if f is the identiy on $\hat{F}$, we choose this homotopy constant) and we let $h_1: \cup c_i \times I \to F$ be this homotopy. On $(F-\cup c_i)\times\{1\}$ we let h_1 be the identity and $h_1|(F-\cup c_i)\times\{0\}=f|F-\cup c_i$. Now h_1 is defined on the boundary of the 3-cell $(F-\cup c_i)\times I$ and can be extended over this 3-cell, since F is aspherical.

Now we show that h_0 and $h_1: F\times I\to F\subset M$ are homotopic $\mathrm{rel}(\{x_1\}\times I)$. Using h_0 and h_1, define this homotopy $K: F\times I\times I\to M$ on $F\times I\times\partial I$. On $\{x_1\}\times I\times I$ let K be constant and let $K(x, 0, t)=f(x)$, $K(x, 1, t)=x (x\in F)$. Now K is defined on $\partial(c_i\backslash x_0\times I\times I)$, where $c_i\backslash x_0$ is a 1-cell. Since M is aspherical we can extend K over $\cup c_i\times I\times I$, and having K defined on the boundary of the 4-cell $(F-\cup c_i)\times I\times I$ we can extend K again over all of $F\times I\times I$.

Finally, we extend the homotopy from $h_0=H|F\times I$ to h_1 to a homotopy from H to a homotopy $G: M\times I\to M$ with $G|F\times I=h_1$.

Lemma 9. *Let F be an essential surface in M, $g: M\to M$ a homeomorphism such that $g^n\simeq 1$ and $\hat{F}=\cup g^i(F)$ is a union of disjoint surfaces.*

(i) *If $\partial F\neq\emptyset$ and if $g^n\cong 1$ $\mathrm{rel}\,\partial M$ then there is a homeomorphism $h\simeq g$ $\mathrm{rel}\,\partial M$ such that $h(\hat{F})=g(\hat{F})$ and $h^n\simeq 1$ $\mathrm{rel}(\hat{F}\cup\partial M)$.*

(ii) *If F is closed and $\mathrm{Obs}(Z_n, \pi, \Psi)=0$, then there is a homeomorphism $h\simeq g$ such that $h^n\simeq 1$ $\mathrm{rel}\,\hat{F}'$ and $h(\hat{F}')\cong g(\hat{F})$ where $\hat{F}'=\cup h^i(F)$ is a system of disjoint surfaces. In either case, if M is a Seifert fiber space and F is a union of fibers, and g is fiber preserving, then h can be taken to be fiber preserving.*

Remark. If M does not fiber over S^1 with F isotopic to a fiber, we can choose $\hat{F}'=\hat{F}$ in case (ii).

Proof. *Case* (i) Let m denote the smallest positive integer such that $g^m(F)=F$ and let $p=n/m$. By lemma 8 (a) there is a homotopy $G:(g^m)^p\simeq 1$ such that $G(\hat{F}\times I)\subset\hat{F}$. By the relative Nielsen Theorem (Prop. 3) there is a homeomorphism (fiber preserving if F is a fibered annulus) $f: F\to F$ such that $f^p=1$ and $f\simeq g^m|F\,\mathrm{rel}\,\partial F$. Let $K:(\hat{F}\cup\partial M)\times I\to\hat{F}\cup\partial M$ be an isotopy such that $K_t|(\hat{F}\backslash F)\cup\partial M$ is the identity and $K|F\times I$ is a homotopy from the identity to $f\circ g^{-m}|F\,\mathrm{rel}\,\partial F$. We can extend K to $M\times I\to M$ so that K is constant outside a regular neighborhood U of F and $K_1: M\to M$ is a homeomorphism. Let $h=K_1\circ g$. Then $(h|F)^m=f$ and $(h|F)^n=1|F$. But then $(h|\hat{F})^n$ is the identity on $\hat{F}$, since $h^{-i}\circ(h^n|h^i(F))\circ h^i|_F=(h|F)^n=1|F$. By lemma 8(a), $h^n\simeq 1$ $\mathrm{rel}(\hat{F}\cup\partial M)$.

Case (ii) First we normalize the homotopy $G: g^n\simeq 1$ by composing it with a suitable cyclic homotopy as follows: Let $x_0\in F$ and let $y_0\in M-\hat{F}$ and temporarily choose $h\simeq g\,\mathrm{rel}\{g^i(x_0)\}_{i=0}^{n-1}$ such that $h(y_0)=y_0$. Thus $h^n\simeq g^n\,\mathrm{rel}\{g^i(x_0)\}_{i=0}^{n-1}$. Consider $h^n\simeq g^n\simeq 1$. Since $\mathrm{Obs}(Z_n, \pi, \Psi)=0$ we can apply lemma 4 to find a cyclic homotopy L such that for the trace γ of y_0 under $K: h^n\simeq g^n\overset{G}{\simeq}1\overset{L}{\simeq}1$ we have $h(\gamma)\simeq\gamma$ (rel y_0). Let τ_i denote the trace of $g^i(x_0)$ under $L\circ G: g^n\simeq 1$ and note that τ_i is also the trace of $g^i(x_0)$ under the homotopy K. For a path λ from x_0 to y_0 we obtain, by restricting K to the path $h^i(\lambda)$, that $\gamma\simeq h^n(\lambda^{-1})\tau_0\lambda$ (rel y_0) and $\gamma\simeq h^{n+i}(\lambda^{-1})\tau_i h^i(\lambda)$ (rel y_0). Applying h^i to the first homotopy we obtain $\gamma\simeq h^i(\gamma)\simeq h^{n+i}(\lambda^{-1})h^i(\tau_0)h^i(\lambda)$, and comparing this with the second homotopy we get $h^i(\tau_0)\simeq\tau_i$ (rel $g^i(x_0)$). Since $g^i\simeq h^i$ rel$\{g^j(x_0)\}_{j=0}^{n-1}$, we have $g^i(\tau_0)\simeq\tau_i$ rel $g^i(x_0)$.

Now we follow the proof of Theorem 7.1 case 4 in [W] and Lemma 3.3 [H. T.] to deform g to a map g' for which there exists an isotopy from g'^n to the identity which carries $\hat{F}$ onto itself at each stage. A sketch of this proof follows: Let $\bar{F}$ be a surface homeomorphic to $g^i(F)$ and let $f:\bar{F}\times I\to M$ be $G|g^i(F)\times I$. We can assume (by a small homotopy of g) that $f^{-1}(g^i(F))\cap U(\bar{F}\times\partial I)=\bar{F}\times\partial I$ (where U is a regular neighborhood of $\bar{F}\times\partial I$). By transversality, there is a homotopy of f, constant on $U(\bar{F}\times\partial I)$, after which $f^{-1}(g^i(F)\cap(\bar{F}\times I-\operatorname{Int}U(\bar{F}\times\partial I))$ is a system of incompressible surfaces, each parallel to $\bar{F}\times 0$. Any two adjacent components bound a domain $\bar{F}\times I'$ and there is a lift $\tilde{f}:\bar{F}\times I\to M'$ of $f|\bar{F}\times I'$, where M' is M cut along $g^i(F)$. Applying Waldhausen's homeomorphism theorem (6.1) of [W], it follows that there is a homotopy of f, constant on $f^{-1}(g^i(F))$, such that either $f(\bar{F}\times I)\subset g^i(F)$ (the desired case) or M is fibered over S^1 with fiber F (cf. [W], p, 84). In the latter case we can deform g so that $g(F)=F$ and $g(x_0)=x_0$, for a base point $x_0\in F$. As before, we can assume that $g(\tau)\simeq\tau$ rel x_0, where g is the trace of x_0 under $G:g^n\simeq 1$. By an isotopy of g which slides F around, we get $g\simeq g'$ and an isotopy $G:g'^n\simeq 1$ whose trace represents an element of $\pi_1(F, x_0)$. Let $q:F\times R^1\to M$ be the infinite cyclic cover corresponding to $\pi_1(F, x_0)$, let $\tilde{f}:F\times I\to F\times R^1$ be the lift of $f=G|F\times I$, and let Γ_s be a strong deformation retraction of $F\times R^1$ onto the componcent $\tilde{F}$ of $p^{-1}(F)$ that contains $\tilde{f}(F\times\{0\})$ and $\tilde{f}(F\times\{1\})$. Then $q\circ\Gamma_s\circ\tilde{f}$ is a homotopy of f, constant on $F\times\partial I$, to a map $f':F\times I\to F\subset M$, as desired.

Thus we can now assume that $G:g^n\simeq 1$ is a homotopy with $G(\hat{F}\times I)\subset\hat{F}$ and such that for the trace τ_i of $g^i(x_0)$ under G we have $g^i(\tau_0)\simeq\tau_i$ rel $g^i(x_0)$. This implies $(g^m|F)^p\simeq 1|F$ where m is the smallest positive integer such that $g^m(F)=F$ and $n=p\cdot m$.

Subcase (a) $m=n$.

Let $U_i=g^i(F)\times[-1, 1]$ a regular neighborhood of $F_i=g^i(F)$ ($i=0, \cdots, n-1$) and let $f_i=g|F_i:F_i\to F_{i+1}$. We can assume that

$$g(x, s)=(f_i(x), s) \qquad \text{for} \quad (x, s)\in U_i,\ i=0, \cdots, n-1$$

and

$$G_t(x, s)=(k(x, t), s) \quad \text{for} \quad (x, s)\in U_0$$

where k is an isotopy $k:g^n\simeq 1$ on $F_0=F$. Note that $g^n(x, s)=(f_{i-1}\cdots f_0f_{n-1}\cdots f_i(x), s)$ on U_i Deform g to g_1 by

$$g_t(x, s)=\begin{cases} g(x, s) & \text{off } U_0 \\ (f_1^{-1}\cdots f_{n-1}^{-1}k(x, (1-|s|)t), s) & \text{on } U_0 \end{cases}$$

observe that $g_1^n(x, s)=(f_i^{-1}\cdots f_{n-1}^{-1}k(f_{n-1}\cdots f_i(x), 1-|s|), s)$ on U_i, in particular $g_1^n|F_i=1|F_i$, and define $H_t: g_1^n\simeq 1$ by

$$H_t|M\backslash\hat{U}=\begin{cases} g_1^n & \text{for } t\leq 1/2 \\ G_{2t-1} & \text{for } t\geq 1/2 \end{cases}$$

and

$$H_t|_{U_i}(x, s)=\begin{cases} f_i^{-1}\cdots f_{n-1}^{-1}k(f_{n-1}\cdots f_i(x), (1-|s|)(1-2t)), s) & \text{for } t\leq 1/2 \\ G_{2t-1}(x, s) & \text{for } t\geq 1/2 \end{cases}$$

Observe that the trace of $g^i(x_0)$ under the homotopy H_t is homotopic (rel $g^i(x_0)$) to the product of the paths $g^i(\tau_0^{-1})\ \tau_i$, which is homotopic to 0 rel $g^i(x_0)$. By lemma 7 we can therefore find a homotopy $K: g_1^n\simeq 1$ with $K(\hat{F}\times I)\subset\hat{F}$ and such that the traces $K(g_1^i(x_0), t)$ are constant. Now lemma 8(b) can be applied to finish the proof in this case.

Ssubcase (b) $m\neq n$ i.e. $p>1$.

By Nielsen's Theorem for 2-manifolds there is a homeomorphism $g': F\to F$ with $(g')^p=1$ and an isotopy $G': F\times I\to F$ from the identity to $g'g^{-m}|F$. We extend G' to an isotopy of M, constant outside a regular neighborhood of F. Let $h'=G_1'g$. Writing again g instead of h we now have $g(\hat{F})=\hat{F}$, $(g|\hat{F})^n=1|\hat{F}$, $G: g^n\simeq 1$, where $G(\hat{F}\times I)\subset\hat{F}$. As before let $U_i=g^i(F)\times[-1, 1]$ and let $f_i=g|F_i: F_i\to F_{i+1}$ for $i=0, \cdots, m-1$, and assume that

$$g(x, s)=(f_i(x), s)$$

$$G_t(x, s)=(h_i(x, t), s) \qquad \text{for} \quad (x, s)\in U_i.$$

Let $k_i=f_{i-1}\cdots f_0f_{m-1}\cdots f_{i+1}f_i$ and observe that $g^n(x, s)=(k_i^p(x), s)=(x, s)$ on U_i. In particular, $h_i: F_i\times I\to F_i$ is a cyclic homotopy and the trace τ_i under h_i lies in the center of $\pi_1(F_i)$. Consider first h_0. If $\tau_0\neq 0$ it follows that F is a torus and $\tau_0\simeq\gamma^r$ for some simple closed curve γ on $F_0=F$. Since $\mathrm{Obs}(Z_n, \pi, \Psi)=0$ we can by lemma 6 assume that $k_0(\tau_0)=g^m(\tau_0)\simeq\tau_0$ rel x_0 and hence $k_0(\gamma)\simeq\gamma$. By lemma 2 we can furthermore choose γ so that $\cup k_0^j(\gamma)$ is a system of simple closed curves. There is a homotopy $B: 1_{F_0}\simeq\beta$, where $\beta: F_0\to F_0$ is a homeomorphism such that $\beta^p=1$, $\beta k_0=k_0\beta$, and the trace of x_0 under B is $\gamma^{+r/p}$. To see this, parametrize $F_0=S^1\times S^1$ such that $\gamma\simeq 1\times S^1$ and $k_0(z_1, z_2)=\left(z_1^az_2^be^{\frac{2\pi i\lambda}{p}},\right.$ $\left.z_1^cz_2^de^{\frac{2\pi i\mu}{p}}\right)$, where $\begin{pmatrix} ab \\ cd \end{pmatrix}^p=1$. Since $k_0(\gamma)\simeq\gamma$ we have $\begin{pmatrix} ab \\ cd \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, hence

$a=\pm 1$, $d=1$, $b=0$. Let $\beta(z_1, z_2)=\left(z_1, z_2 e^{\frac{2\pi i r}{p}}\right)$ and observe that there is an obvious homotopy $B: 1\simeq\beta$ with trace $\gamma^{r/p}$.

Now deform g to $g_1=H_1$ by the homotopy

$$H_t(x, s)=\begin{cases} g(x) & \text{off } U_0 \\ (f_0 B(x, (1-|s|)t), s) & \text{for } (x, s)\in U_0 . \end{cases}$$

Note that $g_1|F_0=f_0\beta$ and $(g_1|F_0)^n=(k_0\beta)^p=k_0^p\beta^p=1|F_0$. Furthermore the homotopy form g to g_1 induces a homotopy $H'=g_1^n\simeq g^n$ with trace of x_0 homotopic to γ^{-r}. Thus the trace of $GH': g_1^n\simeq 1$ is homotopic to $\gamma^{-r}\cdot\gamma^r\simeq 0$. By lemma 7 we can assume that $g_1^n\simeq 1$ with trace of x_0 constant.

Repeat this construction for $i=1, \cdots, m-1$ to get a homeomorphism $h\simeq g$, $(h|\hat{F})^n=1|\hat{F}$ and $h^n\simeq 1$ $\mathrm{rel}(x_0\cup\cdots\cup x_{m-1})$. Again lemma (8a) can now be applied to obtain $h^n\simeq 1$ $\mathrm{rel}\,\hat{F}$.

§ 4. Nielsen's Theorem for Seifert fiber spaces and simple 3-manifolds.

We first prove a relative version of Nielsen's Theorem for Seifert fiber spaces.

Lemma 10. *Let M be a compact, orientable, irreducible Seifert fiber space with nonempty boundary. If g is a fiber preserving map of M with $g\simeq 1$ $\mathrm{rel}\,\partial M$ then there is a homeomorphism h of M with $h\simeq g$ $\mathrm{rel}\,\partial M$ and $h^n=1$.*

Proof. The proof proceeds by induction on $c(M)$, the sum of the number of exceptional fibers in M together with the minimal length of a hierarchy of M defined by fibered annuli reducing M to solid tori.

By [W] we can assume that g^n is isotopic to the identity by a fiber isotopy $\mathrm{rel}\,\partial M$. Thus we obtain a commutative diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d] & M \arrow[d] \\ M^* \arrow[r, "g^*"] & M^* \end{tikzcd}$$

where $(g^*)^n\simeq 1$ $\mathrm{rel}\,\partial M^*\cup E^*$. Now g^* permutes the points of E^* and we assume that g^* is the identity on E^* (the case when g^* is not constant on E^* is similar). Thus we find a regular neighborhood U of E^* such that g maps each component D of U to itself and such that the isotopy $(g^*)^n\simeq 1$ maps D to itself at each stage. Apply the relative Nielsen Theorem (Prop. 3) to $g^*|(M^*\backslash\mathrm{Int}\,U)$ to obtain a homeomorphism $f^*:(M^*\backslash\mathrm{Int}\,U)\rightarrow(M^*\backslash\mathrm{Int}\,U)$ of period n such that $f^*\simeq g^*|(M^*\backslash\mathrm{Int}\,U)$ $\mathrm{rel}\,\partial M^*$. For each component D of U, $f^*|\partial D$ is a rotation and we extend f^* over D by coning from $D\cap E^*$. The isotopy from $f^*|(M^*\backslash$

$\operatorname{Int} U$) to $g^*|(M^*\backslash\operatorname{Int} U)$ can now easily be extended to an isotopy from f^* to g^* that is constant on E^*. By Corollary 5 we can lift f^* to get a fiber preserving map that is homotopic to g by a homotopy constant on ∂M. If we denote this new map again by g, we now have $g^n \simeq 1$ rel ∂M and $g^n_* = 1$.

Applying lemma 1 we lift an essential arc in $M^*\backslash E^*$ to obtain an essential annulus A in M such that $\hat{A} = \{\bigcup_i g^i(A)\}$ is a system of disjoint annuli. By lemma 9 there is a homeomorphism $f \simeq g$ rel ∂M such that $f(\hat{A}) = \hat{A}$ and $f^n \simeq 1$ rel $(\hat{A} \cup \partial M)$.

Let M_i be the components of M split along $\hat{A}$ ($i=1, \cdots, q$) and let $\bigcup_{i=1}^{q} M_j = \bigcup_{i=1}^{r} \hat{M}_j$, where $\hat{M}_j = \left\{\bigcup_{i=1}^{n} f^i(M_j)\right\}$.

If $f|M_i : M_i \to M_i$ then $(f|M_i)^n \simeq 1$ rel ∂M_i and by induction or lemma 11(a) below (since $c(M_i) < c(M)$) there is an isotopy k_1^i of M_i, constant on ∂M_i, such that $(k_1^i \circ f|M_i)^n = 1|M_i$, and we extend k_1^i by the identity on $M\backslash M_i$ to an isotopy of M. Otherwise, let m_i be the smallest positive integer such that $f^{m_i}(M_i) = M_i$ and let $p_i = n/m_i$. Since $((f|M_i)^{m_i})^p \simeq 1$ rel ∂M_i we find again an isotopy k_1^i of M, constant on $M\backslash M_i$, such that $(k_1^i \circ (f|M_i)^{m_i})^p = 1$. Let $h = k_1^r \cdots k_1^1 \circ f$ and observe that $h \simeq f$ rel ∂M and $h^n = 1$ since for each i, j, $h^n|M_i = (k_1^i(f|M_i)^{m_i})^p$ and $h^n|f_1^j(M_i) = f^j(k_1^i f^{m_i})^p f^{-j}|f^j(M_i) = 1|f^j(M_i)$.

Lemma 11. *Let M be a disjoint union of any of the following spaces:*

(a) $D^2 \times S^1$

(b) $S^1 \times S^1 \times I$

(c) *the orientable S^1-bundle over the Moebius band.*

If g is a map of M such that $g^n \simeq 1$ rel ∂M then there is a homeomorphism $h \simeq g$ rel ∂M such that $h^n = 1$. Furthermore, h is fiber preserving for some fibering of M. In case (a), if $g|\partial M$ is fiber preserving for a given Seifert fibering of M then h can be chosen to be fiber preserving.

Proof. As in the proof of lemma 10 it suffices to deal with the case when M is connected. Also we can assume that g is a homeomorphism.

(a) Let α be a meridian curve on ∂M. Since $g(\alpha) \simeq \alpha$ there is by lemma 2 a simple closed curve $\beta \simeq \alpha$ with $\bigcup_i g^i(\beta)$ a system of simple closed curves. Let $\{D_{\lambda(i)}\}$ be a family of pairwise disjoint disks in M such that $D_{\lambda(i)} \cap \partial M = \partial D_{\lambda(i)} = g^i(c)$, where $\lambda(i) = \lambda(j)$ if $g^i(c) = g^j(c)$. Define h by taking $h|\partial M = g|\partial M$, then extending h over the meridian disks $D_{\lambda(i)}$ by coning, and finally extending h over the rest of M which is a disjoint union of open 3-cells. It is easy to see that h can be constructed so as to preserve fibers, if $g|\partial M$ is fiber preserving.

(b) We consider only the case that $g(S^1 \times S^1 \times 0) = S^1 \times S^1 \times 0$, since the other case is similar. We parametrize $S^1 \times S^1 \times \{0\}$ so that for $j=0$ the homeomorphism $g|\partial M$ has the following form:

$$g|_{S^1\times S^1\times\{j\}}: (z_1, z_2, j) \longrightarrow (z_1^a z_2^b \exp(2\pi i\alpha_j/n), z_1^c z_2^d \exp(2\pi i\beta_j/n), j)$$

where $a, b, c, d, \alpha_j, \beta_j$ are integers, z_1, z_2, are points of S^1 considered as unit sphere in the complex plane, and the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has order n.

Since g defines a homotopy between $g|S^1\times S^1\times\{0\}$ and $g|S^1\times S^1\times\{1\}$, we find an isotopic pair of coordinate curves on $S^1\times S^1\times\{1\}$ with respect to which $g|S^1\times S^1\times\{1\}$ has the above form for $j=1$. We extend these parametrizations to a poduct structure on $M=S^1\times S^1\times I$.

Let $l=(1, 1)\times I\subset M$. If h is a homeomorphism of M such that $h|\partial M= g|\partial M$ then observe that $g\simeq h$ rel ∂M if and only if $g(l)\simeq h(l)$ rel ∂M. (This is because $(M, l\cup\partial M)$ has a relative cell-decomposition into open 2- and 3-cells and therefore all obstructions to defining the homotopy $h\simeq g$ rel ∂M vanish.) Using this observation we can now define the desired periodic homeomorphism $h\simeq g$ rel ∂M as follows. Choose integrs p and q so that the arc

$$\left\{\left(\exp 2\pi i\left(\left(p+\frac{\alpha_1-\alpha_0}{n}\right)t+\frac{\alpha_1}{n}\right), \exp\left(2\pi i\left(q+\frac{\beta_1-\beta_0}{n}\right)t+\frac{\beta_1}{n}\right), t\right); t\in I\right\}$$

is homotopic rel ∂M to the arc $g(l)$. Then define h by

$$h(z_1, z_2, t)=\left(z_1^a z_2^b \exp\left(2\pi i\left(\left(p+\frac{\alpha_1-\alpha_0}{n}\right)t+\frac{\alpha_1}{n}\right)\right),\right.$$
$$\left. z_1^c z_2^d \exp\left(2\pi i\left(\left(q+\frac{\beta_1-\beta_0}{n}\right)t+\frac{\beta_1}{n}\right)\right), t\right).$$

Thus, by construction $h|\partial M=g|\partial M$ and $h(l)\simeq g(l)$ rel ∂M. Moreover, $h^n(z_1, z_2, t)$ $=(z_1\exp(2\pi iu(t)), z_2\exp(2\pi iv(t)), t)$ where u, v are linear functions of t. Hence $h^n=1$.

(c) M can be fibered over S^1 with fiber an annulus A. Moreover M contains a Seifert fibered annulus and any two properly embedded essential annuli are isotopic. Thus $g(A)$ is isotopic to A and $g(\partial A)\simeq\partial A$ on ∂M. By lemma 2 we can assume (after an isotopy of A) that $g(\partial A)=\partial A$ or $g(\partial A)\cap\partial A=\emptyset$. Thus we can define a Seifert fibering of M so that A is a union of fibers and $g|\partial M$ is fiber preserving. By [W] we can then deform g relative to ∂M to make it fiber preserving on all of M. Now lemma 10 applies.

Theorem 1. *Let M be a compact, orientable, irreducible Seifert fiber space with nonempty boundary. If g is a map of M such that $g^n\simeq 1$ rel ∂M then there is a homeomorphism h of M such that $h\simeq g$ rel ∂M and $h^n=1$.*

Proof. If M is not one of the three spaces of lemma 11 then by [W], g is isotopic to a fiber preserving homeomorphism. If T is a boundary component of M invariant under g then a fiber on T is isotopic to $g(\alpha)$ on T. Thus by lemma 2 we can assume, after an isotopy of the fibering of M near T, that

$g(\alpha)=\alpha$ or $g(\alpha)\cap\alpha=\emptyset$, and then that $g|T$ is fiber preserving. If T is a boundary component of M not invariant under g and if m is the smallest positive integer such that $g^m(T)=T$, we isotope the fibering near T so that g^m is fiber preserving on T and choose the fibering on $g(T), \cdots, g^{m-1}(T)$ to be induced by g. Thus we can assume that $g|\partial M$ is fiber preserving and by [W] we can further deform g, by an isotopy constant on ∂M, such that afterwards g is fiber preserving on M. Now lemma 10 applies.

We now consider the case that $\partial M=\emptyset$.

Lemma 12. *Let M be one of the following spaces.*

(a) $S^1\times S^1\times S^1$.

(b) *A sufficiently large Seifert fiber space with orbit surface S^2 and with exactly three exceptional fibers.*

(c) *The orientable S^1-bundle over the Klein bottle.*

(d) *The union of two twisted I-bundles over the Klein bottle identified along the two boundary components via an involution interchanging the factors of $S^1\times S^1$.*

Let g be a map of M with $g^n\simeq 1$ $\operatorname{rel}\partial M$. In cases (b) and (c) assume that $\operatorname{Obs}(Z_n, \pi, \Psi)=0$. *Then there is a homeomorphism h of M with $h\simeq g$ $\operatorname{rel}\partial M$ and $h^n=1$.*

Proof. (a) Let the matrix of g_* on $\pi_1(M)=Z\oplus Z\oplus Z$ be (a_{ij}) and define $h: S^1\times S^1\times S^1\to S^1\times S^1\times S^1$ by $h(z_1, z_2, z_3)=(z_1^{a_{11}}z_2^{a_{12}}z_3^{a_{13}}, z_1^{a_{21}}z_2^{a_{22}}z_3^{a_{23}}, z_1^{a_{31}}z_2^{a_{32}}z_3^{a_{33}})$. Then $h_*=g_*$ hence $h\simeq g$, and $h^n=1$.

(b) M contains an incompressible two-sided surface F that is not a fibered torus and therefore does not carry the infinite cyclic center of $\pi_1(M)$. It follows that M can be fibered over S^1 with F as fiber $M\approx M_\phi=F\times I/\phi$ where ϕ_*^k is an inner automorphism for some integer k. By Nielsen's theorem we may assume $\phi^k=1$. If the image of the center z of $\pi_1(M)$ has rank 1 in $H_1(M)$ we can apply [H.T. lemma 6.2] to choose F such that $g(F)\simeq F$. In any other case $\pi_1(F)\cap z\neq 1$ and F is a torus. But periodic homeomorphisms $\phi: S^1\times S^1\to S^1\times S^1$ are classified in [H; 12.3] and an element of $\pi_1(F)\cap z$ is fixed by ϕ_*. The only orientation preserving ϕ for which ϕ_* leaves a nontrivial element fixed is $\phi=1$, which yields $M_\phi=S^1\times S^1\times S^1$, which is not homeomorphic to M. Thus in any case g can be deformed so that $g(F)=F$. By lemma 9 (ii) we can further isotope F and g so that afterwards $g^n\simeq 1$ $\operatorname{rel}\hat{F}$, where $\hat{F}=\bigcup_i g^i(F)$ is a system of surfaces. Now we can apply lemma 11 (b) to M cut along $\hat{F}$ to finish the proof.

(c) M can also be represented as $M=S^1\times S^1\times I/\phi$, where $\phi(x, y)=(\bar{x}, \bar{y})$. Therefore the argument of (b) applies.

(d) This manifold is the Seifert fiber space $\{-1; (n_2, 1); (2, 1), (2, 1)\}$ known as the "Hantzsche-Wendt" manifold and is discussed as Example 1 in Charlap and Vasquez [C. V.]. They show that Out $\pi \cong$ group of Affinities of M, where M is viewed as a flat Riemannian manifold. In fact they calculate Out π explicitely and show that it has order 96. A different proof of this is given in [M].

Theorem 2. *Let M be a compact, orientable, irreducible Seifert fiber space which contains an incompressible fibered torus. Suppose that g is a map of M to itself such that g^n is homotopic to the identity. Then g is homotopic to a homeomorphism h of period n if and only if* $\mathrm{Obs}(Z_n, \pi_1(M), \Psi)=0$. *Furthermore, if M is not one of the exceptional cases considered in lemmas* 11 *and* 12, *then h can be chosen to be fiber preserving.*

Proof. We exclude those manifolds covered by lemmas 11 and 12. Thus we may assume that g is fiber preserving and $g^n \simeq 1$ by a fiber isotopy, by [W]. If $\partial M \neq \emptyset$ we can apply lemma 9 (ii) to obtain a homeomorphism $g_1 \simeq g$ such that $g_1^n \simeq 1$ rel ∂M and the result follows from Theorem 1. Thus assume that M is closed.

Let $p: M \to M^*$ be the projection onto the orbit surface and $E^*=p(E)$ the set of exceptional points. Since g and G are fiber preserving they induce $g^*: M^* \to M^*$ and $G^*: (g^*)^n \simeq 1$ rel E^*. By Nielsen's Theorem we find a homeomorphism $\bar{g}' \simeq g^*$ rel E^* such that $(\bar{g}')^n=1$. Therefore, looking at the lift of $\bar{g}'$ (Corollary 5) we can now assume that $(g^*)^n=1$.

Since M contains an incompressible fibered torus, it follows that $M^* \backslash E^*$ is different from S^2, D^2 and P^2.

Case (1) M has no exceptional fibers, i.e. $E^*=\emptyset$: By lemma 1, we can lift a simple closed curve in M^* to a fibered essential torus F in M such that either

(a) $\hat{F}=\{\cup g^i(F)\}$ is a union of disjoint tori, or

(b) $g^i(F)$ meets F transversally in fibers and $M^* \backslash \cup (g^*)^i(F)$ is a union of disjoint open disks.

In case (a) after applying proposition 9 (ii) we can assume that $g^n \simeq 1$ rel $\hat{F}$. Then splitting M along $\hat{F}$, we obtain a Seifert fiber space M' with boundary and a fiber preserving homeomorphism $g'=g|M'$ with $(g')^n \simeq 1$ rel $\partial M'$. For components of M' that are invariant under g' we apply Theorem 1 or lemma 10 to change g' by a homotopy constant on $\partial M'$ to a (fiber preserving) homeomorphism h' of period n. For the other components we repeat the construction at the end of the proof of lemma 10 to obtain $h' \simeq g'$ rel $\partial M'$ and $(h')^n=1$. In either case we extend h' to the desired homeomorphism h by $h|\hat{F}=g'|\hat{F}$.

In case (b) let $X=\cup\{g^i(F) \cap g^j(F) \mid g^i(F) \neq g^j(F)\}$, a finite set of fibers, and let $\hat{A}=\hat{F} \backslash \hat{X}$, a finite set of disjoint open annuli. By lemma 6 and its proof we can assume that the trace τ of a basepoint $x_0 \in F \cap \hat{X}$ under the fiber isotopy

$G: g^n \simeq 1$ has the property that $g(\tau) \simeq \tau$. Now G induces a cyclic homotopy $G^*: (g^*)^n = 1 \simeq 1$ of M^* with trace $\tau^* = p(\tau)$. Since τ^* represents an element of the center of $\pi_1(M^*)$, it follows that either $\tau^* \simeq 0$ or M^* is a projective plane, a Klein bottle, or a torus. The first case has been excluded and in the second and third case M is an S^1-bundle over a Klein bottle or torus and can be given the structure of a torus bundle over S^1. These latter cases are included in lemma 12(b). Thus we can assume that $\tau^* \simeq 0$ and therefore $\tau \simeq \beta^k$ rel x_0, where β is an (ordinary) fiber of M containing x_0. Since $g(\tau) \simeq \tau$ rel x_0, either $\tau \simeq 0$ or $g(\beta) \simeq \beta$ rel x_0. In the second case we can isotope g along fibers (thus keeping g^* fixed) so that afterwards $\tau \simeq 0$, and by lemma 7 we can now assume that in any case τ is the constant path at x_0. By lemma 8(b) (applied to F and g^n) we can now assume that $G: g^n \simeq 1$ rel x_0 and $G(F \times I) \subset F$. Recall that g^n is fiber preserving and F is a fibered torus. Waldhausen's proof [W] that there is a fiber preserving $H: g^n \simeq 1$ proceeds by deforming G to be fiber preserving on F. Thus we already have this step in Waldhausen's proof and we can therefore assume that G is fiber preserving. Since G leaves the fiber through x_0 invariant we can assume (by a fiber isotopy of G) that all fibers of F remain invariant under G.

Observe that (by the proof of lemma 1) no component of $\hat{X}$ is invariant under g^i for each $i \neq 0 \bmod n$. Thus it is easy to fiber isotope g (without changing g^*) so that $(g|\hat{X})^n = 1|\hat{X}$.

Note that the trace τ' for any base point on $\hat{X}$ is homotopic to β^k, for a fiber β, and is carried by that fiber. Pick a component X of $\hat{X}$ in each orbit $\{X, g(X), \cdots, g^{n-1}(X)\}$. If $g: X \to g(X)$ reverses the orientation of the fiber then $g(\beta) \simeq \beta^{-1}$ and $g(\tau') \simeq \tau'$ implies that $k=0$, hence $\tau' \simeq 0$ and we can find a fiber homotopy G of $g^n \simeq 1$ that is constant on X. If g preserves the fiber orientation, we can slide g around in a fiber neighborhood of X to get afterwards $g^n \simeq 1$ rel$\cup g^i(X)$.

Thus we can now assume that $g^n \simeq 1$ rel $\hat{X}$. Following the proof of theorem 9(i) taking $m=n$, $p=1$, and $\hat{U}$ a "tapered neighborhood" of $\hat{A}$ (see Fig. 1) we can now deform g over the annuli $\hat{A}$ to obtain a (fiber preserving) homeomorphism $h' \simeq g$ and a (fiber) isotopy H' such that $H': (h')^n \simeq 1$ rel $(\hat{X} \cup \hat{A})$. Each component M' of (M split along $\hat{X} \cup \hat{A}$) is a solid torus and H' induces an isotopy $(h'|M')^n \simeq 1$ rel $\partial M'$. Thus by lemma 10 we can further change h' by an isotopy constant on $\partial M'$ to get a homeomorphism $h \simeq h' \simeq g$ with $h^n = 1$.

Case (2) M contains exceptional fibers: Since g is fiber preserving $\hat{\gamma} = \cup\{g^i(\gamma)\}$ is a union of disjoint exceptional fibers, for each exceptional fiber γ of M.

Case a If there is an exceptional fiber γ such that $g(\gamma) = \gamma$, we let U be a fibered solid torus neighborhoood of γ and let F be a torus in U that is parallel to ∂U and such that $F \cup g(F) \subset U$. Since F and $g(F)$ are parallel to ∂U in $M \backslash \gamma$, we find a fiberisotopy of g after which $g(F)=F$. Now F is the boundary of a fibered regular neighborhood V of γ and since the fiber isotopy G leaves γ invariant, we can assume that V is invariant under G. We now repeat the proof of lemma 5.2 of [H. T.] to show that after a deformation of g the isotopy G can be chosen to leave a base point $x_0 \in F$ fixed: First deform g (keeping F invariant) such that $g(x_0)=x_0$. Since $\mathrm{Obs}(Z_n, \pi_1(M), \Psi)=0$ we may assume that $g(\tau) \simeq \tau$ for the trace τ of x_0 under G. If $h \in \pi_1(M)$ denotes the element represented by the fiber β containing x_0 and if $q \in \pi_1(M)$ denotes the element represented by a cross-sectional curve to the fibers in F we have $q^{\alpha}=h^{-\beta}$ for some integers $\alpha > \beta > 0$ and $[\tau]=h^a q^b$. Suppose $b \neq 0$. Now $g_*(h)=h^{\pm 1}$ and $g_*[\tau]=[\tau]$ implies that $g_*(q^b)=q^b h^{(a \pm a)}$. Since $g(F)=F$ it follows that $g_*(q)=q^d h^c$, hence $g_*(q^b)=q^{bd} h^{bc}$. Comparing we see that $d=+1$ i.e. $g_*(q)=qh^c$. It follows that on the orbit surface M^* the path τ_* winds around the circle F^* b times and that $g^*(\tau^*) \simeq \tau^*$ rel x_0^* on F^*. Therefore, deforming g^* along F^* and lifting, we obtain a fiber isotopy of g, keeping F invariant, after which we have $\tau^* \simeq 0$ rel x_0^* in F^*, hence $\tau \simeq \beta^n$ rel x_0 for some integer n. If $b=0$ then we already have $\tau \simeq \beta^n$ rel x_0. Therefore, $g(\beta) \simeq \beta$ and as before we can slide g around β (by a fiber isotopy of M keeping F invariant) so that afterwards we have a fiber isotopy $G: g^n \simeq 1$ rel x_0 such that $G(F \times I) \subset F$. As in the proof of lemma 9 (ii), after applying Nielsen's Theorem for 2-manifolds, we can also assume that $(g|F)^n = 1|F$. By the proof of lemma 8 (a) [[H. T.], (3.4)] (which does not use incompressibility of F), we now obtain an isotopy $G: g^n \simeq 1$ rel F and finish the proof as before, by applying Theorem 1 or lemma 10 to M cut along F.

Case b There is no exceptional fiber γ such that $g(\gamma)=\gamma$. Since G is fiber preserving we find a fibered neighborhood V of γ such that for $F=\partial V$, $G(\hat{F} \times I) \subset \hat{F}$. We use the first paragraph in the proof of lemma 9 (ii) to obtain furthermore $g^i(\tau_0) \simeq \tau_i$ rel $g^i(x_0)$, where τ_j is the trace of $g^j(x_0)$ under the isotopy G, and x_0 is a basepoint on F. Following then that proof from the third paragraph we obtain a homotopy $K: g^n \simeq 1$ rel $(\cup g^i(x_0))$ such that $K(\hat{F} \times I) \subset \hat{F}$ and furthermore $(g|F)^n = 1|F^n$. Because of this last property, the proof of lemma 8 (b) shows that we can in fact choose K so that $K: g^n \simeq 1$ rel $\hat{F}$. Now the proof proceeds as before by splitting M along $\hat{F}$.

Recall that a Haken mainfold is *simple* if every rank 2 free abelian subgroup of $\pi_1(M)$ is peripheral. If ∂M consists of tori only, Thurston's Uniformization Theorem [T] gives a hyperbolic structure of finite volume on the interior of M and Mostow's Rigidity Theorem [T] implies that every homeomorphism g of M for which $g^n \simeq 1$ is isotopic to a homeomorphism h such that $h^n=1$. Using this fact, we now prove a relative version of this result.

Theorem 3. *Let M be a Haken manifold that is either simple and such that each component of M is a torus or a Seifert fiber space. Let F be a system of boundary components and let $g: M \to M$ be a map with $g^n \simeq 1$ rel F. Then there is a homeomorphism $h \simeq g$ rel ∂F such that $h^n = 1$.*

Proof. If a component of ∂M is not incompressible, then M is a solid torus and the theorem follows from lemma 10 (a). If M is a Seifert fiber space this follows from Theorem 1. Thus we assume that M is incompressible, that M is not a Seifert fiber space, and (by [W]) that g is a homeomorphism. We now follow the proof of Theorem 2 in [H. T. II]: We can assume that g leaves a collar neighborhood $U = \partial M \times [0, 1]$ invariant, where $\partial M = \partial M \times \{0\}$, and that the homotopy $G: g^n \simeq 1$ carries U itself at each stage. By Thurston and Mostow there is a homeomorphism h' of $M' = M \setminus \mathring{U}$ that is homotopic to $g | M'$ with $(h')^n = 1$. By [W, Theorem 7.1 (b)] we can further assume that h' is isotopic to $g | M'$. Extending this isotopy to a homotopy constant on ∂M we obtain a map $h: M \to M$ such that $h \simeq g$ rel ∂M, $(h | M' \cup \partial M)^n = 1 | M' \cup \partial M$ and $h^n \simeq 1$ rel ∂M by a homotopy G with $G(U \times I) \subset U$. It now suffices to show that $h'' = h | U$ is homotopic rel ∂U to a map $\bar{h}$ with $\bar{h}^n = 1$. For this it suffices by lemma 8 (b) to construct a homotopy $G'': (h'')^n \simeq 1$ rel ∂U.

For a basepoint x_0 in $\partial M \times \{1\}$, the trace τ under the cyclic homotopy $G | M' \times I: h'^n \simeq 1$ represents an element of the center of $\pi_1(M')$, which is trivial since M is not a Seifert fiber space. By proposition 7 we can assume that $\tau = x_0$. Thus by restricting G to U, it follows that $(h'' | \partial U)^n = 1$, $G'' = G | U: (h'')^n \simeq 1$ rel $\{x_0\}$ and $G''_t(\partial U) = \partial U$. By lemma (8b) we can then assume that in fact $G'': (h'')^n \simeq 1$ rel ∂U.

If some of the boundary components are not tori we obtain the following.

Corollary. *Let M be a simple Haken manifold that contains no essential annulus. Let F be a nonempty system of tori of ∂M and let $g: M \to M$ be a map with $g^n \simeq 1$ rel F Then there is a homeomorphism $h \simeq g$ rel F such that $h^n = 1$.*

Proof. Let $S_1, \cdots, S_k$ be the components of ∂M different from tori. Let $\overline{M}$ be the manifold obtained from two copies of M by identifying the two copies of S_i by the identity $(i = 1, \cdots, k)$. If $\overline{M}$ would not be simple then by the torus theorem [J] (see also [Jo], [S]) there would be an essential torus in $\overline{M}$, or $\overline{M}$ would be a Seifert fiber space. That latter case can not occur since $\overline{M}$ contains an incompressible closed surface S_i different from a torus, but $\partial \overline{M} \neq \emptyset$ (see e. g. [He, Theorem 2]). In the former case any essential torus in $\overline{M}$ would intersect M in annuli at least one of which would be essential. Thus $\overline{M}$ is simple.

There is an involution $i: \overline{M} \to \overline{M}$ that interchanges the two copies of M and g induces a homeomorphism $\bar{g}: \overline{M} \to \overline{M}$ that commutes with i. Since we can assume (by [W]) that G is an isotopy of M it follows that G induces an isotopy

$\bar{G}:\bar{g}^n\simeq 1$ on $\bar{M}$ (that is constant on the two copies $\bar{F}$ of F). By Thurston and Mostow, i is homotopic to an involution j that is an isometry and $\bar{g}$ is homotopic to an isometry $\bar{h}$ such that $\bar{h}^n=1$. Since $\bar{g}$ commutes with i, the isometry $\bar{h}j\bar{h}^{-1}j^{-1}$ is homotopic to the identity and hence (by the uniqueness part of Mostow's theorem) equal to the identity. By Tollefson [To] there is a homeomorphism $\bar{f}\simeq 1$ such that $i=\bar{f}^{-1}j\bar{f}$. Now j is an involution with fix point set $\bar{f}(\hat{S})$ and it follows that $\bar{h}\bar{f}(\hat{S})=\hat{S}$, where $\hat{S}=S_1\cdots S_k$. Let $\bar{t}:\bar{M}\to\bar{M}$ be a homeomorphism isotopic to the identity that maps $f(\hat{S})$ to $\hat{S}$. Then $\bar{t}\bar{h}\bar{t}^{-1}\simeq\bar{g}$ and $\bar{t}\bar{h}\bar{t}^{-1}(\partial M)=\partial M$. By the proof of lemma 9 (ii) or [W, Theorem 7.1] we can find an isotopy $\bar{K}:\bar{t}\bar{h}\bar{t}^{-1}\simeq\bar{g}$ such that $\bar{K}(\hat{S}\times I)\subset\hat{S}$. Thus $\bar{K}$ induces a homotopy between $h'=\bar{t}\bar{h}\bar{t}^{-1}|M$ and $\bar{g}|M$ to give $h'\simeq g$ on M with $(h')^n=1$. Now we repeat the proof of Theorem 3 to obtain a homeomorphism $h\simeq h'$ such that $h^n=1$ and $h\simeq g$ rel F.

§ 5. Nielsen's theorem for closed sufficiently large 3-manifolds.

Theorem 4. *Let M be a closed Haken manifold that is not a Seifert fiber space. Suppose g is a map of M to itself such that g^n is homotopic to the identity. Then g is homotopic to a homeomorphism h with $h^n=1$.*

Proof. As before we can assume that g is a homeomorphism of M. By the splitting Theorem of Jaco-Shalen [J. S.] and Johannson [J] there exists a system F of mutually disjoint incompressible tori that splits M into $\sigma(M)=M_1\cup\cdots\cup M_m\cup N_1\cup\cdots\cup N_q$, where each M_i is a Seifert fiber space and each N_i is simple. Furthermore, F is unique up to isotopy. Thus, after an isotopy of g, we may assume that $g(F)=F$, and by lemma 9 (ii) we can furthermore assume that $(g|F)^n=1|F$ and that there is an isotopy $G:g^n\simeq 1$ rel F.

The case $q=0$ is excluded by hypothesis and the case $m=0$ follows from Thurston and Mostow. Thus assume that $m+q>1$.

The isotopy G induces an isotopy of $(g|M_i)^n\simeq 1$ rel ∂M_i and $(g|N_j)^n\simeq 1$ rel ∂N_j and we can apply Theorem 3 to deform each $g|M_i$ and $g|N_j$ by a homotopy that is constant on F to homeomorphisms h_i and h_j such that $h_i^n=1|M_i$ and $h_j^n=1|N_j$. This gives the desired deformation of g to a homeomorphism h with $h^n=1$.

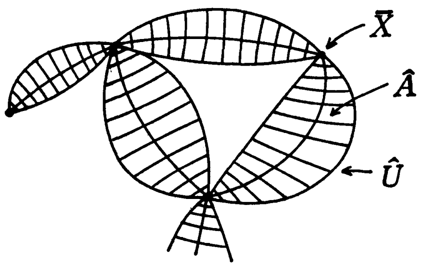

Fig. 1

References

[C.R] P.E. Conner and F. Raymond: Manifolds with few periodic homeomorphisms, Proc. 2nd Conference on Compact Transformation Groups, *Lecture Notes in Mathematics*, **299** (1970), 1-74.

[C.V] Charlap and Vasquez: *Amer. J. Math.* **95** (1973), 471-494.

[He] W. Heil: Almost sufficiently large Seifert fiber spaces, *Michigan Math. J.* **20** (1973), 217-223.

[H.T.] W. Heil and J. Tollefson: Deforming homotopy involutions to involutions, *Topology* **17** (1978), 349-365.

[H.T.II] W. Heil and J. Tollefson: Deforming homotopy involutions to involutions, II, *Topology* (1982),

[H] J. Hempel: 3-manifolds, *Ann. of Math. Studies* **86**, Princeton Univ. Press, 1976.

[J] W. Jaco: Lectures on 3-manifold Topology, *CBMS series* **43**, Amer. Math. Soc. (1980).

[Jo] K. Johannson: Homotopy equivalences of 3-manifolds with boundaries, *Lecture Notes in Math.* **761**, Springer (1979).

[M] A. Miller: Thesis, University of Connecticut, 1981.

[McL] S. MacLane: *Homology*, Springer (1963).

[N] J. Nielsen: Abbildungsklassen endlicher Ordnung, *Acta Math.* **75** (1943), 23-115.

[K] Kerckhoff: The Nielsen Realization Problem, *Bull. Amer. Math. Soc.* 2 (1980), 452-453.

[S] P. Scott: A new proof of the annulus and torus theorems, *Amer. J. Math.* **102** (1980), 241-277.

[S.T.] H. Seifert and W. Threlfall: *A Textbook of Topology*, Academic Press 1980.

[T] W. Thurston: The geometry and Topology of 3-manifolds. Lecture notes Princeton University.

[To] J. Tollefson: Involutions on sufficiently large 3-manifolds, *Topology* **20** (1981), 323-352.

[W] F. Waldhausen: On irreducible 3-manifolds which are sufficiently large, *Ann. Math.* **87** (1968), 56-88.

[Z] H. Zieschang: Finite groups of mappingclasses of surfaces. *Springer Lecture Notes* **875** (1981).

[Z.Z.] H. Zieschang and B. Zimmermann: Endliche Gruppen von Abbildungsklassen gefaserter 3-Mannigfaltigkeiten, *Math. Ann.* **240** (1979), 41-62.

[Zi] B. Zimmermann: Das Nielsensche Realisierugsproblem fur hinreichend grosse Mannigfaltigkeiten. *Math. Z.* **180** (1982), 349-359.

Department of Mathematics
Florida State University
Tallabassee, FL 32306-3027
USA