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A NOTE ON THE CENTRAL LIMIT THEOREM FOR STATIONARY STRONG-MIXING SEQUENCES

By

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1. Introduction and results

Let $\{X_n, n \geqq 1\}$ be a strictly stationary sequence of random variables. Suppose that the sequence $\{X_n\}$ satisfies $EX_1=0$, $EX_1^2<\infty$ and the strong-mixing condition, i.e.

$$\alpha(n) \equiv \sup\{|P(A \cap B)-P(A)P(B)|: A \in \mathscr{F}_0^m, B \in \mathscr{F}_{n+m}^{\infty}\} \downarrow 0 \tag{1}$$

as $n \to \infty$, where $\mathscr{F}_m^n$ denotes the σ-algebra generated by $X_m, X_{m+1}, \cdots, X_n$. Let $S_0=0$, $S_n=X_1+\cdots+X_n$ for $n \geqq 1$, and $s_n^2=ES_n^2$. It is well known (see [3] Theorem 18.4.2) that under the assumption

$$s_n^2=\sigma^2 n(1+o(1)), \qquad 0<\sigma<\infty, \tag{2}$$

$S_n/s_n \xrightarrow{d} N(0, 1)$ if and only if $\{(S_n/s_n)^2, n \geqq 1\}$ is uniformly integrable, i.e.

$$\lim_{a \to \infty} \sup_{n \geqq 1} E\{(S_n/s_n)^2 I(|S_n/s_n|>a)\}=0. \tag{3}$$

Recently in [4] one of the authors gave the following theorem (see [4] Theorem 1): under the assumption (3) $S_n/s_n \to N(0, 1)$ if and only if the sequence $\{s_n^2\}$ satisfies a certain condition which implies $s_n^2 \to \infty$ and is implied by (2). However "only if" part of this theorem is false, as is shown by a simple counterexample (see p. 315 of [3]). This theorem should be corrected as follows: under the assumption on $\{s_n^2\}$ stated above, $S_n/s_n \xrightarrow{d} N(0, 1)$ if and only if (3) holds. The purpose of this note is to strengthen the last statement and prove the following theorem:

Theorem. *Suppose $s_n^2 \to \infty$ as $n \to \infty$. Then in order that $S_n/s_n \xrightarrow{d} N(0, 1)$ it is necessary and sufficient that* (3) *hold. If this is the case then there exists a slowly varying function h on* $(0, \infty)$ *such that $s_n^2=nh(n)$.*

Theorem 2 of [4] should also be read as follows: Suppose that $\{X_n\}$ is strictly stationary ϕ-mixing and satisfies $s_n \to \infty$, then $S_n/s_n \xrightarrow{d} N(0, 1)$ if and only if (3) holds. Obviously this statement is implied by our theorem. It should also be noted that Herrndorf [3] proved that the weak invariance principle for ϕ-mixing stationary sequence $\{X_n\}$ holds if and only if (3) and

$$\lim_{n\to\infty} P\left\{\max_{1\leqq i\leqq n} |X_i| \geqq \varepsilon s_n\right\} = 0$$

hold for every $\varepsilon > 0$.

2. Proof

We prove the theorem applying the following lemma which is a part of Theorem 18.4.1 of [3].

Lemma 1. *Suppose that* $s_n^2 = nh(n)$ *where* h *is a slowly varying function on* $(0, \infty)$, *and suppose that*

$$\lim_{n\to\infty} \frac{n}{p(n)s_n^2} E(S_{p(n)}^2 I(|S_{p(n)}| > \varepsilon s_n)\} = 0 \tag{4}$$

holds for some sequence $\{p(n)\}$ *of positive integers such that* $p(n)\to\infty$, $p(n)=o(n)$, *and there exists a sequence* $\{q(n)\}$ *of positive integers satisfying the following three conditions*:

(a) $$q(n)\to\infty, \qquad q(n)=o(p(n)) \qquad \textit{as} \quad n\to\infty$$

(b) $$\lim_{n\to\infty} n^{1+c} q(n)^{1-c} p(n)^{-2} = 0 \qquad \textit{for some} \quad c>0$$

(c) $$\lim_{n\to\infty} np(n)^{-1}\alpha(q(n)) = 0\,.$$

Then $S_n/s_n \xrightarrow{d} N(0, 1)$.

To apply this result we need the following lemma. In what follows we write $\xi_{m,n} = (S_n - S_m)/s_{n-m}$ for $0\leqq m<n$ and $\xi_n = \xi_{0,n} = S_n/s_n$.

Lemma 2. *If* $s_n^2\to\infty$ *and* (3) *holds, then there exists a sequence* $\{\beta(n)\}$ *satisfying* $0<\beta(n)\to 0$ *and*

$$|E\xi_m \xi_{m+\tau, m+n+\tau}| \leqq \beta(\tau) \tag{5}$$

for m, n, $\tau\geqq 1$.

Proof. For simplicity we writen $\eta_n = \xi_{m+\tau, m+n+\tau}$. Let

$$\xi_n^{(M)} = \xi_n I(|\xi_n| \leqq M)\,, \qquad \bar{\xi}_n^{(M)} = \xi_n - \xi_n^{(M)}$$

and

$$\eta_n^{(M)} = \eta_n I(|\eta_n| \leqq M)\,, \qquad \bar{\eta}_n^{(M)} = \eta_n - \eta_n^{(M)}$$

Define a function γ by

$$\gamma(M) = \sup_{n\geqq 1} E(\bar{\xi}_n^{(M)})^2 \tag{6}$$

for $M>0$. It follows from the uniform integrability of $\{\xi_n^2\}$ that

$$\gamma(M)\downarrow 0 \quad \text{as} \quad M\to\infty . \tag{7}$$

By Theorem 17.2.1 of [3] and by stationarity we have

$$|E\xi_m^{(M)}\xi_n^{(M)}|\leqq 4M^2\alpha(\tau),$$

and

$$E(\bar{\xi}_n^{(M)})^2=E(\bar{\eta}_n^{(M)})^2\leqq\gamma(M),$$

and therefore

$$|E\xi_m^{(M)}\bar{\eta}_n^{(M)}|\leqq\{E(\xi_m^{(M)})^2E(\bar{\eta}_n^{(M)})^2\}^{1/2}\leqq\{\gamma(M)E\xi_m^2\}^{1/2}=\gamma(M)^{1/2}.$$

Similarly we obtain

$$|E\bar{\xi}_m^{(M)}\eta_n^{(M)}|\leqq\gamma(M)^{1/2}$$

and

$$|E\bar{\xi}_m^{(M)}\bar{\eta}_n^{(M)}|\leqq\{E(\bar{\xi}_m^{(M)})^2E(\bar{\eta}_n^{(M)})^2\}^{1/2}\leqq\gamma(M)\leqq\gamma(M)^{1/2}.$$

These inequalities together prove that

$$\begin{aligned}|E\xi_m\eta_n|&\leqq|E\xi_m^{(M)}\eta_n^{(M)}|+|E\xi_m^{(M)}\bar{\eta}_n^{(M)}|+|E\bar{\xi}_m^{(M)}\eta_n^{(M)}|+|E\bar{\xi}_m^{(M)}\bar{\eta}_n^{(M)}|\\&\leqq 4M^2\alpha(\tau)+3\gamma(M)^{1/2},\end{aligned} \tag{8}$$

for any positive integers m, n, τ and for arbitrary $M>0$. Choosing $M=\alpha(\tau)^{-1/2+\delta}$, where $\delta>0$, is an arbitrary constant, we define β by

$$\beta(\tau)=4\alpha(\tau)^{2\delta}+3\{\gamma(\alpha(\tau)^{-1/2+\delta})\}^{1/2}.$$

It follows from (1) and (7) that $\beta(\tau)\to 0$ as $\tau\to\infty$. By (8) we have $|E\xi_m\eta_n|\leqq\beta(\tau)$ for positive integers m, n and τ.

Proof of Theorem. The necessity follows from Theorem 5.4 of [1]. We shall prove the sufficiency. Suppose $s_n^2\to\infty$ and $\{\xi_n^2\}$ is uniformly integrable. In view of Lemma 2 we can apply the proof of Theorem 18.2.3 of [3] to prove that $s_n^2=nh(n)$ with some slowly varying function h (see p. 330 of [3]). It is well known that h admits the following representation:

$$h(x)=c(x)\exp\left(\int_1^x\frac{\phi(u)}{u}du\right),$$

where $\lim_{x\to\infty}c(x)=1$ and $\lim_{x\to\infty}\phi(x)=0$. If we define

$$\psi(x)=\sup_{y\geqq x}|\phi(y)|$$

for $x\geqq 1$, then $\psi(x)$ is nonincreasing and $\lim_{x\to\infty}\psi(x)=0$. Therefore, if we put

$$a(n)=\max\{n^{-1/2},\psi(n^{1/2})\}$$

then

$$\lim_{n\to\infty}\int_{a(n)}^{1}\frac{\psi(nv)}{v}dv=0. \tag{9}$$

Now, let

$$\lambda(n)=\max\{(\alpha[n^{1/4}])^{1/3},\ (\log n)^{-1}\},$$

$$p=\max\left\{\left[\frac{n\alpha([n^{1/4}])}{\lambda(n)}\right],\left[\frac{n^{3/4}}{\lambda(n)}\right], na(n)\right\} \quad\text{and}\quad q=[n^{1/4}].$$

Then sequences $\{p(n)\}$ and $\{q(n)\}$ satisfy conditions (a), (b), (c) in Lemma 1. By (7) and (9) we have for every $\varepsilon>0$

$$\begin{aligned}\frac{n}{p(n)s_n^2}E\{S_{p(n)}^2 I(|S_{p(n)}|>\varepsilon s_n)\}&=\frac{h(p(n))}{h(n)}E\left\{\left(\frac{S_{p(n)}}{s_{p(n)}}\right)^2 I\left(\left|\frac{S_{p(n)}}{s_{p(n)}}\right|>\varepsilon\frac{s_n}{s_{p(n)}}\right)\right\}\\&=\frac{c(p(n))}{c(n)}\exp\left(-\int_{p(n)}^{n}\frac{\phi(u)}{u}du\right)E\left\{\xi_{p(n)}^2 I\left(|\xi_{p(n)}|>\varepsilon\frac{s_n}{s_{p(n)}}\right)\right\}\\&\leqq\frac{c(p(n))}{c(n)}\exp\left(\int_{a(n)}^{1}\frac{\psi(nv)}{v}dv\right)\gamma\left(\frac{\varepsilon s_n}{s_{p(n)}}\right)\sim\gamma\left(\frac{\varepsilon s_n}{s_{p(n)}}\right)\end{aligned}$$

as $n\to\infty$. Since $s_n/s_{p(n)}\to\infty$ as $n\to\infty$, (4) follows from (7). Thus we can apply Lemma 1 to show $\xi_n\xrightarrow{d}N(0,1)$. This proves the theorem.

Added to the proof. Very recently, in [5] Denker obtained the same result by a different method to ours.

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