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GEODESIC SPHERES ON GRASSMANN MANIFOLDS

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§ 1. Preliminaries

For $F=\boldsymbol{R}$, $\boldsymbol{C}$, or $\boldsymbol{H}$ let $G_{p,q}(F)$ denote the real, complex, or quaternion Grassmann manifolds of p-planes in F^{p+q}. Our purpose is to characterise $G_{p,q}(F)$ and its non-compact dual $G_{p,q}^{*}(F)$ by means of a particular parallel tensor field T of type (1, 3) and the Weingarten map on geodesic spheres.

Historically, the problem was first considered by L. Vanhecke and T. J. Willmore who characterised spaces of constant curvature and spaces of constant holomorphic sectional curvature [6]. These results were generalised by D. E. Blair and the author in [1] for $F=\boldsymbol{R}$ and in [4] for $F=\boldsymbol{C}$. The case $G_{2,q}(\boldsymbol{R})$ has also been considered by B. J. Papantoniou [5] using the Hermitian structure which exists in that instance. Since our present purpose is to present a unified treatment we avoid using Hermitian structures. In this respect the conditions imposed here differ from those of [4] but have the advantage of relating closely to those of [1]. In fact they are considerably weaker than those of [1] since, as our proof shows, one of the conditions given there is redundant.

We begin with some general remarks on Jacobi vector fields and geodesic spheres. Let M be a Riemannian manifold of dimension $n>2$ and let B be a normal neighbourhood of a point m in M. We may take B to be a geodesic ball of radius r. Choose an orthonormal basis for the tangent space M_m and let $\{x^i\}$, $i=1, 2, \cdots, n$, be the corresponding normal coordinate system on B. Write N for the unit vector field on $B\backslash\{m\}$ tangent to geodesic rays from m, thus $N=(x^i/s)(\partial/\partial x^i)$ where s denotes geodesic distance from m. Let V be the unit tangent field to a geodesic $\gamma:(-r, r)\to B$ with $\gamma(0)=m$, choose a non-zero vector $W_m=a^i(\partial/\partial x^i)_m$ normal to V_m, and let $Y=a^i s(\partial/\partial x^i)$ on B. Then on $\gamma\backslash\{m\}$ we have $[Y, N]=0$ and $R(N, Y)N=\nabla_N\nabla_Y N=\nabla_N^2 Y$. Consequently, the vector field X on γ defined by $X_{\gamma(\sigma)}=a^i\sigma(\partial/\partial x^i)_{\gamma(\sigma)}$, $-r<\sigma<r$, satisfies

$$(1.1)\qquad \nabla_X N=\nabla_N X$$

on $\gamma\backslash\{m\}$, and, by continuity,

$$(1.2)\qquad R(V, X)V=\nabla_V^2 X \quad \text{on} \quad \gamma .$$

Thus X is the Jacobi vector field on γ for which

$$X_m=0 \quad \text{and} \quad \nabla_{V_m}X=W_m\,. \tag{1.3}$$

In particular, X is normal to V and, for any point q on γ, the subspace of M_q normal to V_q is formed by evaluating all such Jacobi vector fields at q. Now write $A=-\nabla N$. For any geodesic sphere S in B with centre m, the restriction of A to tangent vectors to S is just the Weingarten map with respect to N as unit normal vector field. Also, by (1.1) and (1.2), we have on $\gamma\backslash\{m\}$,

$$\begin{aligned} R(N, X)N &= -\nabla_N AX \\ &= A^2X-(\nabla_N A)X\,. \end{aligned} \tag{1.4}$$

This equation is linear in X, hence, from the above remarks, it is valid for arbitrary vector fields X on $B\backslash\{m\}$, where we note from the definition of A that $AN=0$.

Next suppose M is a Riemannian locally symmetric space. With the previous notation, suppose W_m satisfies

$$R(V_m, W_m)V_m=cW_m\,.$$

Let X be the Jacobi vector field on γ satisfying (1.3), and W the parallel vector field on γ with initial value W_m. Now $R(V, W)V=cW$ since $\nabla R=0$, hence fW is a Jacobi vector field on γ with the same initial conditions (1.3) as X when we choose

$$f(\sigma)=\begin{cases} |c|^{-1/2}\sin(|c|^{1/2}\sigma) & \text{if} \quad c<0 \\ c^{-1/2}\sinh(c^{1/2}\sigma) & \text{if} \quad c>0 \\ \sigma & \text{if} \quad c=0 \end{cases}$$

Thus $X=fW$. Then, as a consequence of (1.1) and the definition of A,

$$AW=-\frac{N(f)}{f}W\,. \tag{1.5}$$

Since the Riemannian curvature at m is bounded, the set of eigenvalues c of $R(V_m, \text{—})V_m$ taken over all unit vectors V_m is bounded, say $|c|<k^2$, $k>0$. Thus if B is a geodesic ball of radius $<\pi/k$ then f is nowhere zero on $\gamma\backslash\{m\}$. Equation (1.5) now has the following immediate consequence.

Proposition 1.1. *Let m be a point in a Riemannian locally symmetric space of dimension >2. Then m has a normal neighbourhood B such that, for each unit vector $V_m\in M_m$ and corresponding geodesic γ, the parallel translate of an eigenspace of the linear map $R(V_m, \text{—})V_m$ along γ is contained in an eigenspace of the Weingarten map for each geodesic sphere in B with centre m.*

§ 2. Statement of main theorem

We consider $G_{p,q}(F)$ as the homogeneous Riemannian symmetric space $SO(p+q)/SO(p)\times SO(q)$, $SU(p+q)/S(U_p\times U_q)$ or $Sp(p+q)/Sp(p)\times Sp(q)$ for $F=\boldsymbol{R}$,

$\boldsymbol{C}$, or $\boldsymbol{H}$ respectively. The tangent space at any point $m \in G_{p,q}(F)$ can be identified with the vector space $F_{p,q}$ of all $p \times q$ matrices over F considered as a real vector space with inner product

$$g(X, Y) = \operatorname{re} \operatorname{tr} X\bar{Y}^t , \tag{2.1}$$

where $\bar{Y}$ denotes conjugation of Y in $F_{p,q}$ and $\bar{Y} = Y$ when $F = \boldsymbol{R}$. The corresponding Riemannian curvature tensor R at m is given by

$$R(X, Y)Z = X\bar{Y}^tZ + Z\bar{Y}^tX - Y\bar{X}^tZ - Z\bar{X}^tY . \tag{2.2}$$

Similarly, for the non-compact dual $G^*_{p,q}(F)$, the curvature tensor is just the negative of this, and it will be sufficient to consider the compact case. Of course the metric g can be replaced by any metric homothetic to it without affecting R.

The tensor T of type $(1, 3)$ defined at m by

$$T(X, Y, Z) = X\bar{Y}^tZ \tag{2.3}$$

is invariant by the isotropy group so extends to a parallel tensor field on $G_{p,q}(F)$, also denoted by T. We define real linear endomorphisms T_{XY}, T^{XY}, and T^Y_X at m by

$$T_{XY}Z = T(X, Y, Z) , \quad T^{XY}Z = T(Z, X, Y) , \quad T^Y_XZ = T(X, Z, Y).$$

Then it is easily verified that T has the following properties at m, hence on $G_{p,q}(F)$:

P_1: $g(T(X, Y, Z), W) = g(T(Z, W, X)Y) = g(T(Y, X, W), Z)$;

P_2: $T(T(X, Y, Z), U, V) = T(X, T(U, Z, Y), V) = T(X, Y, T(Z, U, V))$;

P_3: there exist positive real numbers μ, ν, ω such that for each unit vector X

(a) $\operatorname{tr} T^{XX} = \mu$, (b) $\operatorname{tr} T_{XX} = \nu$, (c) $\operatorname{tr} T^X_X = 2 - \omega$, (d) $\operatorname{tr}(T^{X^2}_X) = \omega$;

P_4: $\dim G_{p,q}(F) = \mu\nu/\omega$.

We note from (2.3) that the values of μ, ν, ω are given by $\mu = \omega p$, $\nu = \omega q$, and $\omega = \dim F$. However, for the purpose of later references these relations are omitted from the above properties.

Particular use will be made of unit vectors X at m satisfying $T(X, X, X) = X$. Such vectors can be characterised as in the following lemma which is easily proved using elementary matrix methods or equivalence under the isotropy group.

Lemma 2.1. *Suppose $X \in F_{p,q}$ satisfies $\operatorname{re} \operatorname{tr} X\bar{X}^t = 1$. Then $X\bar{X}^tX = X$ if and only if $X = (x_i y_\alpha)$ for $x_1, \cdots, x_p, y_1, \cdots, y_q \in F$.*

As an easy consequence of (2.2) and Lemma 2.1 we see that if V is a unit vector at m such that $T(V, V, V) = V$ then the self-adjoint linear map $R(V, —)V$ acting on $V^\perp$, the orthogonal complement of V, has three eigenspaces. These are $T^V_V(V^\perp)$, $(T_{VV} + T^{VV} + 2T^V_V)(V^\perp)$, and their orthogonal complement in $V^\perp$. From this and Proposition 1.1. the next lemma is immediate.

Lemma 2.2. *Let $m \in G_{p,q}(F)$, choose a normal neighbourhood B of m as in Proposition* 1.1, *and let $\gamma \subset B$ be any geodesic ray from m with unit tangent vector field V satisfying $T(V, V, V) = V$. Write $V^\perp$ for the distribution of orthogonal complements* *to V along γ. Then the Weingarten map A has the following property.*

P_5: *At each point of $\gamma\backslash\{m\}$ each of the three subspaces $T_V^V(V^\perp)$, $(T_{VV}+T^{VV}+2T_V^V)(V^\perp)$, and their orthogonal complement in $V^\perp$ is contained in an eigenspace of A.*

We remark that $T_V^V V^\perp$ is trivial when $F=\boldsymbol{R}$. The main theorem can now be stated as follows.

Theorem 2.3. *Let M be a non-flat, complete, simply connected Riemannian manifold of dimension ≥ 3 and let T be a parallel tensor field of type $(1, 3)$ on M satisfying P_1, P_2, P_3. Then in P_3, $\omega=1$, 2, or 4 and $\mu=\omega p$, $\nu=\omega q$ for some positive integers p and q. Now suppose $\dim M=\omega pq$ as in P_4, and that each $m\in M$ has a normal neighbourhood in which each geodesic ray γ from m with unit tangent vector field V satisfying $T(V, V, V)=V$, has property P_5. Then M is homothetic to $G_{p,q}(F)$ or $G_{p,q}^*(F)$ where $F=\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$ corresponds to $\omega=1$, 2, 4.*

§3. A characterisation of T on $F_{p,q}$

The proof of the theorem depends largely on a characterisation of the structure described earlier on the tangent space to $G_{p,q}(F)$ at any point. For this purpose, we require the following result.

Proposition 3.1. *Let Λ be a real finite dimensional vector space with inner product $\langle\ ,\ \rangle$, and let T be a tensor of type $(1, 3)$ on Λ satisfying P_1, P_2, P_3 with $\langle\ ,\ \rangle$ replacing g. Then $\omega=1$, 2, or 4, and $\mu=\omega p$, $\nu=\omega q$ for some positive integers p, q. Now suppose $\dim\Lambda=\omega pq$, and write $F=\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$ corresponding to $\omega=1$, 2, 4. Then there is a linear isomorphism of Λ onto $F_{p,q}$, considered as a real vector space, such that, under identification,*

$$T(X, Y, Z)=X\bar{Y}^tZ \quad \textit{and} \quad \langle X, X\rangle=\operatorname{tr}X\bar{X}^t\,.$$

The proof requires several lemmas which refer to Λ under the above assumptions. The first of these lemmas provides a useful duality; the proof is immediate.

Lemma 3.2. *Define a tensor S on Λ by $S(X, Y, Z)=T(Z, Y, X)$, and write $S_{XY}=Y^{YX}$, $S^{XY}=T_{YX}$, and $S_Y^X=T_X^Y$. Then P_1, P_2 are satisfied when T is replaced by S, and P_3 is satisfied with T^{XX} and T_{XX} are replaced by S_{XX} and S^{XX} respectively.*

This shows that any property of T_{XY} (resp. T_X^Y) has a dual for T^{YX} (resp. T_Y^X) and conversely, provided μ and ν are exchanged in P_3. In what follows we make frequent use of this duality by stating or proving lemmas for one case only. Also we remark that P_1 and P_2 may be used occasionally without reference.

Lemma 3.3. *For each non-zero $X\in\Lambda$, the linear endomorphisms T_{XX}, T^{XX}, and T_X^X are self-adjoint and $T(X, X, X)\neq 0$.*

Proof. The self-adjoint properties are clear from P_1. Also, from P_3 (b), there exists Y such that $T(X, X, Y) \neq 0$. Then from P_1 and P_2,

$$0 < \langle T(X, X, Y), T(X, X, Y)\rangle = \langle T(X, X, T(X, X, Y)), Y\rangle$$
$$= \langle T(T, X, X, X), X, Y), Y\rangle .$$

Thus $T(X, X, X) \neq 0$.

Lemma 3.4. *Suppose $X, Y \in \Lambda$ are non-zero and $T(X, X, Y) = \lambda Y$. Then* im T_{YY} *is contained in the λ-eigenspace of T_{XX}. If $T(X, X, X) = \lambda X$ then λ is the only non-zero eigenvalue of T_{XX}.*

Proof. For any $Z \in \Lambda$,

$$T(X, X, T(Y, Y, Z)) = T(T(X, X, Y), Y, Z)$$
$$= \lambda T(Y, Y, Z)$$

and the result follows. The last part of the lemma is just a particular case.

From now on we use the following notation. Define $D \subset \Lambda$ by $X \in D$ if and only if $X = 0$ or rk $T_{XX} = \min\{\text{rk } T_{YY}\colon Y \in \Lambda \text{ and } Y \neq 0\}$. Then for each non-zero $X \in D$ write $\Lambda_X = \text{im } T_{XX}$. Dually, we define $D' \subset \Lambda$ by replacing T_{XX}, T_{YY} above by T^{XX}, T^{YY} and writing $\Lambda^X = \text{im } T^{XX}$ for $X \neq 0$. However, $D' = D$ as (iii) of the next lemma shows. Finally, we write $\Lambda_X^X = \Lambda_X \cap \Lambda^X$.

Lemma 3.5. *Let X and Y be non-zero vectors such that $X \in D$ and $Y \in \Lambda_X$. Then*
(i) $\Lambda_X \subset D$;
(ii) $\Lambda_X = \Lambda_Y$;
(iii) $T(X, X, X) = k\|X\|^2 X$ *where* $k\|X\|^2 \text{rk } T_{XX} = v$ *and* $k = \max\{\theta \mid T(Z, Z, Z) = \theta Z$ *and* $\|Z\| = 1\}$, *conversely, any vector V satisfying this equation belongs to D.*
(iv) $T_{XX} \mid \Lambda_X = k\|X\|^2 I$ *where I is the identity map on Λ_X, and $T_{XX}(\Lambda_X^\perp) = 0$ where $\Lambda_X^\perp$ is the orthogonal complement of Λ_X in Λ.*

Proof. We may assume that X and Y, as given in the lemma, are unit vectors. As a consequence of Lemmas 3.3 and 3.4, T_{XX} has exactly one non-zero eigenvalue, say λ, possibly with multiplicity > 1. Since $T(X, X, Y) = \lambda Y$ then, from the definition of X and Lemma 3.4, T_{YY} has a unique non-zero eigenvalue θ and im $T_{XX} =$ im T_{YY} which proves (i) and (ii). The last equation also shows that $T(Y, Y, Y) = \theta Y$. Furthermore, T_{XX} and T_{YY} are self-adjoint and have the same trace v. Hence $\theta = \lambda$. Next, let X_1 be the orthogonal projection of X onto the λ-eigenspace of T_{XX}, noting from Lemma 3.3 that $X_1 \neq 0$. Then

$$\lambda^3 \|X_1\|^2 X_1 = \lambda^2 T(X_1, X_1, X_1)$$
$$= T(T(X, X, X), T(X, X, X), X_1)$$
$$= T(X, X, T(X, T(X, X, X), X_1))$$
$$= T(X, X, T(X, X, T(X, X, X_1))$$

$$= T^3_{XX} X_1$$
$$= \lambda^3 X_1 .$$

Thus $\|X_1\| = 1$ so $X = X_1$ and $T(X, X, X) = \lambda X$. Since rk $T_{XX} = \nu/\lambda$ and is a minimum then the first part of (iii) follows. Conversely, if $T(V, V, V) = k\|V\|^2 V$ then rk $T_{VV} =$ rk T_{XX} so $V \in D$ as required. Finally, (iv) is immediate since T_{XX} is self-adjoint and Λ_X is the k-eigenspace of T_{XX}.

Lemma 3.6. *If $Y \in \Lambda_X$ and $V, W \in \Lambda$ then $T(Y, V, W) \in \Lambda_X$.*

Proof. $Y = T(X, X, Z)$ for some $Z \in \Lambda$. Hence, from P_2,

$$T(Y, V, W) = T(T(X, X, Z), V, W)$$
$$= T(X, X, T(Z, V, W)) \in \Lambda_X .$$

In the rest of this section let U be a unit vector in D. Now for any $X, Y \in \Lambda^U_U$,

$$T(X, X, Y) = T(Y, X, X) = k\|X\|^2 Y ,$$

and linearization gives, in particular,

$$T(X, Y, X) + T(Y, X, X) = 2k\langle X, Y\rangle X .$$

These equations imply

Lemma 3.7. *For all $X, Y \in \Lambda^U_U$,*

$$T^X_X Y = 2k\langle X, Y\rangle X - k\|X\|^2 Y .$$

On the other hand we have the following result for $(\Lambda^U_U)^\perp$.

Lemma 3.8. *If $X \in \Lambda^U_U$ and $Y \in (\Lambda^U_U)^\perp$ then $T^X_X Y = 0$.*

Proof. Since T^X_X is self adjoint it is sufficient to prove $T^{X2}_X Y = 0$. Let $Z \in \Lambda$. Then $T^{X^2}_X Z = T_{XX} T^{XX} Z = T^{XX} T_{XX} Z$ so $T^{X^2}_X Z \in \Lambda^U_U$. Hence

$$\langle T^{X^2}_X Y, Z\rangle = \langle Y, T^{X^2}_X Z\rangle = 0$$

which proves the lemma.

Lemma 3.9. (i) *For any non-zero vector $X \in \Lambda^U_U$, $\Lambda^X_X = \Lambda^U_U = T^X_X(\Lambda^X_X)$;*
(ii) $k = 1$;
(iii) *if $Y \in D$ is non-zero then* $\dim \Lambda^Y_Y = \omega$.

Proof. From Lemma 3.5 (ii) and its dual, $\Lambda^X_X = \Lambda^U_U$ and $\Lambda^U_U = T^X_X(\Lambda^X_X)$ from Lemma 3.7, this proves (i). From Lemmas 3.7 and 3.8, the non-zero eigenvalues of T^U_U are k and $-k$ with multiplicity 1 and $d-1$, where $d = \dim \Lambda^U_U$. Hence, using P_3, we have $k(2-d) = 2-\omega$ and $k^2 d = \omega$ from which $(k-1)(kd+2) = 0$. Now from Lemma 3.5, k is positive, hence $k = 1$ and $\dim \Lambda^U_U = \omega$. This proves (ii) and (iii) follows since the choice of unit vector $U \in D$ is arbitrary.

Lemma 3.10. *Λ_U^U admits a multiplication with respect to which it is isomorphic to* $\boldsymbol{R}$, $\boldsymbol{C}$, *or* $\boldsymbol{H}$. *In particular* $\omega=1$, 2, *or* 4.

Proof. Define a bilinear binary operation on Λ_U^U by $X\cdot Y=T(X, U, Y)$. We show that Λ_U^U becomes a real associative division algebra and the lemma follows using Frobenius' Theorem. Clearly U is a unit because $k=1$. Also multiplication is associative since

$$\begin{aligned}(X\cdot Y)\cdot Z&=T(T(X, U, Y), U, Z)\\&=T(X, U, T(Y, U, Z))\\&=X\cdot(Y\cdot Z)\,.\end{aligned}$$

Next, for any $X\in\Lambda_U^U$, define $\bar{X}=T(U, X, U)$. Then $\bar{X}\in\Lambda_U^U$ and

$$\begin{aligned}X\cdot\bar{X}&=T(X, U, T(U, X, U))=T(T(X, U, U), X, U)\\&=T(X, X, U)=\|X\|^2U\,.\end{aligned}$$

Hence any non-zero X has an inverse $\|X\|^{-2}\bar{X}$ and the proof is complete.

Lemma 3.11. *Suppose X and Y are unit vectors in Λ_U with Y orthogonal to Λ_X^X. Then*

(i) $\langle\Lambda_X^X, \Lambda_Y^Y\rangle=0$

(ii) $T(\Lambda_X^X, \Lambda_Y^Y, \Lambda)=\{0\}$.

Proof. *Let* $V\in\Lambda_X^X$ and $W\in\Lambda_Y^Y$. Then from Lemma 3.6 and its dual,

$$\begin{aligned}\langle T(X, V, X), T(Y, W, Y)\rangle&=\langle T(W, Y, T(X, V, X), Y\rangle\\&=\langle T(W, T(V, X, Y), X), Y\rangle\\&=0\,,\end{aligned}$$

and (i) follows using Lemma 3.9 (i). Next, for $V\in\Lambda$,

$$\begin{aligned}\langle T(X, Y, V), T(X, Y, V)\rangle&=\langle T(Y, X, T(X, Y, V)), V\rangle\\&=\langle T((T(Y, X, X), Y, V), V\rangle\,.\end{aligned}$$

Now $T(X, X, Y)=Y$ so from Lemma 3.8, $T(Y, X, X)=T(T(X, X, Y), X, X)=T_X^{X^2}Y=0$. Hence $T(X, Y, V)=0$, and (ii) follows using (i).

Proof of Proposition 3.1. From Lemmas 3.9 and 3.11 together with their duals, Λ_U (resp. Λ^U) is an othogonal direct sum of subspaces of the form Λ_X^X, $X\in\Lambda_U$ (resp. $X\in\Lambda^U$), each of dimension ω. Since $k=1$, we obtain using Lemma 3.5 and its dual, $\dim\Lambda_U=\operatorname{rk}T_{UU}=\nu=\omega q$ and $\dim\Lambda^U=\operatorname{rk}T^{UU}=\mu=\omega p$ for some positive integers p and q.

For convenience of notation, write $U=e=e_{11}$. From Lemma 3.10, we may consider Λ_U^U as a 1-dimensional right vector space over F with vectors ef, $f\in F$. Next, from the above remarks, we may choose sets of orthogonal unit vectors

$\{e_{11}, \cdots, e_{1q}\} \subset \Lambda_e$ and $\{e_{11}, \cdots, e_{p1}\} \subset \Lambda^e$ such that $\Lambda_e = \Lambda_1^1 \oplus \cdots \oplus \Lambda_1^q$ and $\Lambda^e = \Lambda_1^1 \oplus \cdots \oplus \Lambda_p^1$ where $\Lambda_1^\alpha = \Lambda_{e_{1\alpha}}^{e_{1\alpha}}$, $\Lambda_i^1 = \Lambda_{e_{i1}}^{e_{i1}}$ for $\alpha = 1, \cdots, q$, $i = 1, \cdots, p$, and the direct sums are orthogonal. Now define $e_{i\alpha} = T(e_{i1}, e, e_{1\alpha})$ for $i = 1, \cdots, p$, $\alpha = 1, \cdots, q$ noting consistency when $i = 1$ or $\alpha = 1$. Then $e_{i\alpha} \in \Lambda_{e_{i1}} \cap \Lambda^{e_{1\alpha}} \subset D$ and we write $\Lambda_{e_{i\alpha}}^{e_{i\alpha}} = \Lambda_i^\alpha$. From Lemma 3.9 (iii), each Λ_i^α has dimension ω. Also, we note from Lemma 3.11 (ii) and its dual form that for $\alpha \neq \beta$ and $i \neq j$

$$T(e_{1\alpha}, e_{1\beta}, \Lambda_1^1) = T(\Lambda_1^1, e_{i1}, e_{j1}) = \{0\}\,,$$

and it follows easily that Λ_i^α and Λ_j^β are orthogonal if $\alpha \neq \beta$ or $i \neq j$. Since $\dim \Lambda = \omega pq$, Λ is the orthogonal direct sum of subspaces Λ_i^α, $i = 1, \cdots, p$, $\alpha = 1, \cdots, q$.

Next write $\bar{f}$ for the conjugate of any $f \in F$ and define $e_{i\alpha}f = T(e_{i1}, e\bar{f}, e_{1\alpha})$, noting that consistency for $i = \alpha = 1$ follows from the definition of $\bar{X}$ in Lemma 3.10. Then for $f, g, h \in F$,

$$\begin{aligned}
T(e_{i\alpha}f, e_{j\beta}g, e_{k\gamma}h) &= T(T(e_{i1}, e\bar{f}, e_{1\alpha}), e_{j\beta}g, e_{k\gamma}h) \\
&= T(e_{i1}, T(T(e_{j1}, e\bar{g}, e_{1\beta}), e_{1\alpha}, e\bar{f}), e_{k\gamma}h) \\
&= T(e_{i1}, T(e_{j1}, e\bar{g}, T(e_{1\beta}, e_{1\alpha}, e\bar{f})), e_{k\gamma}h) \\
&= \delta_{\alpha\beta} T(e_{i1}, T(e_{j1}, e\bar{g}, e\bar{f}), e_{k\gamma}h) \\
&= \delta_{\alpha\beta} T(e_{i1}, e\bar{f}, T(e\bar{g}, e_{j1}, e_{k\gamma}h)) \\
&= \delta_{\alpha\beta} T(e_{i1}, e\bar{f}, T(e\bar{g}, e_{j1}, T(e_{k1}, e\bar{h}, e_{1\gamma}))) \\
&= \delta_{\alpha\beta} T(e_{i1}, e\bar{f}, T(T(e\bar{g}, e_{j1}, e_{k1}), e\bar{h}, e_{1\gamma})) \\
&= \delta_{\alpha\beta}\delta_{jk} T(e_{i1}, e\bar{f}, T(e\bar{g}, e\bar{h}, e_{1\gamma})) \\
&= \delta_{\alpha\beta}\delta_{jk} T(e_{i1}, T(e\bar{h}, e\bar{g}, e\bar{f}), e_{1\gamma})\,.
\end{aligned}$$

Now

$$\begin{aligned}
e\bar{h}g\bar{f} &= T(T(e\bar{h}, e, eg), e, e\bar{f}) \\
&= T(e\bar{h}, T(e, eg, e), e\bar{f}) \\
&= T(e\bar{h}, e\bar{g}, e\bar{f})\,.
\end{aligned}$$

Hence

$$\begin{aligned}
T(e_{i\alpha}f, e_{j\beta}g, e_{k\gamma}h) &= T(e_{i1}, e\bar{h}g\bar{f}, e_{1\gamma})\delta_{\alpha\beta}\delta_{jk} \\
&= e_{i\gamma} f\bar{g}h\delta_{\alpha\beta}\delta_{jk}\,.
\end{aligned} \tag{3.1}$$

Since each $\Lambda_i^\alpha = T(e_{i1}, \Lambda_1^1, e_{1\alpha}) = e_{1\alpha}F$, it follows that Λ can be considered as a right vector space over F with basis $\{e_{i\alpha}\}$ $i = 1, \cdots, p$, $\alpha = 1, \cdots, q$. Then, by considering $F_{p,q}$ as a right vector space over F, we have an F-linear isomorphism

$$\phi\colon \Lambda \longrightarrow F_{p,q}\,; \qquad \sum_{i,\alpha} e_{i\alpha}x_{i\alpha} \longmapsto (x_{i\alpha})\,.$$

We may, of course, consider ϕ as a real linear isomorphism by regarding Λ and $F_{p,q}$ as real vector spaces. From (3.1),

$$T(e_{i\alpha}x_{i\alpha}, e_{j\beta}y_{j\beta}, e_{k\gamma}z_{k\gamma}) = e_{i\gamma}x_{i\alpha}\bar{y}_{j\alpha}z_{j\gamma},$$

where, as usual, we use the summation convention. Thus, if elements of Λ are represented by their corresponding matrices then $T(X, Y, Z)$ corresponds to $X\bar{Y}^t Z$. Finally, using Lemma 3.10,

$$\begin{aligned}\langle e_{i\alpha}x_{i\alpha}, e_{j\beta}x_{j\beta}\rangle &= \langle T(e_{i1}, e\bar{x}_{i\alpha}, e_{1\alpha}), e_{j\beta}x_{j\beta}\rangle \\ &= \langle T(e_{1\alpha}, e_{j\beta}x_{j\beta}, e_{i1}), e\bar{x}_{i\alpha}\rangle \\ &= \langle e\bar{x}_{i\alpha}, e\bar{x}_{i\alpha}\rangle \\ &= x_{i\alpha}\bar{x}_{i\alpha} \\ &= \operatorname{tr} X\bar{X}^t\end{aligned}$$

and the proof is complete.

Remark 3.12. Proposition 3.1 has a dual form obtained essentially by exchanging p, q and replacing T by S as defined in Lemma 3.2. Thus, write each basis vector $e_{i\alpha}$ as $\varepsilon_{\alpha i}$ and write any $X \in \Lambda$ as $\varepsilon_{\alpha i}x_{\alpha i}$. Then an F-linear isomorphism $\psi: \Lambda \to F_{q,p}$ is defined by $\varepsilon_{\alpha i}x_{\alpha i} \longmapsto (x_{\alpha i})$; clearly $\psi = t \circ \phi$ where $t: F_{p,q} \to F_{q,p}$ is the transpose. If elements of Λ are represented by their corresponding matrices in $F_{q,p}$ then $S(X, Y, Z) = T(Z, Y, X)$ corresponds to $X\bar{Y}^t Z$ and $\langle \varepsilon_{\alpha i}x_{\alpha i}, \varepsilon_{\beta j}x_{\beta j}\rangle = \operatorname{tr} X\bar{X}^t$.

§4. Proof of main theorem

Before proving the main theorem, one further lemma is required. We use the previous notation except possibly for the restriction on U.

Lemma 4.1. *Let R be a tensor of type* (1, 3) *on Λ with the symmetry properties of a Riemannian curvature tensor. Suppose for all X, $Y \in \Lambda$ and $Z \in D$*

(i) $R(Z, X)Z = 0$ *and* (ii) $R(X, Y)T = 0$.

Then $R = 0$.

Proof. For non-zero $Z \in D$ consider R acting on vectors $V \in \Lambda$ and U, $W \in \Lambda_Z$ (or U, $W \in \Lambda^Z$.) Then (i) implies

$$R(U, V)W + R(W, V)U = 0. \tag{4.1}$$

Also (4.1) and the Bianchi identity imply

$$2R(U, V)W = R(U, W)V. \tag{4.2}$$

It follows from (4.1) and (4.2) that if U, V, $W \in \Lambda_Z$ or U, V, $W \in \Lambda^Z$ then

$$R(U, V)W = 0. \tag{4.3}$$

Next let X, $Y \in \Lambda$ and let $U \in D$ be a unit vector. Then from (ii),

$$\begin{aligned} R(X, Y)U = R(X, Y)T(U, U, U) \\ &= T(R(X, Y)U, U, U) + T(U, R(X, Y)U, U) \\ &\quad + T(U, U, R(X, Y)U) . \end{aligned}$$

Consequently, from Lemma 3.6 and its dual together with Lemma 3.7, we have

$$R(X, Y)U \in \Lambda_U + \Lambda^U . \tag{4.4}$$

Now each subspace $\Lambda_{e_{i\alpha}}$ (resp. $\Lambda^{e_{i\alpha}}$) of Λ is independent of α (resp. i) and we write it as Λ_i (resp. Λ^α). Then from (4.3) and (4.4),

$$R(\Lambda_i, \Lambda_i)\Lambda_i = R(\Lambda^\alpha, \Lambda^\alpha)\Lambda^\alpha = 0 \tag{4.5}$$

and

$$R(\Lambda, \Lambda)\Lambda_i^\alpha \subset \Lambda_i + \Lambda^\alpha ,$$

for all i, α.

Suppose now that $i \neq j$ and $\alpha \neq \beta$. Since $\Lambda_i + \Lambda^\alpha$ and Λ_j^β are orthogonal then from (4.6)

$$\langle R(\Lambda, \Lambda)\Lambda_i^\alpha, \Lambda_j^\beta \rangle = 0 , \tag{4.7}$$

equivalently

$$R(\Lambda_i^\alpha, \Lambda_j^\beta) = 0 . \tag{4.8}$$

Also from (4.1), (4.4), (4.8), and the Bianchi identity we see that if $i \neq j$ and $\alpha \neq \beta$ then

$$\begin{aligned} \langle R(\Lambda, \Lambda_i^\alpha)\Lambda_j^\beta, \Lambda_j + \Lambda^\beta \rangle &= \langle R(\Lambda, \Lambda_j)\Lambda_i^\alpha, \Lambda_j + \Lambda^\beta \rangle \\ &= \langle R(\Lambda_j^\beta, \Lambda)(\Lambda_j + \Lambda^\beta), \Lambda_i^\alpha \rangle \\ &= \langle R(\Lambda_j + \Lambda^\beta, \Lambda)\Lambda_j^\beta, \Lambda_i^\alpha \rangle \\ &= 0 . \end{aligned}$$

This relation together with (4.6) gives

$$R(\Lambda, \Lambda_i^\alpha)\Lambda_j^\beta = 0 . \tag{4.9}$$

As a consequence of (4.5), (4.8), and (4.9), $R(\Lambda_i^\alpha \Lambda_j^\beta)\Lambda_k^\gamma = 0$ for all $i, j, k = 1, \cdots, p$ and $\alpha, \beta, \gamma = 1, \cdots, q$, so $R = 0$ as required.

Proof of Theorem 2.3. From the conditions of the theorem, there exist functions f_1, f_2, f_3 along $\gamma \backslash \{m\}$ such that if X is any vector field along γ normal to V then, at each point p of $\gamma \backslash \{m\}$, $X_1 = T(V, X, V)$, $X_2 = T(V, V, X) + T(X, V, V) + 2T(V, X, V)$, and $X_3 = X + X_1 - X_2$ are orthogonal eigenvectors of A with eigenvalues $f_1(p)$, $f_2(p)$, $f_3(p)$ respectively. We note that $X_1 = 0$ when $\omega = 1$ in P_4, that is when $F = \boldsymbol{R}$, so in this case take $f_1 = 0$. For all remaining cases it is clear from the representation of T in Proposition 3.1 that there exists a parallel vector field X along γ for which X_1, X_2, and X_3 are non-zero. Then, by taking inner products $g(AX_i, X_i)$, $i=1, 2, 3$, we see that f_1, f_2, f_3 are smooth functions. Next, it is immediate from equation (1.4) that along $\gamma\backslash\{m\}$, X_1, X_2, and X_3 are eigenvectors of $R(V, —)V$. The corresponding eigenfunctions $f_i^2 - V(f_i)$, $i=1, 2, 3$, are independent of X, and it follows by continuity that, at m, each of the subspaces $T_V^V(V^\perp)$, $(T_{VV}+T^{VV}+2T_V^V)(V^\perp)$, and their orthogonal complement in $V^\perp$ is contained in an eigenspace of $R(V, —)$. Also, $V^\perp$ is an orthogonal direct sum of these three subspaces. Clearly these relations hold for any non-zero vector V at m satisfying $T(V, V, V)=g(V, V)V$. As before, write D for the set of all such vectors including zero, also write D_1 for the subset of unit vectors in D. The above properties of the curvature tensor R at m can be described equivalently as follows. There exist functions u, v, w on D_1 such that if $V\in D_1$ and $X\in M_m$ is orthogonal to V then

$$(4.10)\qquad R(V, X)V=u(V)X+v(V)T(V, X, V) +w(V)(T(V, V, X)+T(X, V, V)).$$

Next, we prove that $u=0$ and v, w are constant on D_1. This requires Proposition 3.1 so, for convenience of notation, we write M_m as Λ. First assume $pq>1$ and choose $U\in D_1$. Then, as shown in the proof of Proposition 3.1, Λ can be considered as a right vector space over F with an orthonormal basis $\{e_{i\alpha}\}$ of vectors in D_1 for which $U=e_{11}$. If $p\geq 2$ and $q\geq 2$ then from (4.10), $R(U, e_{22})U=u(U)e_{22}$. But T is parallel so $R(X, Y)T=0$ for all $X, Y\in\Lambda$. Hence, as in the proof of Lemma 4.1, $R(X, Y)U\in\Lambda_U+\Lambda^U$. Since e_{22} is orthogonal to $\Lambda_U+\Lambda^U$, it follows that $u(U)=0$, so $u=0$ on D_1. If $p=1$ or $q=1$ then, in (4.10), $X=T(V, V, X)+T(X, V, V)+T(V, X, V)$ and we may assume $u=0$.

Before considering w and v we note that in (4.10) the condition that X and V should be orthogonal can be removed by replacing X by $X-g(X, V)V$. Then properties P_1, P_2 and the symmetry of $g(R(X, Y)X, Y)$ in X and Y imply that for all $X, Y\in D_1$,

$$(4.11)\qquad (v(X)-v(Y))(g(T(X, Y, X), Y)-(g(X, Y))^2) +(w(X)-w(Y))(g(T(X, X, Y), Y)+g(T(Y, X, X), Y) -2(g(X, Y))^2)=0\,.$$

To prove that v and w are constant, assume $pq>1$ and apply (4.11). We obtain $(w(e_{j\alpha}x)-w(e_{k\beta}y))(\delta_{jk}+\delta_{\alpha\beta})=0$ for all $j, k=1, \cdots, p$, $\alpha, \beta=1, \cdots, q$, and $|x|=|y|=1$; so $w(e_{j\alpha}x)=w(e_{j\beta}y)=w(e_{k\beta}y)$. Next choose $X\in D_1$; thus by Lemma 2.1, $X=e_{i\alpha}x_i y_\alpha$. Assume that x_i, y_α, y_β are non-zero for some i, α, β with $\alpha\neq\beta$, and write $Y=e_{i\alpha}z$ where $z=x_i y_\alpha|x_i y_\alpha|^{-1}$ (not summed). Then in (4.11), $v(X)-v(Y)$ and $w(X)-w(Y)$ have coefficients zero and $|x_i|^2\sum_\gamma|y_\gamma|^2+|y_\alpha|^2\sum_j|x_j|^2-2|x_i|^2|y_\alpha|^2\geq|x_i|^2|y_\beta|^2>0$ respectively, from which $w(X)=w(Y)=w(e_{i\alpha}Z)=w(e_{11})$. A similar proof applies if X has components with some x_i, x_j, y_α non-zero, and it follows that w is constant on D_1. In considering v, we may assume $\omega=2$ or 4, as remarked earlier. Now choose $i\in F$ such that $i^2=-1$, and, for any x, $y\in F$ with $|x|=|y|=1$, write $Z=((e_{j\alpha}ix+e_{j\beta}iy)1/\sqrt{2})$, $\alpha\neq\beta$. Clearly $Z\in D_1$ and then (4.11) gives $v(e_{j\alpha}x)=v(Z)=v(e_{j\beta}y)$. Similarly, $v(e_{j\beta}y)=v(e_{k\beta}y)$, from which $v(e_{j\alpha}x)=v(e_{j\beta}y)=v(e_{k\beta}y)$ for all j, k, α, β. Then for any $X=e_{j\alpha}x_jy_\alpha\in D_1$ with some x_k, $y_\beta\neq 0$, write $Y=e_{k\beta}z$ where $z=ix_ky_\beta|x_ky_\beta|^{-1}$ (not summed). From (4.11), $v(X)=v(Y)=v(e_{11})$ so v is constant on D_1.

Finally, suppose $p=q=1$. This case arises only when $\omega=4$ since $\dim M\geq 3$. Also, in (4.10), $X=T(V, V, X)=T(X, V, V)=-T(V, X, V)$ so the equation reduces to the form

$$R(V, X)V=w(V)X\,.$$

If Y is a unit vector orthogonal to V and X then clearly $w(V)=w(Y)=w(X)$ so w is constant as required. Moreover, (M, g) is a space of constant curvature and the theorem is proved for $p=q=1$ and $\omega=4$. We have established that in all cases $u=0$ and v, w are constants.

Next we may assume $q>1$ and choose $U\in D_1$ and a basis $\{e_{i\alpha}\}$ with $U=e_{11}$ as before. We prove that the constants v, w satisfy $v+2w=0$ by considering only the subspace Λ_U of Λ. First, as an easy consequence of (4.10) and the Bianchi identity, for all X, Y, $Z\in\Lambda_U$,

$$3R(X, Y)Z=(v-w)P(X, Y)Z+(v+2w)Q(X, Y)Z \tag{4.12}$$

where P and Q are tensors of type (1, 3) on Λ defined by

$$P(X, Y)Z=T(X, Y, Z)+T(Z, Y, X)-T(Y, X, Z)-T(Z, X, Y)\,,$$

$$Q(X, Y)Z=g(X, Z)Y-g(Y, Z)X\,.$$

Clearly P and Q correspond to the curvature tensors on $G_{p,q}(F)$ and a space of constant curvature respectively. In particular, this observation or a direct calculation shows that, for all X, $Y\in\Lambda$, $P(X, Y)T=0$. Also $R(X, Y)T=0$ since T is parallel on M. Now $T(X, Y, Z)\in\Lambda_U$ for all X, Y, $Z\in\Lambda_U$ so the above comments and (4.12) show that either $v+2w=0$ or, for all X, $Y\in\Lambda_U$, $Q(X, Y)T_U=0$ where T_U is the restriction of T to vectors in Λ_U. But

$$\begin{aligned} e_{12}&=Q(e_{11}, e_{12})e_{11}\\ &=Q(e_{11}, e_{12})(T_U(e_{11}, e_{11}, e_{11}))\\ &=(Q(e_{11}, e_{12})T_U)(e_{11}, e_{11}i, e_{11}i)\,, \end{aligned}$$

so $v+2w=0$. Then from (4.10) the tensor $L=R-(v/2)P$ on Λ satisfies (i) and (ii) of Lemma 4.1, hence $R=(v/2)P$ on Λ. Clearly this relation holds everywhere on M, v now being regarded as a real-valued function on M. Since $G_{p,q}(F)$ is an Einstein space with curvature tensor of the form P, it follows that (M, g) is also an Einstein space and v is a constant, say $2a$. Since P is parallel then so is R so (M, g) is a symmetric space where we assume, as in the theorem, that (M, g) is complete, simply connected and non-flat, that is $a \neq 0$.

It remains only to obtain (2.2) for a metric $\bar{g}$ on M homothetic to g. Define $\bar{g} = |a|g$ and $\bar{T}(X, Y, Z) = |a|T(X, Y, Z)$ on M. Then $P_1 - P_5$ are satisfied for $\bar{g}$ and $\bar{T}$. Thus the conditions of the theorem still apply and, since the curvature tensor is unchanged, we have

$$(4.13) \qquad R(X, Y)Z = \frac{a}{|a|}(\bar{T}(X, Y, Z) + \bar{T}(Z, Y, X) - \bar{T}(Y, X, Z) - \bar{T}(Z, X, Y))$$

for all vector fields X, Y, Z on M. Now assume $a > 0$. Given M, g, and T as in the theorem, the real numbers ω, p and q are uniquely determined. Then it is immediate from Proposition 3.1 and equations (2.1), (2.2), and (4.13) that the tangent spaces at any two points of $G_{p,q}(F)$ and M are related by a linear isomorphism which preserves inner products and the curvature tensors. Hence $G_{p,q}(F)$ and M are isometric since each is complete and simply connected. A corresponding result applies to $G^*_{p,q}(F)$ if $a < 0$, and the proof is complete.

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