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APPROXIMATION OF LENGTH OF CURVES IN RIEMANNIAN MANIFOLDS BY GEODESIC POLYGONS

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Let γ be a smooth curve of finite length $L(\gamma)$ in an m-dimensional Riemannian manifold M. Let P_n be any geodesic polygon with n sides which is inscribed in γ. Let $L(P_n)$ be the length of P_n. We denote the geodesic curvature of γ by κ. In this paper we prove the following inequality:

Theorem.

$$\lim_{n\to\infty} n^2(L(\gamma)-L(P_n)) \geqq \frac{1}{24}\left[\int_\gamma \kappa^{2/3}\right]^3$$

We also construct an inscribed polygon P_n for each n such that the equality in the above inequality holds.

This is a generalization of a theorem by A. M. Gleason [2] which gives the inequality for curves in Euclidean spaces.

Let s be the arc-length parameter of γ.

Lemma 1. *Fix s_0 and let $L(t)$ be the length of $\gamma|[s_0, s_0+t]$. Let $D(t)=d_M(\gamma(s_0), \gamma(s_0+t))$ where d_M is the distance in M. Then*

$$D(t)=L(t)-\frac{1}{24}\kappa(s_0)^2L(t)^3+o(L(t)^3)$$

Proof. Since γ is parameterized by arc-length, $L(t)=t$. Let $f(t)=D(t)^2$. Let $\{e_i\colon i=1, \cdots, m\}$ be an orthonormal basis for $T_{\gamma(s_0)}M$, the tangent space of M at $\gamma(s_0)$. Let $(x_1, \cdots, x_m)$ be the normal coordinate system associated with $\{e_i\}$: To each point p in a neighborhood of $\gamma(s_0)$ we assign the coordinates $(x_1, \cdots, x_m)$ if $p=\exp_{\gamma(s_0)}(\sum_{i=1}^m x_ie_i)$. We set

$$g_{ij}=\left\langle \frac{\partial}{\partial x_i}\ \frac{\partial}{\partial x_j}\right\rangle \quad \text{and} \quad \nabla_{\partial/\partial x_i}\frac{\partial}{\partial x_j}=\sum_{k=1}^m \Gamma_{ij}^k\frac{\partial}{\partial x_k},$$

where $\langle\, ,\rangle$ is the Riemannian metric on M and ∇ is the Levi-Civita connection associated with $\langle\, ,\rangle$. g_{ij} and Γ_{ij}^k satisfy the following conditions for all i, j and k at $\gamma(s_0)$:

$$g_{ij}(\gamma(s_0))=\delta_{ij} \tag{1}$$

$$\Gamma^k_{ij}(\gamma(s_0))=0\,. \tag{2}$$

(See [1], for instance.)

Fix $(a_1, \cdots, a_m)\in \boldsymbol{R}^m$. We define a curve $\sigma(u)$ on M by $x_i(\sigma(u))=ua_i$. The tangent vector of $\sigma(u)$ is

$$\sigma'(u)=\sum_{i=1}^m \frac{d}{du}x_i(\sigma(u))\frac{\partial}{\partial x_i}=\sum_{i=1}^m a_i\frac{\partial}{\partial x_i}\,. \tag{3}$$

Since $\sigma(u)$ is a geodesic on M,

$$\nabla_{\sigma'}\sigma'=\sum_{i,j,k} a_ia_j\Gamma^k_{ij}\frac{\partial}{\partial x_k}=0\,. \tag{4}$$

Since $a_i=x_i/u$, it follows from (4) that

$$\sum_{i,j} x_ix_j\Gamma^k_{ij}=0 \tag{5}$$

for all k.

We set $x_i(t)=x_i(\gamma(s_0+t))$. Then $f(t)=\sum_i x_i(t)^2$. Since $x_i(0)=0$, we have

$$f(0)=0 \tag{6}$$

$$f'(0)=0 \tag{7}$$

$$f''(0)=2\sum_i x'_i(0)^2 \tag{8}$$

$$f'''(0)=6\sum_i x'_i(0)x''_i(0) \tag{9}$$

$$f''''(0)=6\sum_i x''_i(0)^2+8\sum_i x'_i(0)x'''_i(0)\,. \tag{10}$$

Let T be the tangent vector of γ so that $T=\sum_i x'_i(t)\partial/\partial x_i$. Since γ is parameterized by arc-length, $\langle T, T\rangle\equiv 1$. This implies that

$$\sum_{i,j} x'_i(t)x'_j(t)g_{ij}(t)\equiv 1 \tag{11}$$

where $g_{ij}(t)=g_{ij}(\gamma(s_0+t))$. From (1), (8) and (11) we have

$$f''(0)=2\,. \tag{12}$$

We set $\Gamma^k_{ij}(t)=\Gamma^k_{ij}(\gamma(s_0+t))$. Since

$$\nabla_T T=\sum_k (x''_k(t)+x'_i(t)x'_j(t)\Gamma^k_{ij}(t))\frac{\partial}{\partial x_k}\,,$$

(2) implies that

(13) $$\nabla_T T\,|_{\gamma(s_0)} = \sum_i x_i''(0)\frac{\partial}{\partial x_i}\,.$$

Since $\langle \nabla_T T, T\rangle = 0$, using (1), (9) and (10), we have

(14) $$f'''(0) = 0\,.$$

Differentiating (11) twice and evaluating at $t=0$, we obtain

(15) $$\sum_i (2x_i'''(0)x_i'(0) + 2x_i''(0)^2) + \sum_{i,j} 4x_i''(0)x_j'(0)g_{ij}'(0) + \sum_{i,j} x_i'(0)x_j'(0)g_{ij}''(0) = 0\,.$$

Since $g_{ij}'(t) = \sum_{k,l} x_k'(t)(\Gamma_{ki}^l(t)g_{lj}(t) + \Gamma_{kj}^l(t)g_{li}(t))$, (2) implies that

(16) $$g_{ij}'(0) = 0\,.$$

Since

$$g_{ij}''(t) = \sum_{k,l} [x_k''(t)(\Gamma_{ki}^l(t)g_{lj}(t) + \Gamma_{kj}^l(t)g_{li}(t)) + x_k'(t)\{(\Gamma_{ki}^l)'(t)g_{lj}(t) + \Gamma_{ki}^l(t)g_{lj}'(t) + (\Gamma_{kj}^l)'(t)g_{li}(t) + \Gamma_{kj}^l(t)g_{li}'(t)\}]\,,$$

it follows from (1) and (2) that

(17) $$g_{ij}''(0) = \sum_k [x_k'(0)(\Gamma_{ki}^j)'(0) + x_k'(0)(\Gamma_{kj}^i)'(0)]\,.$$

Differentiating (5) three times and evaluating at $t=0$, we obtain

(18) $$\sum_{i,j} x_i'(0)x_j'(0)(\Gamma_{ij}^k)'(0) = 0$$

for all k. From (17) and (18) we have

(19) $$\sum_{i,j} x_i'(0)x_j'(0)g_{ij}''(0) = 0\,.$$

Combining (15), (16) and (19), we have

(20) $$\sum_i x_i'''(0)x_i'(0) = -\sum_i x_i''(0)^2\,.$$

Since $\kappa(s_0)^2 = \sum_i x_i''(0)^2$ by (1) and (13), it follows from (10) and (20) that

(21) $$f''''(0) = -2\kappa(s_0)^2\,.$$

Using (6), (7), (12), (14) and (21), we obtain the following Taylor expansion for f:

(22) $$f(t) = t^2 - \frac{1}{12}\kappa(s_0)^2 t^4 + o(t^4)$$

Since $f(t) = D(t)^2$, (22) implies that

(23) $$D(t) = t - \frac{1}{24}\kappa(s_0)^2 t^3 + o(t^3)\,.$$

This completes the proof of Lemma 1.

Proof of the theorem. We will prove the following statement from which the theorem follows immediately.

For any $\varepsilon>0$ there is an integer r such that

$$(n+r)^{2/3}(L(\gamma)-L(P_n))^{1/3}>24^{-1/3}\int_\gamma \kappa^{2/3}ds-\varepsilon L(\gamma) \tag{24}$$

for all polygons P_n having n sides.

Let $\tilde{\gamma}=\gamma\,|\,[s_1, s_2]$ be a subarc of γ. Let σ be the geodesic segment which joins $\gamma(s_1)$ and $\gamma(s_2)$. By Lemma 1, for any $\varepsilon>0$ there exists a $\delta>0$ such that

$$(L(\tilde{\gamma})-L(\sigma))^{1/3}-24^{-1/3}\kappa(s)^{2/3}L(\tilde{\gamma})>-\varepsilon L(\tilde{\gamma}) \tag{25}$$

holds for any $\tilde{\gamma}$ with $L(\tilde{\gamma})<\delta$ and $s\in[s_1, s_2]$. Let r be an integer such that $L(\gamma)/r<\delta$. We can find at most r points on γ and adjoin them to the vertices of P_n to construct a new polygon Q_t $(n\leqq t\leqq n+r)$ each of whose sides has length $<\delta$. Let $\gamma(s_0)$, $\gamma(s_1), \cdots, \gamma(s_{t-1})$, $\gamma(s_t)$ be the vertices of Q_t and $\gamma_i=\gamma\,|\,[s_{i-1}, s_i]$. Let σ_i be a side of Q_t which joins $\gamma(s_{i-1})$ and $\gamma(s_i)$. For each i there is an $s_i'\in[s_{i-1}, s_i]$ such that

$$\int_{s_{i-1}}^{s_i} \kappa(s)^{2/3}ds=\kappa(s_i')^{2/3}L(\gamma_i)\,. \tag{26}$$

Using (25) and (26), we have

$$\begin{aligned}\sum_{i=1}^{t}(L(\gamma_i)-L(\sigma_i))^{1/3}&>\sum_{i=1}^{t}[24^{-1/3}\kappa(s_i')^{2/3}L(\gamma_i)-\varepsilon L(\gamma_i)]\\ &=\sum_{i=1}^{t}\left[24^{-1/3}\int_{s_{i-1}}^{s_i}\kappa(s)^{2/3}ds-\varepsilon L(\gamma_i)\right]\\ &=24^{-1/3}\int_\gamma \kappa(s)^{2/3}ds-\varepsilon L(\gamma)\,.\end{aligned} \tag{27}$$

On the other hand, Hölder's inequality gives the following inequality:

$$\begin{aligned}\sum_{i=1}^{t}(L(\gamma_i)-L(\sigma_i))^{1/3}&=\sum_{i=1}^{t}1(L(\gamma_i)-L(\sigma_i))^{1/3}\\ &\leqq\left(\sum_{i=1}^{t}1^{3/2}\right)^{2/3}\left(\sum_{i=1}^{t}(L(\gamma_i)-L(\sigma_i))\right)^{1/3}\\ &=t^{2/3}(L(\gamma)-L(Q_t))^{1/3}\,.\end{aligned} \tag{28}$$

Since $t\leqq n+r$ and $L(P_n)\geqq L(Q_t)$, it follows from (27) and (28) that

$$24^{-1/3}\int_\gamma \kappa(s)^{2/3}ds-\varepsilon L(\gamma)<t^{2/3}(L(\gamma)-L(Q_t))^{1/3}\leqq(n+r)^{2/3}(L(\gamma)-L(P_n))^{1/3}\,. \tag{29}$$

This completes the proof of the theorem.

Construction of a polygon which gives the best approximation.

Now we construct a geodesic polygon inscribed in γ which gives the best approximation to the length of γ. More precisely, we will show that for any positive number $\eta>0$ there exists a geodesic polygon P_n inscribed in γ such that

$$n^2(L(\gamma)-L(P_n))<\frac{1}{24}\left(\int_\gamma \kappa(s)^{2/3}ds\right)^3+\eta\,. \tag{30}$$

As in the proof of the theorem, let $\tilde{\gamma}=\gamma\,|\,[s_1, s_2]$ be a subarc of γ and σ be the geodesic segment which joins $\gamma(s_1)$ and $\gamma(s_2)$. Let $\varepsilon>0$ be given. Then, by Lemma 1, there exists a $\delta>0$ such that

$$L(\tilde{\gamma})-L(\sigma)<\frac{1}{24}\kappa(s)^2L(\tilde{\gamma})^3+\varepsilon L(\tilde{\gamma})^3 \tag{31}$$

for any $\tilde{\gamma}$ with $L(\tilde{\gamma})<\delta$ and any $s\in[s_1, s_2]$.

We write $\gamma=\gamma_0+\gamma_1+\gamma_2$, where $\gamma_0=\{\gamma(s)\colon \kappa(s)=0\}$, $\gamma_1=\{\gamma(s)\colon 0<\kappa(s)<\varepsilon^{1/3}\}$ and $\gamma_2=\{\gamma(s)\colon \varepsilon^{1/3}\leqq\kappa(s)\}$.

Choose an integer l such that $l>L(\gamma_1)/\delta$. Let $\gamma_{11}, \gamma_{12}, \cdots, \gamma_{1l}$ be a subdivision of γ_1 such that $L(\gamma_{1i})$ are all equal. Let σ_i be the geodesic segment which joins the endpoints of γ_{1i} and let Q_l be the polygon consisting of σ_i's. Then we have $L(\gamma_{1i})=L(\gamma_1)/l<\delta$ for each i and hence, using (31), we have

$$\begin{aligned}L(\gamma_{1i})-L(\sigma_i)&<\frac{1}{24}\kappa(s)^2L(\gamma_{1i})^3+\varepsilon L(\gamma_{1i})^3<\frac{1}{24}\varepsilon^{2/3}L(\gamma_{1i})^3+\varepsilon L(\gamma_{1i})^3\\&=\frac{L(\gamma_1)^3}{l^3}\left(\frac{1}{24}\varepsilon^{2/3}+\varepsilon\right).\end{aligned} \tag{32}$$

Summing on i, we obtain

$$L(\gamma_1)-L(Q_l)<\frac{L(\gamma_1)^3}{l^2}\left(\frac{1}{24}\varepsilon^{2/3}+\varepsilon\right). \tag{32}$$

Choose an integer m such that $m>(1/\delta\varepsilon^{2/9})(\int_{\gamma_2}\kappa^{2/3}ds)$. We can find a subdivision $\{\gamma_{2j}; j=1, \cdots, m\}$ of γ_2 and points $\gamma(s_j')$ in each γ_{2j} such that $S_j=\kappa(s_j')^{2/3}L(\gamma_{2j})$ are equal for all j and $S_j=(1/m)\int_{\gamma_2}\kappa(s)^{2/3}ds$. Let τ_j be the geodesic segment which joins the endpoints of γ_{2j} and let R_m be the polygon consisting of all τ_j's. Since we have $L(\gamma_{2j})<\delta$ for each j, it follows from (31) that

$$\begin{aligned}L(\gamma_{2j})-L(\tau_j)&<\frac{1}{24}\kappa(s_j')^2L(\gamma_{2j})^3+\varepsilon L(\gamma_{2j})^3=\frac{1}{24}S_j^3+\varepsilon L(\gamma_{2j})^3\\&=\frac{1}{24m^3}\left(\int_{\gamma_2}\kappa^{2/3}ds\right)^3+\varepsilon L(\gamma_{2j})^3\\&<\frac{1}{24m^3}\left(\int_\gamma \kappa^{2/3}ds\right)^3+\frac{\varepsilon^{1/3}}{m^3}\left(\int_\gamma \kappa^{2/3}ds\right)^3.\end{aligned} \tag{33}$$

Here the last inequality follows from

$$\int_{\gamma_2} \kappa(s)^{2/3} ds < \int_{\gamma} \kappa(s)^{2/3} ds \tag{34}$$

and

$$L(\gamma_{2j}) = S_j \kappa(s'_j)^{-2/3} = \frac{1}{m}\left(\int_{\gamma_2} \kappa^{2/3} ds\right) \kappa(s'_j)^{-2/3} < \frac{1}{m} \varepsilon^{-2/9} \int_{\gamma_2} \kappa^{2/3} ds\,. \tag{35}$$

Summing on j, we obtain the following enequality:

$$L(\gamma_2) - L(R_m) < \left(\frac{1}{24m^2} + \frac{\varepsilon^{1/3}}{m^2}\right)\left(\int_{\gamma} \kappa^{2/3} ds\right)^3. \tag{36}$$

Let $\gamma_{01}, \gamma_{02}, \cdots, \gamma_{0k}$ be the connected components of γ_0. We construct a geodesic polygon P_n $(n = k + l + m)$ with $k + l + m$ sides, $\gamma_{01}, \cdots, \gamma_{0k}, \gamma_{11}, \cdots, \gamma_{1l}, \gamma_{21}, \cdots, \gamma_{2m}$. Now we choose l and m in the follwoing way: Let

$$l > \max\left\{\frac{L(\gamma_1)}{\delta}, \frac{1}{\delta} \varepsilon^{1/36} \int_{\gamma} \kappa^{2/3} ds, \frac{k}{\varepsilon}\right\} \quad \text{and} \quad m = \left[\frac{l}{\varepsilon^{1/4}}\right] + 1\,.$$

Note that the condition $m > (1/\delta\varepsilon^{2/9}) \int_{\gamma_2} \kappa^{2/3} ds$ is satisfied in this choice. Then we have

$$\begin{aligned} n^2(L(\gamma) - L(P_n)) &= (k + l + m)^2 (L(\gamma_1) - L(Q_l) + L(\gamma_2) - L(R_m)) \\ &< (k + l + m)^2 \left\{\frac{L(\gamma_1)^3}{l^2}\left(\frac{1}{24}\varepsilon^{2/3} + \varepsilon\right) + \left(\frac{1}{24m^2} + \frac{\varepsilon^{1/3}}{m^2}\right)\left(\int_{\gamma} \kappa^{2/3} ds\right)^3\right\} \\ &< l^2 (1 + \varepsilon^{-1/4} + \varepsilon)^2 \left\{\frac{L(\gamma_1)^3}{l^2}\left(\frac{1}{24}\varepsilon^{2/3} + \varepsilon\right) + \left(\frac{\varepsilon^{1/2}}{24 l^2} + \frac{\varepsilon^{5/6}}{l^2}\right)\left(\int_{\gamma} \kappa^{2/3} ds\right)^3\right\} \\ &= (1 + \varepsilon^{1/4} + \varepsilon^{5/4})^2 \left\{L(\gamma_1)^3 \left(\frac{1}{24}\varepsilon^{1/6} + \varepsilon\right) + \left(\frac{1}{24} + \varepsilon^{1/3}\right)\left(\int_{\gamma} \kappa^{2/3} ds\right)^3\right\} \end{aligned} \tag{37}$$

The inequality (37) shows that if we choose ε small enough, we can construct a geodesic polygon P_n which satisfies the condition (30) for any given η.

References

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