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ON LOCALLY W^*-ALGEBRAS

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We consider in this paper a more general class than that of the classical W^*-algebras; namely the class of *locally* W^*-*algebras* defined as inverse limits of W^*-algebras (Definition 1.1). Among the examples of this sort of algebras we work out, is the locally C^*-algebra $L(H)$, H a "locally Hilbert space" [8; Section 5]. As it is apparent from what follows, $L(H)$ represents, in effect, the most general case of a locally W^*-algebra. That is, defining the respective σ-weak (operator) topology on $L(H)$, we prove that every locally W^*-algebra E equipped with the inverse limit topology σ (Proposition 1.3) coincides (within an isomorphism of topological algebras) with a σ-weakly closed *-subalgebra of some $L(H)$, H a locally Hilbert space (Theorem 2.1). In this respect, one gets that the center $Z(E)$ of a locally W^*-algebra E, is a σ-closed *-subalgebra of (E, σ), hence also a locally W^*-algebra (cf. Corollary 2.2 and Example 1.4.1, Proposition). On the other hand, one always obtains that every locally W^*-algebra admits an "Arens-Michael-type decomposition" consisting of W^*-algebras (Proposition 1.3). An application of the above gives information concerning the inner derivations of a locally W^*-algebra E. More precisely, one has that each inner derivation δ_x, $x=(x_\alpha)\in E$, of E is an inverse limit of inner derivations δ_{x_α}, $\alpha\in A$, corresponding to the W^*-algebra factors $\bar{E}_\alpha^{\sigma_\alpha}$, $\alpha\in A$ of E. Thus, the set $\delta_0(E)$ of all inner derivations of E, becomes a complete locally convex space (Theorem 3.1), so that if (q_α), $\alpha\in A$, is a defining family of seminorms for $\delta_0(E)$, one gets that for every $\delta_x\in\delta_0(E)$

$$q_\alpha(\delta_x)=2\inf\{p_\alpha(x-z)\colon z\in Z(E)\}\ , \qquad \alpha\in A\ ,$$

where (p_α), $\alpha\in A$, is a family of (submultiplicative) C^*-seminorms defining the topology of E. The latter extends in our case a previous result of L. Zsidó [16] concerning an estimation of the norm of an (inner) derivation acting on an abstract W^*-algebra. Similar estimations referred to (inner) derivations of the C^*-algebra $\mathscr{L}(H)$ of all bounded linear operators on a Hilbert space H, or of a W^*-algebra acting on a separable Hilbert space, have been also given earlier by J. G. Stampfli [14] and P. Gajendragadkar [5] respectively. Furthermore we use Zsidó's technique of Theorems 1 and 2 in [16] in order to estimate the above numbers $q_\alpha(\delta_x)$, $\alpha\in A$, by means of a certain map $\Phi\colon E\to Z(E)$, at the cost however of some particular restriction on the locally W^*-algebra E (Theorem 3.5). A further application of the latter leads to a new information even for the normed case, according to which

Zsido's continuous map $v: \delta_0(E) \to E$ [16; Theor. 3] is, in effect, a relatively open section of a continuous open linear surjection $u: E \to \delta_0(E)$ (Theorem 4.3).

1. The topological algebras considered throughout are all over the field $\boldsymbol{C}$ of complex numbers and have an identity element.

By an *lmc* (locally *m*-convex) **-algebra*, we mean a *-algebra E endowed with a topology defined by a family (p_α), $\alpha \in A$ (a directed index set), of *-preserving submultiplicative seminorms. A complete lmc *-algebra E is called a *locally C^*-algebra* [8] if, in addition, $p_\alpha(x)^2 = p_\alpha(x^* x)$, for any $\alpha \in A$, $x \in E$. Given an lmc *-algebra E, set $N_\alpha \equiv \ker(p_\alpha)$ and denote by $\tilde{E}_\alpha$ the completion of the normed algebra $E_\alpha \equiv E/N_\alpha$, with norm $\dot{p}_\alpha$ defined by $\dot{p}_\alpha(x_\alpha) = p_\alpha(x)$, $x_\alpha = x + N_\alpha \in E_\alpha$, $x \in E$ and $\alpha \in A$ (cf. [1], [11]). Then, if E is complete one gets $E = \varprojlim_\alpha \tilde{E}_\alpha$, within a topological algebraic isomorphism (ibid.), the latter expression being called an *Arens-Michael decomposition* of E [10; Def. III, 3.1]. In particular, if E is a locally C^*-algebra, E_α is always complete, hence a C^*-algebra, i.e., $E_\alpha = \tilde{E}_\alpha$ for every $\alpha \in A$ (cf. [13; Folg. 5.4]), a fact being actually true even if E has no unit (cf. [4; Prop. 2.1, (ii)]), E given thus as an inverse limit of C^*-algebras.

Now, let $(F_\alpha, f_{\alpha\beta})$, $\alpha \in A$, be an inverse system of W^*-algebras, where the connecting maps $f_{\alpha\beta}$, $\alpha \leq \beta$ in A, are considered continuous with respect to the uniform (norm) topology of F_α's. The canonical map of the inverse limit $\varprojlim_\alpha F_\alpha$ into F_α, will be denoted by f_α, $\alpha \in A$. In this regard, we now set the next.

Definition 1.1 (A. Mallios). An algebra E is said to be a *locally W^*-algebra*, if it is given as an inverse limit of W^*-algebras, i.e., $E = \varprojlim_\alpha F_\alpha$, where each F_α, $\alpha \in A$, is a W^*-algebra.

Scholium 1.2. *Every locally W^*-algebra E is a locally C^*-algebra equipped with the inverse limit topology τ, induced on it by the uniform topology of its W^*-algebra factors F_α, $\alpha \in A$. In addition, the Arens-Michael decomposition of (E, τ) is given exactly by the *-subalgebras $f_\alpha(E)$ of F_α's, $\alpha \in A$.*

In fact, if $\|\cdot\|_\alpha$ denotes the C^*-norm defining the uniform topology of F_α, $\alpha \in A$, one gets by [10; Lemma III, 3.2] that

$$\varprojlim_\alpha F_\alpha = (E, \tau) = \varprojlim_\alpha f_\alpha(E) = \varprojlim_\alpha \overline{f_\alpha(E)}, \tag{1.1}$$

where "—" means $\|\cdot\|_\alpha$-closure. Now the relation

$$p_\alpha = \|\cdot\|_\alpha \circ f_\alpha, \qquad \alpha \in A, \tag{1.2}$$

defines a (*-preserving submultiplicative) C^*-seminorm on E, and it is clear by (1.1) that τ *is defined by the family* (p_α), $\alpha \in A$, *and makes E into a locally C^*-algebra.* Now, considering the Arens-Michael factor E_α, $\alpha \in A$, of (E, τ), one has by (1.2) that

$$\dot{p}_\alpha(x_\alpha) = p_\alpha(x) = \|f_\alpha(x)\|_\alpha, \qquad x \in E, \qquad \alpha \in A,$$

consequently the map

$$E_\alpha \equiv E/N_\alpha \longrightarrow f_\alpha(E): x_\alpha \longmapsto f_\alpha(x)\,, \qquad x\in E\,, \qquad \alpha\in A\,, \tag{1.3}$$

is a topological algebraic isomorphism, and since $E_\alpha=\tilde{E}_\alpha$, $\alpha\in A$ (cf. discussion before Definition 1.1), one also gets $f_\alpha(E)=\overline{f_\alpha(E)}$, $\alpha\in A$. Thus, we finally have

$$\varprojlim_\alpha (F_\alpha, \|\cdot\|_\alpha)=(E,\tau)=\varprojlim_\alpha (E_\alpha \cong f_\alpha(E), \dot{p}_\alpha)\,, \tag{1.4}$$

within isomorphisms of topological algebras.

Note that, because of (1.3), the connecting maps of the inverse system (E_α), $\alpha\in A$, coincide with the restrictions of $f_{\alpha\beta}$, $\alpha\le\beta$, on $f_\beta(E)$. Thus, from now on, we agree to keep the symbols $f_{\alpha\beta}$, $\alpha\le\beta$, f_α, $\alpha\in A$, for the connecting maps, respectively, canonical maps of the inverse system (E_α), $\alpha\in A$, as well.

Now, given a locally W^*-algebra $E=\varprojlim_\alpha F_\alpha$, for each $\alpha\in A$, there is a Banach space M_*^α the dual of which is F_α, $\alpha\in A$ [12; Defs. 1.1.2, 1.1.3]. We denote by σ_α the weak $*$-topology $\sigma((M_*^\alpha)^*, M_*^\alpha)$ on $F_\alpha\cong(M_*^\alpha)^*$ and by $(M_*^\alpha)^*_{\sigma_\alpha}$ the W^*-algebra F_α endowed with σ_α, $\alpha\in A$. In this respect, we now have the next.

Proposition 1.3. *Let $E=\varprojlim_\alpha F_\alpha$ be a locally W^*-algebra, in such a way that the connecting maps $f_{\alpha\beta}$, $\alpha\le\beta$ in A, of the respective inverse system (F_α), $\alpha\in A$, are weakly $*$-continuous. Then, E is endowed with the inverse limit topology σ, coarser than the inverse limit lmc C^*-topology τ (cf. Scholium* 1.2). *In addition, (E,σ) admits an Arens-Michael-type decomposition consisting of W^*-algebras, in the sense that*

$$(E,\sigma)=\varprojlim_\alpha \bar{E}_\alpha^{\sigma_\alpha}\,, \tag{1.5}$$

within a topological algebraic isomorphism, where "$-\sigma_\alpha$" means σ_α-closure in F_α, $\alpha\in A$.

Proof. By the above comments and the weak $*$-continuity of $f_{\alpha\beta}$, $\alpha\le\beta$, one concludes that $((M_*^\alpha)^*_{\sigma_\alpha}, f_{\alpha\beta})$, $\alpha\le\beta$, is a projective system of topological vector spaces, in such a way that

$$E=\varprojlim_\alpha (M_*^\alpha)^*_{\sigma_\alpha}\,, \tag{1.6}$$

within an isomorphism of vector spaces. So that by (1.6) E is obviously equipped with the inverse limit topology $\sigma=\varprojlim_\alpha \sigma_\alpha$, which is coarser than the lmc C^*-topology τ defined by the family (p_α), $\alpha\in A$ (cf. (1.2)), as this follows by the next commutative diagram

$$\begin{tikzcd}
(E,\tau) \arrow[r, "f_\alpha"] \arrow[d, "\mathrm{id}_E"'] & (F_\alpha, \|\cdot\|_\alpha) \arrow[d, "\mathrm{id}_{F_\alpha}"] \\
(E,\sigma) \arrow[r, "f_\alpha"'] & (F_\alpha,\sigma_\alpha)=(M_*^\alpha)^*_{\sigma_\alpha}\,,
\end{tikzcd}$$

and the fact that $\sigma_\alpha \leq \|\cdot\|_\alpha$ on F_α for every $\alpha \in A$.

Now, denote by $\bar{E}_\alpha^{\sigma_\alpha}$ the σ_α-closure of E_α into F_α, $\alpha \in A$ (cf. Scholium 1.2). Then, $\bar{E}_\alpha^{\sigma_\alpha}$ is a W^*-algebra as a σ_α-closed $*$-subalgebra of the W^*-algebra F_α, $\alpha \in A$, [12; Def. 1.1.4]. On the other hand, since $f_{\alpha\beta}$, $\alpha \leq \beta$ are weakly $*$-continuous, they are uniquely extended to $\tilde{f}_{\alpha\beta}: \bar{E}_\beta^{\sigma_\beta} \to \bar{E}_\alpha^{\sigma_\alpha}$, $\alpha \leq \beta$, in such a way that $(\bar{E}_\alpha^{\sigma_\alpha}, \tilde{f}_{\alpha\beta})$, $\alpha \in A$, is a projective system too. Thus, one gets

$$E = \varprojlim_\alpha E_\alpha \subset \varprojlim_\alpha \bar{E}_\alpha^{\sigma_\alpha} \hookrightarrow \varprojlim_\alpha F_\alpha = E\,,$$

which implies

$$(E, \sigma) = \varprojlim_\alpha \bar{E}_\alpha^{\sigma_\alpha}\,,$$

within an isomorphism of topological algebras. □

Motivated by the above when we speak in the sequel about the σ-topology of a locally W^*-algebra, we shall always mean that the connecting maps of the respective inverse system are weakly $*$-continuous.

1.4. Examples of locally W^*-algebras.

1. The first example of a locally W^*-algebra is given by the next.

Proposition. *Every σ-closed $*$-subalgebra G of a locally W^*-algebra $E = \varprojlim_\alpha F_\alpha$, is also a locally W^*-algebra.*

Proof. Since $\sigma \leq \tau$ on E, G is also τ-closed, hence a locally C^*-subalgebra of (E, τ). Thus, (cf. [1], [11])

$$(G, \tau|_G) = \varprojlim_\alpha G_\alpha\,,$$

within a topological algebraic isomorphism, where $G_\alpha \equiv G/\ker(p_\alpha|_G)$, $\alpha \in A$, is a C^*-algebra (cf. discussion before Definition 1.1). In particular, $G_\alpha \cong f_\alpha(G)$, $\alpha \in A$ (cf. (1.3)), so that, since G is σ-closed one gets by [10; Lemma III, 3.2]

$$(G, \sigma|_G) = \varprojlim_\alpha \bar{G}_\alpha^{\sigma_\alpha}\,, \tag{1.7}$$

within an isomorphism of topological algebras, where each $\bar{G}_\alpha^{\sigma_\alpha}$ is a W^*-algebra as a σ_α-closed $*$-subalgebra of the W^*-algebra $\bar{E}_\alpha^{\sigma_\alpha}$, $\alpha \in A$. □

2. Let (E_n), $n \in N$ (natural numbers), be a descending sequence of W^*-algebras with non-trivial intersection (cf. [3]). Let also that uniform, respectively, weak $*$-topologies on E_n, $n \in N$, form an ascending sequence (for any $n \leq m$ in N, $\|\cdot\|_n|_{E_m} \leq \|\cdot\|_m$, as well as $\sigma_n|_{E_m} \leq \sigma_m$). *Then,*

$$E \equiv \bigcap_n E_n\,,$$

is a locally W^*-algebra. It is easily seen that

$$\varprojlim_n E_n = \bigcap_n E_n ,$$

within an algebraic isomorphism. Moreover, the canonical injections

$$j_{nm} : E_m \hookrightarrow E_n , \qquad n \leq m \quad \text{in} \quad N ,$$

are norm, respectively, weakly $*$-continuous, so that $\varprojlim_n E_n$ is endowed with the inverse limit topologies $\tau = \varprojlim_n \| \cdot \|_n$, $\sigma = \varprojlim_n \sigma_n$, where $\sigma \leq \tau$ since $\sigma_n \leq \| \cdot \|_n$, $n \in N$. Thus, E is a locally W^*-algebra by Definition 1.1. In particular (cf. also Proposition 1.3),

$$\varprojlim_n (E_n, \sigma_n) = (E, \sigma) \cong \varprojlim_n \bar{F}_n^{\sigma_n} ,$$

where

$$F_n \equiv (E, \| \cdot \|_{n|E}) , \qquad n \in N ,$$

is a C^*-algebra corresponding to the Arens-Michael decomposition of (E, τ), and $\bar{F}_n^{\sigma_n}$ a W^*-subalgebra of E_n, $n \in N$.

3. Let (H_λ), $\lambda \in \Lambda$, be a directed family of Hilbert spaces, with $H_\lambda \subset H_\mu$ and $\langle\, , \rangle_\lambda = \langle\, , \rangle_\mu$ on H_λ for any $\lambda \leq \mu$ in Λ. Then, $H = \varinjlim_\lambda H_\lambda$ endowed with the respective locally convex inductive limit topology, is called a *locally Hilbert space* (cf. [8; Def. 5.2]). Thus, if

$$L(H) = \{T \in \mathscr{L}(H)\text{: for every } \lambda \leq \mu \,|_\Lambda,\ T_\mu \circ i_{\mu\lambda} = i_{\mu\lambda} \circ T_\lambda. \text{ where } T_\lambda = T|_{H_\lambda}$$
$$\in \mathscr{L}(H_\lambda),\ \lambda \in \Lambda, \text{ and } i_{\mu\lambda} \text{ the canonical injection of } H_\lambda \text{ into } H_\mu\} ,$$

$L(H)$ is, in fact, a locally C^*-algebra (cf. [8; Prop. 5.1]). *In particular, $L(H)$ is a locally W^*-algebra, in such a way that the connecting maps of the inverse system of W^*-algebras corresponding to $L(H)$, are weakly $*$-continuous.* The topology of $L(H)$ is defined by a family of ($*$-preserving submultiplicative) C^*-seminorms (p_λ), $\lambda \in \Lambda$, such that $p_\lambda(T) = \|T_\lambda\|$, $\lambda \in \Lambda$ (ibid). Now, considering the C^*-algebra $L(H)/N_\lambda$, $\lambda \in \Lambda$, corresponding to the Arens-Michael decomposition of $L(H)$, we conclude that the map

$$L(H)_\lambda \equiv L(H)/N_\lambda \longrightarrow \mathscr{L}(H_\lambda) : T + N_\lambda \longmapsto T_\lambda = T|_{H_\lambda} , \qquad \lambda \in \Lambda ,$$

is an isomorphism of topological algebras, so that

$$L(H = \varinjlim_\lambda H_\lambda) = \varprojlim_\lambda \mathscr{L}(H_\lambda) , \tag{1.8}$$

within a topological algebraic isomorphism, where each $\mathscr{L}(H_\lambda)$, $\lambda \in \Lambda$, is a W^*-algebra; hence, by Definition 1.1 $L(H)$ is a locally W^*-algebra.

We shall now show that the connecting maps $f_{\lambda\mu}$, $\lambda \leq \mu$ in Λ, of the inverse system $(L(H)_\lambda \cong \mathscr{L}(H_\lambda))$, $\lambda \in \Lambda$, are weakly $*$-continuous. For each $\lambda \in \Lambda$ $\mathscr{L}(H_\lambda) = (\mathscr{L}_*(H_\lambda))^*$, with $\mathscr{L}_*(H_\lambda)=\mathscr{L}\mathscr{C}(H_\lambda)^*$ the dual of the compact operators of $\mathscr{L}(H_\lambda)$, $\lambda\in\Lambda$. Moreover, the weak *-topology $\sigma(\mathscr{L}(H_\lambda), \mathscr{L}_*(H_\lambda))$, $\lambda\in\Lambda$, is the σ-weak (operator) topology of $\mathscr{L}(H_\lambda)$, defined by the following family of seminorms

$$p_{(\xi_\lambda^n),(\eta_\lambda^n)}(T_\lambda)=\left|\sum_{n=1}^{\infty}\langle T_\lambda\xi_\lambda^n, \eta_\lambda^n\rangle_\lambda\right|, \quad T_\lambda\in\mathscr{L}(H_\lambda), \quad \lambda\in\Lambda, \tag{1.9}$$

for any sequences (ξ^n), (η_λ^n) in H_λ with $\sum_{n=1}^{\infty}\|\xi_\lambda^n\|^2<\infty$, $\sum_{n=1}^{\infty}\|\eta_\lambda^n\|^2<\infty$ [15; p. 67], $\lambda\in\Lambda$. Thus, if (T_μ^α) is a net in $\mathscr{L}(H_\mu)$ such that $T_\mu^\alpha\to 0$ with respect to $\sigma(\mathscr{L}(H_\mu), \mathscr{L}_*(H_\mu))$, then since $T_\lambda^\alpha=T_\mu^\alpha|_{H_\lambda}$ and $\langle\, ,\rangle_\lambda=\langle\, ,\rangle_\mu|_{H_\lambda}$, one also gets that $T_\lambda^\alpha\to 0$ with respect to $\sigma(\mathscr{L}(H_\lambda), \mathscr{L}_*(H_\lambda))$, $\lambda\leq\mu$ in Λ.

By the above and Proposition 1.3 we now have that $L(H)$ is endowed with the inverse limit topology

$$\sigma=\varprojlim_\lambda \sigma_\lambda, \qquad \text{with} \quad \sigma_\lambda\equiv\sigma(\mathscr{L}(H_\lambda), \mathscr{L}_*(H_\lambda)). \tag{1.10}$$

The topology σ on $L(H)$ will be called, in the sequel, *σ-weak (operator) topology.*

2. In this Section we shall show that every locally W^*-algebra E equipped with the inverse limit topology σ (Proposition 1.3) is identified (within a topological algebraic isomorphism) with a σ-weakly closed *-subalgebra of some $L(H)$, H a locally Hilbert space (Theorem 2.1). The latter constitutes in our case, an analogue of the respective classical situation for W^*-algebras (cf., for instance, [15; Theor. III, 3.5]).

Theorem 2.1. *Every locally W^*-algebra E endowed with the inverse limit topology σ coincides, within an isomorphism of topological algebras, with a σ-weakly closed *-subalgebra of some $L(H)$, H a locally Hilbert space.*

Proof. $E=\varprojlim_\alpha F_\alpha$, where each F_α, $\alpha\in A$, is a W^*-algebra, hence [15; Theor. III, 3.5] there is a faithful representation

$$\varphi_\alpha\colon F_\alpha\longrightarrow\mathscr{L}(H_\alpha), \qquad \alpha\in A, \tag{2.1}$$

H_α, $\alpha\in A$, a Hilbert space, bicontinuous with respect to the topologies $\sigma_\alpha\equiv\sigma(F_\alpha, M_*^\alpha)$, $\sigma(\mathscr{L}(H_\alpha), \mathscr{L}_*(H_\alpha))$ of F_α, $\mathscr{L}(H_\alpha)$, $\alpha\in A$, respectively (cf. comments before Proposition 1.3 and (1.10)). Now, set (cf. also [8; Theor. 5.1])

$$\mathscr{H}_\lambda=\bigoplus_{\alpha\leq\lambda} H_\alpha,$$

where $\oplus$ means orthogonal direct sum. Then, for any $\lambda\leq\mu$ $\mathscr{H}_\lambda\subset\mathscr{H}_\mu$ and $\langle\, ,\rangle_\lambda=\langle\, ,\rangle_\mu$ on $\mathscr{H}_\lambda$, so that

$$H=\varinjlim_\lambda \mathscr{H}_\lambda,$$

is a locally Hilbert space (cf. Example 1.4.3). In this regard, define

$$\varphi\colon E\longrightarrow L(H)\cong\varprojlim_\lambda \mathscr{L}(\mathscr{H}_\lambda)\colon x\longmapsto\varphi(x)\colon \varphi(x)|_{\mathscr{H}_\lambda}=\varphi(x)_\lambda,$$

with

$$\varphi(x)_\lambda(\xi_\lambda)=\bigoplus_{\alpha\le\lambda}\varphi_\alpha(x_\alpha)(\xi_\alpha), \qquad \text{for every} \quad \xi_\lambda=(\xi_\alpha)_{\alpha\le\lambda} \quad \text{in} \quad \mathscr{H}_\lambda .$$

It is easily seen that φ is a 1-1 algebraic morphism. On the other hand, if π_λ is the canonical map of $L(H)$ onto $\mathscr{L}(\mathscr{H}_\lambda)$, φ is continuous if, and only if, each $\pi_\lambda\circ\varphi$ is continuous. Thus, if (x_δ) is a net in E with $x_\delta \xrightarrow[\sigma]{} 0$, we have to show that $\pi_\lambda(\varphi(x_\delta))=\varphi(x_\delta)_\lambda \xrightarrow[\sigma_\lambda]{} 0$, for all λ. According to the definition of σ_λ (cf. (1.9)), for any sequences (ξ^n_λ), (η^n_λ) in $\mathscr{H}_\lambda$, such that

$$\sum_{n=1}^{\infty}\|\xi^n_\lambda\|^2<\infty , \qquad \sum_{n=1}^{\infty}\|\eta^n_\lambda\|^2<\infty , \tag{2.1}$$

we must show

$$\left|\sum_{n=1}^{\infty}\langle\varphi(x_\delta)_\lambda\xi^n_\lambda, \eta^n_\lambda\rangle_\lambda\right| \longrightarrow 0 . \tag{2.2}$$

But,

$$\left|\sum_{n=1}^{\infty}\langle\varphi(x_\delta)_\lambda\xi^n_\lambda, \eta^n_\lambda\rangle_\lambda\right| = \left|\sum_{n=1}^{\infty}\sum_{\alpha\le\lambda}\langle\varphi_\alpha(x^\delta_\alpha)\xi^n_{\lambda,\alpha}, \eta^n_{\lambda,\alpha}\rangle_\alpha\right| , \tag{2.3}$$

where $(\xi^n_{\lambda,\alpha})$, $(\eta^n_{\lambda,\alpha})$ are sequences in H_α, $\alpha\le\lambda$, such that

$$\sum_{n=1}^{\infty}\|\xi^n_{\lambda,\alpha}\|^2<\infty \qquad \sum_{n=1}^{\infty}\|\eta^n_{\lambda,\alpha}\|^2<\infty . \tag{2.4}$$

On the other hand, $x_\delta \xrightarrow[\sigma]{} 0 \Leftrightarrow x^\delta_\alpha \xrightarrow[\sigma_\alpha]{} 0$, for all α, which by the weak $*$-continuity of φ_α, $\alpha\in A$, yields that

$$\varphi_\alpha(x^\delta_\alpha) \longrightarrow 0 , \quad \text{with respect to} \quad \sigma(\mathscr{L}(H_\alpha), \mathscr{L}_*(H_\alpha)) , \tag{2.5}$$

for all $\alpha\in A$. But, (2.5) means that for any sequences as in (2.4)

$$\left|\sum_{n=1}^{\infty}\langle\varphi_\alpha(x^\delta_\alpha)\xi^n_{\lambda,\alpha}, \eta^n_{\lambda,\alpha}\rangle_\alpha\right| \longrightarrow 0 , \qquad \alpha\le\lambda .$$

Thus, (2.2) follows now from the fact that the right-hand side of (2.3) is less than or equal to $\sum_{\alpha\le\lambda}|\sum_{n=1}^{\infty}<\varphi_\alpha(x^\lambda_\alpha)\xi^n_{\lambda,\alpha}, \eta^n_{\lambda,\alpha}>_\alpha|$.

Conversely, let $(\varphi(x_\delta))$ be a net in $\varphi(E)\subset L(H)$, with

$$\varphi(x_\delta) \xrightarrow[\sigma]{} 0 , \tag{2.6}$$

where σ is now given by (1.10). We must show that

$$(f_\alpha\circ\varphi^{-1})(\varphi(x_\delta))=x^\delta_\alpha \xrightarrow[\sigma_\alpha]{} 0 , \qquad \text{for all} \quad \alpha\in A ,$$

or equivalently

$$(2.7) \qquad \varphi_\alpha(x_\alpha^\delta) \longrightarrow 0, \quad \text{with respect to} \quad \sigma(\mathscr{L}(H_\alpha), \mathscr{L}_*(H_\alpha)), \qquad \alpha \in A.$$

By (2.6) we have that $\varphi(x_\delta)_\lambda \xrightarrow[\sigma_\lambda]{} 0$, for all λ, which means that (2.2) is true for all sequences (ξ_λ^n), (η_λ^n), in $\mathscr{H}_\lambda$, fulfilling (2.1). Moreover, (2.7) will have been proved, if for all sequences (ξ_α^n), (η_α^n) in H_α with $\sum_{n=1}^\infty \|\xi_\alpha^n\|^2 < \infty$, $\sum_{n=1}^\infty \|\eta_\alpha^n\|^2 < \infty$, one gets

$$(2.8) \qquad \left| \sum_{n=1}^{\infty} \langle \varphi_\alpha(x_\alpha^\delta)\xi_\alpha^n, \xi_\alpha^n \rangle_\alpha \right| \longrightarrow 0, \qquad \alpha \in A.$$

But each H_α is imbedded to some $\mathscr{H}_\lambda$, $\alpha \leq \lambda$, so that the sequences (ξ_α^n), (η_α^n) as before, define sequences (ξ_λ^n), (η_λ^n) which satisfy (2.1). Thus, (cf. also (2.3))

$$\left| \sum_{n=1}^{\infty} \langle \varphi_\alpha(x_\alpha^\delta)\xi_\alpha^n, \eta_\alpha^n \rangle_\alpha \right| = \left| \sum_{n=1}^{\infty} \langle \varphi(x_\delta)_\lambda \xi_\lambda^n, \eta_\lambda^n \rangle_\lambda \right| \longrightarrow 0,$$

$\alpha \leq \lambda$, which proves (2.8).

We now show that $\varphi(E)$ is σ-weakly closed in $L(H)$. Let $\varphi(x_\delta)$ be a net in $\varphi(E)$ with

$$\varphi(x_\delta) \xrightarrow[\sigma]{} T \in L(H).$$

Then, $\varphi(x_\delta)_\lambda \xrightarrow[\sigma_\lambda]{} T_\lambda$, for all λ (cf. (1.10)), so that applying the argument of (2.3) and the notation σ_α for $\sigma(\mathscr{L}(H_\alpha), \mathscr{L}_*(H_\alpha))$, $\alpha \in A$, too, we conclude that

$$\varphi_\alpha(x_\alpha^\delta) \xrightarrow[\sigma_\alpha]{} T_\alpha, \qquad \text{for all} \quad \alpha \in A,$$

where $T_\alpha \in \varphi_\alpha(F_\alpha)$, $\alpha \in A$, since $\varphi_\alpha(F_\alpha)$ is σ_α-closed. Thus, $T_\alpha = \varphi_\alpha(y_\alpha)$, $y_\alpha \in F_\alpha$, $\alpha \in A$, where, in particular, $f_{\alpha\beta}(y_\beta) = y_\alpha$, for any $\alpha \leq \beta$ in A. On the other hand,

$$y_\alpha = \varphi_\alpha^{-1}(\varphi_\alpha(y_\alpha)) = \varphi_\alpha^{-1}(\lim_\delta^{\sigma_\alpha} \varphi_\alpha(x_\alpha^\delta)) = \lim_\delta^{\sigma_\alpha} x_\alpha^\delta, \qquad \text{for every} \quad \alpha \in A.$$

Hence,

$$\lim_\delta^\sigma x_\delta = y \in E \quad \text{and} \quad T = \lim_\delta^\sigma \varphi(x_\delta) = \varphi(\lim_\delta^\sigma x_\delta) = \varphi(y) \in \varphi(E). \quad \square$$

Corollary 2.2. *The center $Z(E)$ of a locally W^*-algebra E is also a locally W^*-algebra. In particular,*

$$(2.9) \qquad (Z(E), \sigma|_{Z(E)}) = Z((E, \sigma) = \varprojlim_\alpha \bar{E}_\alpha^{\sigma_\alpha}) = \varprojlim_\alpha Z(\bar{E}_\alpha^{\sigma_\alpha}),$$

within a topological algebraic isomorphism, where (E_α), $\alpha \in A$, corresponds to the Arens-Michael decomposition of E.

Proof. It is easily seen that multiplication of $L(H)$, H a locally Hilbert space, is separately continuous with respect to the σ-weak topology. Thus, by the preceding theorem, one also gets that multiplication of E is separately continuous with respect to σ. A consequence of the latter is that $Z(E)$ is a σ-closed *-subalgebra of (E, σ), therefore a locally W^*-algebra according to Example 1.4.1, Proposition. Moreover, by (1.7) (cf. also Proposition 1.3)

$$(2.10)\qquad (Z(E), \sigma|_{Z(E)}) = \varprojlim_{\alpha} \overline{Z(E)_\alpha}^{\sigma_\alpha},$$

within an isomorphism of topological algebras, where $(Z(E)_\alpha)$, $\alpha \in A$, is the Arens-Michael decomposition of $Z(E)$, as a locally C^*-subalgebra of (E, τ). Now,

$$(2.11)\qquad Z(E)_\alpha = Z(E_\alpha), \qquad \alpha \in A,$$

within the topological algebraic isomorphism

$$Z(E)_\alpha \longrightarrow Z(E_\alpha): z + \ker(p_\alpha|_{Z(E)}) \longmapsto z + N_\alpha, \qquad \alpha \in A,$$

while,

$$(2.12)\qquad Z(\bar{E}_\alpha^{\sigma_\alpha}) = Z(E_\alpha) = \overline{Z(E_\alpha)}^{\sigma_\alpha}, \qquad \alpha \in A,$$

as this follows by the separate continuity of the multiplication of $\bar{E}_\alpha^{\sigma_\alpha}$ with respect to σ_α, $\alpha \in A$ (cf., for instance, [15; Theor. III, 3.5]). Thus, (2.9) follows now by (2.10), (2.11), (2.12). □

2.3. The connecting maps of the inverse system $(Z(\bar{E}_\alpha^{\sigma_\alpha}) = Z(E_\alpha))$, $\alpha \in A$ (cf. (2.12)), are those of $(\bar{E}_\alpha^{\sigma_\alpha})$ restricted on $Z(\bar{E}_\alpha^{\sigma_\alpha})$, $\alpha \in A$. Thus, according to Section 1 (cf., in particular, proof of Proposition 1.3 and discussion before it), when no confusion is likely to result, we shall keep the symbols $f_{\alpha\beta}$, $\alpha \leq \beta$, f_α, $\alpha \in A$, of the connecting, respectively, canonical maps of the projective system (F_α), $\alpha \in A$, for both of the projective systems $(\bar{E}_\alpha^{\sigma_\alpha})$, $\alpha \in A$ and $(Z(\bar{E}_\alpha^{\sigma_\alpha}) = Z(E_\alpha))$, $\alpha \in A$.

Remark 2.4. If $L(H)$, H a locally Hilbert space, is the locally W^*-algebra of Example 1.4.3, one also defines on $L(H)$ the respective of the classical weak (operator) topology. In fact, if $T = (T_\lambda) \in L(H = \varinjlim_\lambda H_\lambda) \cong \varprojlim_\lambda \mathscr{L}(H_\lambda)$ (cf. (1.8)) and ξ, $\eta \in H$, there is $\lambda \in \Lambda$ such that $\xi, \eta \in H_\lambda$. Thus, if

$$(2.13)\qquad p_{\xi,\eta}(T) = |\langle T_\lambda \xi, \eta \rangle_\lambda|,$$

the seminorms $(p_{\xi,\eta})$, $\xi, \eta \in H$, define a locally convex topology on $L(H)$, which is called *weak (operator) topology* and *it is denoted by w*.

Furthermore, for any $\xi, \eta \in H$, one defines

$$\omega_{\xi,\eta}: L(H) \longrightarrow \boldsymbol{C}: T \longmapsto \omega_{\xi,\eta}(T) = \langle T_\lambda \xi, \eta \rangle_\lambda,$$

where λ is that index in Λ with $\xi, \eta \in H_\lambda$. Then, $\omega_{\xi,\eta} \in L(H)'$ (topological dual of $(L(H), w)$), for any $\xi, \eta \in H$. Thus, if $L\mathscr{F}(H)$ is the linear subspace of $L(H)$ generated by $\omega_{\xi,\eta}$, $\xi, \eta \in H$, the pair $(L(H), L\mathscr{F}(H))$ forms a dual system, in such a way that one particularly obtains

$$(2.14)\qquad w = \sigma(L(H), L\mathscr{F}(H)).$$

On the other hand, if w_λ denotes the weak (operator) topology on $\mathscr{L}(H_\lambda)$, $\lambda \in \Lambda$, and $\mathscr{L}\mathscr{F}(H_\lambda)$ the respective to $L\mathscr{F}(H)$ linear subspace of $\mathscr{L}(H_\lambda)'$, $\lambda \in \Lambda$, one has (cf.

also [15; p. 68])

$$w_\lambda=\sigma(\mathscr{L}(H_\lambda), \mathscr{L}\mathscr{F}(H_\lambda)), \qquad \lambda\in\Lambda. \tag{2.15}$$

In particular, the connecting maps $f_{\lambda\mu}$, $\lambda\le\mu$, of the inverse system $(\mathscr{L}(H_\lambda))$, $\lambda\in\Lambda$, are weakly continuous, so that one finally gets

$$w=\varprojlim_\lambda w_\lambda=\varprojlim_\lambda \sigma(\mathscr{L}(H_\lambda), \mathscr{L}\mathscr{F}(H_\lambda))=\sigma(L(H), L\mathscr{F}(H)). \tag{2.16}$$

Moreover (cf. [15; p. 68] as well as (1.10)),

$$w_\lambda\le\sigma_\lambda\equiv\sigma(\mathscr{L}(H_\lambda), \mathscr{L}_*(H_\lambda)), \qquad \lambda\in\Lambda, \tag{2.17}$$

consequently,

$$w\le\sigma=\varprojlim_\lambda \sigma_\lambda, \tag{2.18}$$

with σ the σ-weak topology on $L(H)$. Thus, a consequence of (2.18) and Example 1.4.1, Proposition is that *every w-closed *-subalgebra of $L(H)$, H a locally Hilbert space, is a locally W^*-algebra.*

3. In Sections 3 and 4 we apply the theory developed above, in order to get some information about the inner derivations of a locally W^*-algebra. Some of our results extend in the more general case of locally W^*-algebras, previous results of L. Zsidó [16] concerning the norm of a derivation of an abstract W^*-algebra.

A *derivation* of an algebra E is a linear map $\delta: E\to E$ such that $\delta(xy)=\delta(x)y+x\delta(y)$ for any x, y in E. A derivation δ of E is called *inner*, if $\delta=\delta_x$ for some x in E, with $\delta_x(y)=xy-yx$, y in E.

Now, let $E=\varprojlim_\alpha F_\alpha$ be a locally W^*-algebra. Since each F_α, $\alpha\in A$, is a W^*-algebra, every derivation of F_α, $\alpha\in A$, is norm-continuous and inner [12]. Thus, if $x=(x_\alpha)\in E$ and δ_x, δ_{x_α}, $\alpha\in A$, are the inner derivations of E, F_α, $\alpha\in A$, respectively defined by x, x_α, $\alpha\in A$, one gets

$$f_{\alpha\beta}\circ\delta_{x_\beta}=\delta_{x_\alpha}\circ f_{\alpha\beta},$$

for any $\alpha\le\beta$ in A and $x_\beta\in F_\beta$ with $f_{\alpha\beta}(x_\beta)=x_\alpha$. Hence, there is a unique continuous linear map

$$\delta=\varprojlim_\alpha \delta_{x_\alpha}: E\longrightarrow E,$$

such that $f_\alpha\circ\delta=\delta_{x_\alpha}\circ f_\alpha$, $\alpha\in A$, and $\delta=\delta_x$, $x=(x_\alpha)\in E$.

In this respect, if $\delta_0(E)$ *is the set of all inner derivations of* E and $\delta(F_\alpha)$, $\alpha\in A$, *the Banach space of all* (inner) *derivations of* F_α, $\alpha\in A$, one has the following.

Theorem 3.1. *For every locally W^*-algebra E, $\delta_0(E)$ is a complete locally convex space, in such a way that*

$$\delta_0(E)=\varprojlim_{\alpha} \delta(\bar{E}_\alpha^{\sigma_\alpha}),$$

where (E_α), $\alpha\in A$, *corresponds to the Arens-Michael decomposition of* E.

Proof. Each $E_\alpha\equiv E/N_\alpha$, $\alpha\in A$, is complete [13; Folg. 5.4] (cf. also discussion before Definition 1.1) therefore the connecting maps $f_{\alpha\beta}: E_\beta\to E_\alpha$, $\alpha\leq\beta$ in A, are onto, which yields that

$$f_{\alpha\beta}(Z(E_\beta))\subset Z(E_\alpha), \qquad \alpha\leq\beta \quad \text{in} \quad A.$$

Hence, according to (2.12) one also gets (cf. besides 2.3 and comments after Proposition 1.3)

$$f_{\alpha\beta}(Z(\bar{E}_\beta^{\sigma_\beta}))\subset Z(\bar{E}_\alpha^{\sigma_\alpha}), \qquad \alpha\leq\beta \quad \text{in} \quad A.$$

Thus, for any $\alpha\leq\beta$ in A, we may define the continuous morphisms

$$g_{\alpha\beta}: \bar{E}_\beta^{\sigma_\beta}/Z(\bar{E}_\beta^{\sigma_\beta}) \longrightarrow \bar{E}_\alpha^{\sigma_\alpha}/Z(\bar{E}_\alpha^{\sigma_\alpha}): \dot{x}_\beta\equiv x_\beta+Z(\bar{E}_\beta^{\sigma_\beta}) \longmapsto \widehat{f_{\alpha\beta}(x_\beta)}^{\,\cdot}. \tag{3.1}$$

Now, every $\bar{E}_\alpha^{\sigma_\alpha}$, $\alpha\in A$, is a W^*-algebra (cf. Proposition 1.3), so that

$$\|\delta_{x_\alpha}\|=2\inf\{\|x_\alpha-z_\alpha\|_\alpha: z_\alpha\in Z(\bar{E}_\alpha^{\sigma_\alpha})\}, \qquad \alpha\in A, \tag{3.2}$$

for any $\delta_{x_\alpha}\in\delta(\bar{E}_\alpha^{\sigma_\alpha})$, $x_\alpha\in\bar{E}_\alpha^{\sigma_\alpha}$, $\alpha\in A$ (cf. [16; p. 148, Corol.]). Thus, since moreover $\delta_{x_\alpha}=0$, for all $x_\alpha\in Z(\bar{E}_\alpha^{\sigma_\alpha})$, $\alpha\in A$, it follows that the map

$$u_\alpha: \bar{E}_\alpha^{\sigma_\alpha}/Z(\bar{E}_\alpha^{\sigma_\alpha}) \longrightarrow \delta(\bar{E}_\alpha^{\sigma_\alpha}): \dot{x}_\alpha \longmapsto \delta_{x_\alpha}, \qquad \alpha\in A, \tag{3.3}$$

is a topological vector space isomorphism. A consequence of (3.1), (3.3) is now that the pair $(\delta(\bar{E}_\alpha^{\sigma_\alpha}), h_{\alpha\beta})$, $\alpha\leq\beta$ in A, is a projective system of Banach spaces, where

$$h_{\alpha\beta}: \delta(\bar{E}_\alpha^{\sigma_\beta}) \longrightarrow \delta(\bar{E}_\alpha^{\sigma_\alpha}): \delta_{x_\beta} \longmapsto \delta_{x_\alpha}, \qquad \text{with} \quad x_\alpha=f_{\alpha\beta}(x_\beta), \qquad \alpha\leq\beta.$$

Hence, taking also into account the discussion before Theorem 3.1 we obtain

$$\delta_0(E)=\varprojlim_{\alpha} \delta(\bar{E}_\alpha^{\sigma_\alpha}),$$

so that $\delta_0(E)$ becomes a complete locally convex space, for which a defining family of seminorms is given by

$$q_\alpha(\delta_x)=\|\delta_{x_\alpha}\|, \qquad \alpha\in A, \tag{3.4}$$

for any $\delta_x\in\delta_0(E)$, $x=(x_\alpha)\in E$. $\square$

An interesting result in this direction would be, of course, that every derivation of a locally W^*-algebra is inner.

The next theorem provides in our case an analogue of a known result concerning the norm of a derivation of a W^*-algebra, see [16; p. 148, Corol.] as well as [5; Theor. 1].

Theorem 3.2. *Let E be a locally W^*-algebra. Then,* (3.5) $$q_\alpha(\delta_x)=2\inf\{p_\alpha(x-z): z\in Z(E)\}\ , \qquad \alpha\in A\ ,$$

for any $\delta_x\in\delta_0(E)$, $x=(x_\alpha)\in E$.

Proof. By Proposition 1.3 $E\cong\varprojlim_\alpha \bar{E}_\alpha^{\sigma_\alpha}$, so that if $x=(x_\alpha)$ in E, $x_\alpha=f_\alpha(x)$ for every $\alpha\in A$ (cf. also 2.3). Thus, by Theorem 3.1, $\delta_x=(\delta_{f_\alpha(x)})$ with $\delta_{f_\alpha(x)}\in\delta(\bar{E}_\alpha^{\sigma_\alpha})$, $\alpha\in A$. Moreover, by (2.12), (2.11) $Z(\bar{E}_\alpha^{\sigma_\alpha})=Z(E_\alpha)\cong Z(E)_\alpha=Z(E)/\ker(p_\alpha|_{Z(E)})=f_\alpha(Z(E))$, $\alpha\in A$. Hence, according to (3.2), (3.4), (1.2) we conclude that

$$\begin{aligned} q_\alpha(\delta_x)=\|\delta_{f_\alpha(x)}\| &=2\inf\{\|f_\alpha(x)-z_\alpha\|_\alpha : z_\alpha\in Z(\bar{E}_\alpha^{\sigma_\alpha})\}\\ &=2\inf\{\|f_\alpha(x)-f_\alpha(z)\|_\alpha : z\in Z(E)\}\\ &=2\inf\{p_\alpha(x-z): z\in Z(E)\}\ , \qquad \alpha\in A \end{aligned}$$

for any $\delta_x\in\delta_0(E)$, $x\in E$. □

As a corollary to the preceding theorem we now get Stampfli's result for $L(H)$, H a locally Hilbert space, referred to the norm of a derivation acting on the C^*-algebra $\mathscr{L}(H)$, H a Hilbert space (cf. [14; Theor. 4]).

Corollary 3.3. *Let H be a locally Hilbert space and $L(H)$ the locally W^*-algebra of Example* 1.4.3. *Let also δ_T in $\delta_0(L(H))$, $T=(T_\lambda)$ in $L(H)$. Then,*

(3.6) $$q_\lambda(\delta_T)=2\inf\{p_\lambda(T-z\mathrm{id}_H): z\in \boldsymbol{C}\}\ ,$$

for all $\lambda\in\Lambda$.

Proof. Working out the Example 1.4.3, we saw that each factor $L(H)_\lambda$, $\lambda\in\Lambda$, of the Arens-Michael decomposition of $L(H)$ coincides (algebraically-topologically) with the W^*-algebra $\mathscr{L}(H_\lambda)$, H_λ, $\lambda\in\Lambda$, a Hilbert space. Moreover, by (2.11) $Z(L(H))\cong\varprojlim_\lambda Z(\mathscr{L}(H_\lambda))$, where $Z(\mathscr{L}(H_\lambda))=\{z_\lambda\mathrm{id}_{H_\lambda}: z_\lambda\in\boldsymbol{C}\}\equiv \boldsymbol{C}_\lambda$, $\lambda\in\Lambda$, with $\boldsymbol{C}_\lambda\cong\boldsymbol{C}$, for all $\lambda\in\Lambda$. Hence, (cf. [2; p. 77, Exemple 2)]), $Z(L(H))=\{z\mathrm{id}_H: z\in\boldsymbol{C}\}$, so that the assertion now follows by Theorem 3.2. □

Corollary 3.4. *Let E be a locally W^*-algebra. Then, $\delta_0(E)=E/Z(E)$ within an isomorphism of locally convex spaces or, equivalently, the sequence*

$$0 \longrightarrow Z(E)\longrightarrow E\xrightarrow{u}\delta_0(E)\longrightarrow 0\ ,$$

is topologically exact.

Proof. The map $u: E\to\delta_0(E): x\longmapsto u(x)=\delta_x$ is a linear surjection with $\ker(u)=Z(E)$. Moreover, $q_\alpha(u(x))\leq 2p_\alpha(x)$, for any $x\in E$, $\alpha\in A$ (cf. (3.5)), therefore u is also continuous. Thus, taking the induced from u linear bijection $\bar{u}: E/Z(E)\to\delta_0(E): \dot{x}\equiv x+Z(E)\longmapsto\bar{u}(\dot{x})=\delta_x$, this is, in fact, an isomorphism of locally convex spaces by (3.5). Hence, the continuous linear surjection u is a "topological homomorphism" [7; p. 106, Def. 2], which is equivalent to the fact that u is moreover open [7; p.

106, Theor. 1]. $\square$

L. Zsidó realized the norm of a derivation acting on an abstract W^*-algebra E by means of a certain particular map $\Phi: E \to Z(E)$ [16; Theorems 1, 2]. Using Zsidó's technique we shall also try to estimate the numbers $q_\alpha(\delta_x)$, $\alpha \in A$, (cf. (3.5)) via of a corresponding to our case map Φ, at the cost however of some extra restriction for the locally W^*-algebra involved (cf. Theorem 3.5). Thus, let E be a locally W^*-algebra. Then (Proposition 1.3),

$$E \cong \varprojlim_{\alpha} \bar{E}_\alpha^{\sigma_\alpha}.$$

Denote by

$$\mathfrak{M}_\alpha \equiv \mathfrak{M}(Z(\bar{E}_\alpha^{\sigma_\alpha})), \qquad \alpha \in A,$$

the *spectrum* (Gel'fand space) of $Z(\bar{E}_\alpha^{\sigma_\alpha})$, $\alpha \in A$, where

$$Z(\bar{E}_\alpha^{\sigma_\alpha}) \cong \mathscr{C}(\mathfrak{M}_\alpha), \qquad \alpha \in A,$$

by the Gel'fand-Naimark theorem. Thus, if n is a positive integer and $h_\alpha \in \mathfrak{M}_\alpha$ set

$$\text{(3.7)} \qquad M_\alpha = \left\{ \sum_{i=1}^{n} m_\alpha^i y_\alpha^i : m_\alpha^i \in \ker(h_\alpha),\ y_\alpha^i \in \bar{E}_\alpha^{\sigma_\alpha} \right\}^- \equiv [\ker(h_\alpha)], \qquad \alpha \in A,$$

where "—" means norm-closure in $\bar{E}_\alpha^{\sigma_\alpha}$, $\alpha \in A$. Then, M_α is the smallest closed 2-sided ideal of $\bar{E}_\alpha^{\sigma_\alpha}$ containing $\ker(h_\alpha)$, $\alpha \in A$, so that it is primitive by [6; Theor. 4.7]. Therefore, every $\bar{E}_\alpha^{\sigma_\alpha}/M_\alpha$, has a faithful irreducible representation ψ_α in some Hilbert space H_α, $\alpha \in A$. In this regard, L. Zsidó shows in [16; Theor. 1] the existence of a unique norm continuous map

$$\text{(3.8)} \qquad \Phi_\alpha: \bar{E}_\alpha^{\sigma_\alpha} \longrightarrow Z(\bar{E}_\alpha^{\sigma_\alpha}) \cong \mathscr{C}(\mathfrak{M}_\alpha), \qquad \alpha \in A,$$

defined by the relation

$$\text{(3.9)} \qquad \Phi_\alpha(x_\alpha)(h_\alpha) = \text{center of the operator } \psi_\alpha(x_\alpha + M_\alpha),$$

for any $x_\alpha \in \bar{E}_\alpha^{\sigma_\alpha}$, $h_\alpha \in \mathfrak{M}_\alpha$, $\alpha \in A$, where by the *center of an operator* T in $\mathscr{L}(H_\alpha)$ is meant the unique complex number z_T for which

$$\text{(3.10)} \qquad \|T - z_T\| = \inf\{\|T - z\mathrm{id}_H\| : z \in \boldsymbol{C}\}$$

(cf. [14]). By means of Φ_α, L. Zsidó estimates the norm of a derivation in $\bar{E}_\alpha^{\sigma_\alpha}$, $\alpha \in A$ (cf. [16; Theor. 2]). Namely, for any $x_\alpha \in \bar{E}_\alpha^{\sigma_\alpha}$ and $z_\alpha \in Z(\bar{E}_\alpha^{\sigma_\alpha})$, $\alpha \in A$, the map (3.8) has the following properties:

$$\text{(3.11)} \qquad \Phi_\alpha(x_\alpha + z_\alpha) = \Phi_\alpha(x_\alpha) + z_\alpha, \qquad \Phi_\alpha(x_\alpha z_\alpha) = \Phi_\alpha(x_\alpha) z_\alpha,$$

$$\text{(3.12)} \qquad \|x_\alpha - \Phi_\alpha(x_\alpha)\|_\alpha = \inf\{\|x_\alpha - z_\alpha\|_\alpha : z_\alpha \in Z(\bar{E}_\alpha^{\sigma_\alpha})\},$$

$$\text{(3.13)} \qquad \|\delta_{x_\alpha}\| = 2\|x_\alpha - \Phi_\alpha(x_\alpha)\|_\alpha, \qquad \delta_{x_\alpha} \in \delta(\bar{E}_\alpha^{\sigma_\alpha}).$$

Following the preceding notation, as well as the notation of 2.3, we consider the primitive ideal $M_\beta \equiv [\ker(h_\beta)]$ in $\bar{E}_\beta^{\sigma_\beta}$ (cf. (3.7)), with $h_\beta = h_\alpha \circ f_{\alpha\beta}$ in $\mathfrak{M}_\beta$, $\alpha \le \beta$, h_α in $\mathfrak{M}_\alpha$, $\alpha \in A$. Thus, for any $\alpha \le \beta$ in A we can define the map

$$(3.14)\quad \lambda_{\alpha\beta}\colon \bar{E}_\beta^{\sigma_\beta}/M_\beta \longrightarrow \bar{E}_\alpha^{\sigma_\alpha}/M_\alpha\colon x_\beta + M_\beta \longmapsto \lambda_{\alpha\beta}(x_\beta + M_\beta) = f_{\alpha\beta}(x_\beta) + M_\alpha\,.$$

In this respect, if π_α denotes the quotient map of $\bar{E}_\alpha^{\sigma_\alpha}$ onto $\bar{E}_\alpha^{\sigma_\alpha}/M_\alpha$, $\alpha \in A$, one has

$$\lambda_{\alpha\beta} \circ \pi_\beta = \pi_\alpha \circ f_{\alpha\beta}\,, \qquad \text{for any} \quad \alpha \le \beta \text{ in } A\,,$$

so that (3.14) is continuous. On the other hand, the faithful irreducible representation ψ_α of the primitive algebra $\bar{E}_\alpha^{\sigma_\alpha}/M_\alpha$, $\alpha \in A$, is in fact, an isometric representation, since $\bar{E}_\alpha^{\sigma_\alpha}/M_\alpha$, $\alpha \in A$, is a C^*-algebra. Besides, supposing that $\lambda_{\alpha\beta}$, $\alpha \le \beta$, is 1–1, this also becomes an isometry. Hence, considering the diagram

$$(3.15)\qquad \begin{array}{ccc} \bar{E}_\beta^{\sigma_\beta}/M_\beta & \xrightarrow{\lambda_{\alpha\beta}} & \bar{E}_\alpha^{\sigma_\alpha}/M_\alpha \\ \Big\uparrow{\scriptstyle \psi_\beta^{-1}} & & \Big\downarrow{\scriptstyle \psi_\alpha} \\ \operatorname{Im}(\psi_\beta) \subset \mathscr{L}(H_\beta) & \xrightarrow[\psi_\alpha \circ \lambda_{\alpha\beta} \circ \psi_\beta^{-1}]{} & \operatorname{Im}(\psi_\alpha) \subset \mathscr{L}(H_\alpha) \end{array} \qquad \alpha \le \beta \text{ in } A$$

the map $\psi_\alpha \circ \lambda_{\alpha\beta} \circ \psi_\beta^{-1}$ is an isometry too. Note that $\lambda_{\alpha\beta}$, $\alpha \le \beta$, becomes 1–1 for any locally W^*-algebra $E = \varprojlim_\alpha F_\alpha$, with connecting maps $f_{\alpha\beta}\colon F_\beta \to F_\alpha$, $\alpha \le \beta$, bijections, hence norm and weakly $*$-bicontinuous (for the latter see [12; Corol. 4.1.23]).

In this concern, one now gets the next theorem, which is an analogue in our case of [16; Theorems 1, 2].

Theorem 3.5. *Let E be a locally W^*-algebra, in such a way that the map* (3.14) *is* 1–1. *Then, there is a unique τ-continuous* (*cf. Scholium* 1.2) *map $\Phi\colon E \to Z(E)$, with the properties*:

i) $\Phi(x+z) = \Phi(x) + z$, $\Phi(xz) = \Phi(x)z$, *for any* $x \in E$, $z \in Z(E)$.

ii) $p_\alpha(x - \Phi(x)) = \inf\{p_\alpha(x - z)\colon z \in Z(E)\}$, *for any* $x \in E$, $\alpha \in A$.

iii) $q_\alpha(\delta_x) = 2p_\alpha(x - \Phi(x))$, *for any* $\delta_x \in \delta_0(E)$, $(x \in E)$, $\alpha \in A$.

Proof. We shall first show that the maps (Φ_α), $\alpha \in A$, (cf. (3.8)) form an inverse system. Thus, we consider the diagram

$$(3.16)\qquad \begin{array}{ccc} \bar{E}_\beta^{\sigma_\beta} & \xrightarrow{\Phi_\beta} & Z(\bar{E}_\beta^{\sigma_\beta}) \cong \mathscr{C}(\mathfrak{M}_\beta) \\ \Big\downarrow{\scriptstyle f_{\alpha\beta}} & & \Big\downarrow{\scriptstyle {}^t({}^tf_{\alpha\beta})} \\ \bar{E}_\alpha^{\sigma_\alpha} & \xrightarrow[\Phi_\alpha]{} & Z(\bar{E}_\alpha^{\sigma_\alpha}) \cong \mathscr{C}(\mathfrak{M}_\alpha) \end{array} \qquad \alpha \le \beta \text{ in } A$$

where ${}^tf_{\alpha\beta}$ denotes the transpose of $f_{\alpha\beta}$, $\alpha \le \beta$. Then, for any $x_\beta \in \bar{E}_\beta^{\sigma_\beta}$, $h_\alpha \in \mathfrak{M}_\alpha$, $\alpha \le \beta$ in A, one has (cf. also (3.9), (3.10))

$$(({}^t({}^tf_{\alpha\beta})\circ\Phi_\beta)(x_\beta))(h_\alpha)=(\phi_\beta(x_\beta)\circ{}^tf_{\alpha\beta})(h_\alpha)=\phi_\beta(x_\beta)(h_\alpha\circ f_{\alpha\beta})$$
$$=\phi_\beta(x_\beta)(h_\beta)=z_{\psi_\beta(x_\beta+M_\beta)}\,.$$

But, since $\lambda_{\alpha\beta}$, $\alpha\leq\beta$, is 1–1, the map $\psi_\alpha\circ\lambda_{\alpha\beta}\circ\psi_\beta^{-1}$ of diagram (3.15) is an isometry, so that

$$z_{\psi_\beta(x_\beta+M_\beta)}=z_{(\psi_\alpha\circ\lambda_{\alpha\beta}\circ\psi_\beta^{-1})(\psi_\beta(x_\beta+M_\beta))}$$
$$=z_{\psi_\alpha(\lambda_{\alpha\beta}(x_\beta+M_\beta))}=z_{\psi_\alpha(f_{\alpha\beta}(x_\beta)+M_\alpha)}$$
$$=(\Phi_\alpha\circ f_{\alpha\beta})(x_\beta)(h_\alpha)\,,\qquad \alpha\leq\beta\,.$$

Thus, the diagram (3.16) is commutative, which yields the existence of a unique τ-continuous map (cf. also Proposition 1.3 and Corollary 2.2)

$$\Phi=\varprojlim_\alpha \Phi_\alpha\colon E\cong\varprojlim_\alpha \bar{E}_\alpha^{\sigma_\alpha}\longrightarrow Z(E)\cong\varprojlim_\alpha Z(\bar{E}_\alpha^{\sigma_\alpha})\,, \tag{3.17}$$

such that

$$f_\alpha\circ\Phi=\Phi_\alpha\circ f_\alpha\,,\qquad\text{for all}\quad \alpha\in A\,.$$

Now, the definition of Φ and (3.11) imply i). On the other hand, (cf. also (1.2) and (3.12)), for every $x=(x_\alpha)\in E$

$$p_\alpha(x-\Phi(x))=\|x_\alpha-\Phi_\alpha(x_\alpha)\|_\alpha\leq\|x_\alpha-z_\alpha\|_\alpha\,,\qquad \alpha\in A\,,$$

for all $z_\alpha=f_\alpha(z)\in Z(\bar{E}_\alpha^{\sigma_\alpha})\cong f_\alpha(Z(E))$, $\alpha\in A$ (cf. proof of Theorem 3.2). Thus,

$$p_\alpha(x-\Phi(x))\leq\inf\{p_\alpha(x-z):z\in Z(E)\}\leq p_\alpha(x-\Phi(x))\,,$$

for any $x\in E$, $\alpha\in A$.

Concerning iii), this is a consequence of (3.4), (3.13) and (1.2). □

4. *If* $(E,(p))$ is a locally convex space and Z a closed linear subsapce of E, denote by π the quotient map of E onto E/Z and by r the seminorm of E/Z derived by p. In this respect, using the properties of the map (3.17) of Theorem 3.5, we are led to the following general statement.

Theorem 4.1. *Let $(E,(p))$ be a locally convex space and Z a closed linear subspace of E. Then, $\psi\colon E\to Z$ is a continuous map with the properties $\Psi(x+z)=\Psi(x)+z$, $x\in E$, $z\in Z$, and $p(x-\Psi(x))=\inf\{p(x-z):z\in Z\}$, $x\in E$, for every p if, and only if, π admits a continuous relatively open section s, in the sense that $\pi\circ s=\mathrm{id}_{E/Z}$ and $p(s(\dot{x}))=r(\dot{x})$, $\dot{x}\equiv x+Z$, $x\in E$, for any p, r.*

Proof. For any $x\in E$ and $y\in\dot{x}$, $\Psi(x)=x-y+\Psi(y)$, so that one defines $s\colon E/Z\to E\colon \dot{x}\longmapsto s(\dot{x})=x-\Psi(x)$, where $\pi\circ s=\mathrm{id}_{E/Z}$ and $p(x-\Psi(x))=r(\dot{x})$, $x\in E$, for all p, r. Conversely, if s is a section of π with $p(s(\dot{x}))=r(\dot{x})$, $x\in E$, for any p, r, we first have that $\pi(x-s(\dot{x}))=0$, hence, one defines $\Psi\colon E\to Z\colon x\longmapsto\Psi(x)=x-s(\dot{x})$. Then, $p(\Psi(x))\leq 2p(x)$, $x\in E$, for every p, so that Ψ is continuous, having besides the required by Theorem 4.1 properties. □

A direct consequence of Theorems 4.1, 3.5 is now the next.

Corollary 4.2. *Let E be a locally W^*-algebra for which the map* (3.14) *is* 1–1. *Then, $\pi: E \to E/Z(E)$ has a continuous relatively open section s, with $s(\dot{x}) = x - \Phi(x)$, $x \in E$, where Φ is the map* (3.17) □

Now, according to Corollary 3.4, for every locally W^*-algebra E, one defines the "topological homomorphism"

$$u: E \longrightarrow \delta_0(E): x \longmapsto u(x) = \delta_x , \tag{4.1}$$

in such a way that the induced linear bijection

$$\bar{u}: E/Z(E) \longrightarrow \delta_0(E): \dot{x} \longmapsto \bar{u}(\dot{x}) = u(x) , \tag{4.2}$$

becomes an isomorphism of locally convex spaces.

The next theorem gives an extension and strengthening as well of a previous result of L. Zsidó [16; Theor. 3]. More precisely, Theorem 4.3 below, gives a new information, even in the norm case, ensuring that Zsidó's continuous map of Theorem 3 in [16] (cf., for instance, the map v of the next theorem) is, in addition, a relatively open section of the topological homomorphism (4.1). That is, one has.

Theorem 4.3. *Let E be a locally W^*-algebra for which the map* (3.14) *is* 1–1. *Let also Φ be the map* (3.17) *and $v: \delta_0(E) \to E: \delta_x \longmapsto v(\delta_x) = x - \Phi(x)$. Then, the continuous open linear surjection u* (cf. (4.1)) *has v as a continuous relatively open section. In particular,*

$$2p_\alpha(v(\delta_x)) = q_\alpha(\delta_x) ,$$

for any $\delta_x \in \delta_0(E)$, $(x \in E)$, $\alpha \in A$.

Proof. If δ_x, $\delta_y \in \delta_0(E)$, $(x, y \in E)$, with $\delta_x = \delta_y$, one gets that $x - y \in Z(E)$, consequently (cf. Theorem 3.5, i)) $\Phi(x) = \Phi(x - y + y) = x - y + \Phi(y)$, which yields that v is well defined. On the other hand, $(u \circ v)(\delta_x) = \delta_x - \delta_{\Phi(x)} = \delta_x$, for every $\delta_x \in \delta_0(E)$, $x \in E$; moreover, $v = s \circ \bar{u}^{-1}$, with s, $\bar{u}$ as in Corollary 4.2, respectively (4.2). Hence, v is, in effect, a continuous relatively open section of u, having besides the property (cf. Theorem 3.5) $q_\alpha(\delta_x) = 2p_\alpha(v(\delta_x))$, for any $\alpha \in A$, $\delta_x \in \delta_0(E)$, $(x \in E)$. □

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