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ON GENERALIZED ITÔ TYPE STOCHASTIC INTEGRAL EQUATION

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ABSTRACT. The object of the present paper is to study a more general class of Itô type stochastic Volterra integral equations with continuous sample paths with probability one. Our main results deal with the problems of existence and uniqueness of the solutions which generalize similar results on Itô type stochastic differential and integral equations in the literature.

1. Introduction

The study of the influence of random forces on nonlinear systems is a very practical problem. Many problems in almost all phases of physics, control theory and other areas of applied mathematics lead to the study of stochastic dynamical systems which are described by mathematical models involving time-varying parameters. Generally one may assume that these time-varying parameters are stochastic processes with known stastistics. A typical example of interesting applications which lead to the study of such stochastic systems is governed by the system

$$\ddot{x}+f(x, \dot{x})\dot{x}+g(x)=\sigma(x, \dot{x})\dot{\xi}(t)\,, \tag{1.1}$$

where $f(0, z)\equiv g(0)\equiv\sigma(0, z)\equiv 0$ and $\xi(t)$ is a Wiener process, which is obtained from the action of "white noise" random forces on the system described by Liénard's equation

$$\ddot{x}+f(x, \dot{x})\dot{x}+g(x)=0$$

which is characteristic of many automatic control processes. The question of stability in the large of trajectories of system (1.1) is examined by Nevel'son [12] by writing the system (1.1) in the form of two stochastic equations such that

$$\begin{aligned} &dx(t)=z(t)dt \\ &dz(t)=[-g(x(t))-z(t)f(x(t), z(t))]dt+\sigma(x(t), z(t))d\xi(t)\,. \end{aligned} \tag{1.2}$$

The system (1.2) with $x(0)=z(0)=c$ is more conveniently written in the form of the following stochastic integral equation

$$z(t)=c+\int_0^t\left[-g\left(c+\int_0^s z(\tau)d\tau\right)-z(s)f\left(c+\int_0^s z(\tau)d\tau,\, z(s)\right)\right]ds + \int_0^t \sigma\left(c+\int_0^s z(\tau)d\tau,\, z(s)\right)d\xi(s). \tag{1.3}$$

Systems of this type frequently arise in control engineering and in many other fields and the study of the questions of existence and uniqueness of the solutions of more general form of system (1.3) becomes interesting and challenging which provides one of the motivation for our present work. We are also influenced by the recent work of Berger and Mizel [2] on general forms of Itô type stochastic equations.

In this paper, we wish to study the generalized version of equation (1.3) in the form

$$x(t)=\phi(t)+\int_0^t F(t, s, x(s), A_1x(s), A_2x(s))ds + \int_0^t H(t, s, x(s), B_1x(s), B_2x(s))d\beta(s) \qquad 0\leq t\leq T, \tag{1.4}$$

where

$$A_1x(t)=\int_0^t f_1(t, s, x(s))ds,\; A_2x(t)=\int_0^t f_2(t, s, x(s))d\beta(s),$$

$$B_1x(t)=\int_0^t h_1(t, s, x(s))ds,\; B_2x(t)=\int_0^t h_2(t, s, x(s))d\beta(s),$$

which in turn contains as special cases the more general stochastic differential and integral equations recently studied by many authors in the literature (see [1–3, 5–10, 13–15]). Here $\beta(t)$ $(t\geq 0)$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathscr{F}, P)$. Let $\mathscr{F}_t$ be an increasing family of σ-fields such that $\mathscr{F}_{t_1}\subset\mathscr{F}_{t_2}$ if $t_1\leq t_2$, $\sigma(\beta(s), 0\leq s\leq t)$ be in $\mathscr{F}_t$ and $\sigma(\beta(t+s)-\beta(t), s\geq 0)$ be independent of $\mathscr{F}_t$ for all $t\geq 0$. We will write the stochastic process $\{x(t, \omega), t\in[0, T]\}$ as $\{x(t), t\in[0, T]\}$ by suppressing the argument ω. In our discussion, continuous process means that the sample paths are continuous functions with probability one. In equation (1.4) $\phi(t)$ is a given $\mathscr{F}_t$-adapted continuous process, $f_1(t, s, x)$, $h_1(t, s, x)$ and $f_2(t, s, x)$, $h_2(t, s, x)$ are $\mathscr{F}_t$-measurable and $\mathscr{F}_s$-measurable respectively and continuous random functions defined for $0\leq s\leq t\leq T$ and $-\infty<x<\infty$ and F and H are $\mathscr{F}_t$-adapted and $\mathscr{F}_s$-adapted continuous processes respectively and defined on $\{0\leq s\leq t\leq T\}^2\times R^3$.

The paper is organised as follows. In Section 2 we present the basic lemmas needed in our subsequent discussion. The main theorem concerning the existence and uniqueness of the solutions of the equation (1.4) is dealt in Section 3. Finally, in Section 4 we establish the stronger version of our theorem on existence and uniqueness of the solution of the equation (1.4) given in Section 3.

2. Basic lemmas

In this section, we establish the basic lemmas needed in our subsequent discussion. For convenience we use the following notations.

$$I(t)=\int_0^t [F(t, s, x_1(s), A_1x_1(s), A_2x_1(s))-F(t, s, x_2(s), A_1x_2(s), A_2x_2(s))]ds\,,$$

$$I_1(t)=\int_0^t F(t, s, x(s), A_1x(s), A_2x(s))ds\,,$$

$$J(t)=\int_0^t [H(t, s, x_1(s), B_1x_1(s), B_2x_1(s))-H(t, s, x_2(s), B_1x_2(s), B_2x_2(s))]d\beta(s)\,,$$

$$J_1(t)=\int_0^t H(t, s, x(s), B_1x(s), B_2x(s))d\beta(s)\,.$$

Throughout this paper, we make use of the following assumptions. For $x, y, z\in R$, $x_i, y_i, z_i\in R$, $i=1, 2$, $t_1, t_2\in[0, T]$ and $0\le s\le t\le T$ there exists a constant K such that

(A1) $$|f_i(t, s, x)|^2+|h_i(t, s, x)|^2\le K^2(1+|x|^2)\,,\quad \text{a.s.}\,,\qquad i=1, 2\,,$$

(A2) $$|f_i(t, s, x_1)-f_i(t, s, x_2)|^2+|h_i(t, s, x_1)-h_i(t, s, x_2)|^2 \le K^2|x_1-x_2|^2\,,\quad \text{a.s.}\,,\qquad i=1, 2\,,$$

(A3) $$|f_2(t_1, s, x)-f_2(t_2, s, x)|^2+|h_2(t_1, s, x)-h_2(t_2, s, x)|^2\le K^2|t_1-t_2|^2\,,\quad \text{a.s.}\,,$$

(A4) $$|F(t, s, x, y, z)|^2+|H(t, s, x, y, z)|^2\le K^2(1+|x|^2+|y|^2+|z|^2)\,,\quad \text{a.s.}\,,$$

(A5) $$|F(t, s, x_1, y_1, z_1)-F(t, s, x_2, y_2, z_2)|^2+|H(t, s, x_1, y_1, z_1)-H(t, s, x_2, y_2, z_2)|^2 \le K^2(|x_1-x_2|^2+|y_1-y_2|^2+|z_1-z_2|^2)\,,\quad \text{a.s.}\,,$$

and

(A6) $$|F(t_1, s, x, y, z)-F(t_2, s, x, y, z)|^2+|H(t_1, s, x, y, z)-H(t_2, s, x, y, z)|^2 \le K^2|t_1-t_2|^2\,,\qquad \text{a.s.}\,.$$

Let $x(t)$ be a $\mathscr{F}_t$-adapted continuous process with $\sup_{0\le t\le T} E[x^2(t)]<\infty$, then from the assumption (A1) it follows that $E[\int_0^t|f_2(t, s, x(s))|^2ds]<\infty$, and $E[\int_0^t|h_2(t, s, x(s))|^2ds]<\infty$. Thus the integrals $A_2x(t)$ and $B_2x(t)$ are well defined and they are $\mathscr{F}_t$-measurable. By using the assumptions (A1) and (A4), stochastic integral isometry and Schwarz inequality it follows that

$$E\left[\int_0^t |F(t, s, x(s), A_1x(s), A_2x(s))|^2ds\right]<\infty$$

and

$$E\left[\int_0^t |H(t, s, x(s), B_1x(s), B_2x(s))|^2 ds\right] < \infty .$$

Therefore both the integrals on right side of (1.4) exist and the symbols $I_1(t)$ and $J_1(t)$ are well defined and they are $\mathscr{F}_t$-measurable. Further, by using assumptions (A1) and (A4) and assuming $\sup_{0 \le t \le T} E[|x(t)|^4] < \infty$ we observe that

$$E\left[\int_0^t |f_2(t, s, x(s))|^4 ds\right] < \infty , \qquad E\left[\int_0^t |h_2(t, s, x(s))|^4 ds\right] < \infty ,$$

and

$$E\left[\int_0^t |H(t, s, x(s), B_1x(s), B_2x(s))|^4 ds\right] < \infty ,$$

for $0 \le s \le t \le T$.

We shall now establish the following lemmas in which we assume, without further mention, that $\sup_{0 \le t \le T} E[|x(t)|^4] < \infty$.

Lemma 1. *Suppose that the assumptions* (A1), (A3), (A4) *and* (A6) *hold, and let* $0 \leqq u \leqq t \leqq T$. Then

$$\left.\begin{array}{l} E|I_1(t) - I_1(u)|^4 \\ E|J_1(t) - J_1(u)|^4 \end{array}\right\} \le C_1(K, T)\left[1 + \sup_{0 \le s \le T} E|x(s)|^4\right](t-u)^2$$

with a constant $C_1(K, T) > 0$ *depending on* K *and* T.

Using assumptions (A1), (A3), (A4) and (A6), Kolmogorov's lemma and standard results on stochastic integrals, along the lines of Arnold [1], Berger and Mizel [2], Friedman [5], Gikhman and Skorokhod [6], Itô [7, 8], McKean [11] the lemma can be easily proved. We omit the details.

Lemma 2. *Suppose that the assumptions* (A2), (A5) *and* (A6) *hold. Then*

$$\left.\begin{array}{l} P\left(\sup_{0 \le t \le T} |J(t)| \ge \lambda\right) \\ P\left(\sup_{0 \le t \le T} |I(t)| \ge \lambda\right) \end{array}\right\} \le \lambda^{-4} D_1(K, T)\left[\sup_{0 \le s \le T} E|x_1(s) - x_2(s)|^4 + \sup_{0 \le s \le T} E|x_1(s) - x_2(s)|^2\right]$$

with a constant $D_1(K, T) > 0$ *depending on* K *and* T.

The following lemma from Billingsley [4, Theorem 12.2] (see also Itô [7, p. 14]) is needed in the proof of Lemma 2.

Lemma. *Let* $S_0 = 0$ *and let* $S_k = \xi_1 + \xi_2 + \cdots + \xi_k$ $(k = 1, 2, \cdots, m)$ *be a sequence of partial sums of random variables* $\xi_1, \xi_2, \cdots, \xi_m$. *If* $\gamma \ge 0$ *and* $\alpha > 1$, *and if there exist* *nonnegative numbers* $u_1, u_2, \cdots, u_m$ *such that*

$$P(|S_j - S_i| \geq \lambda) \leq \frac{1}{\lambda^\gamma}\left(\sum_{l=i+1}^{j} u_l\right)^\alpha, \qquad 0 \leq i \leq j \leq m, \tag{2.1}$$

holds for all positive λ, *then, for all positive* λ,

$$P\left(\sup_{0 \leq k \leq m} |S_k| \geq \lambda\right) \leq \frac{K_0}{\lambda^\gamma}(u_1 + \cdots + u_m)^\alpha,$$

where K_0 *is an absolute constant depending only on* γ *and* α.

Remark (See Itô [7, p. 14]). We note that the condition

$$E[|S_j - S_i|^\gamma] \leq \left(\sum_{l=i+1}^{j} u_l\right)^\alpha \tag{2.2}$$

implies the condition (2.1).

The details of the proof of Lemma 2 can be easily obtained by using Arnold [1, Theorem (5.1.1) (d), p. 81, replacing the first exponent $k-1$ by k], the known lemma from Billingsley [4, Theorem 12.2] stated above, assumptions (A2), (A5)–(A6), Schwarz inequality and stochastic integral isometry and following the same method as in the proof of Itô [8, Lemma 2.2] or Itô [7, Lemma 1]. We do not discuss it here.

Lemma 3. *Suppose that the assumptions* (A1), (A3), (A4) *and* (A6) *hold. Then*

$$\left.\begin{array}{l} P\left(\sup_{0 \leq t \leq T} |I_1(t)| \geq \lambda\right) \\ P\left(\sup_{0 \leq t \leq T} |J_1(t)| \geq \lambda\right) \end{array}\right\} \leq \lambda^{-4} C_2(K, T)\left[1 + \sup_{0 \leq s \leq T} E|x(s)|^4\right].$$

with a constant $C_2(K, T) > 0$ *depending on* K *and* T.

The proof follows from Lemma 1 and argument similar to that given in Itô [7, Lemma 1] or Itô [8, Lemma 2.2] with suitable modifications. We omit the details.

3. Existence and uniqueness

In this section we establish our main result on the existence and uniqueness of the solution of the equation (1.4).

Theorem 1. *Let the assumptions* (A1), (A2) *and* (A4)–(A6) *hold. If* $\phi(t)$ *is a* $\mathscr{F}_t$-*adapted continuous process with* $\sup_{0 \leq t \leq T} E[\phi^4(t)] < \infty$. *Then there exits a unique continuous solution* $x(t)$ *of* (1.4) *satisfying* $\sup_{0 \leq t \leq T} E[x^2(t)] < \infty$.

Proof. Define

$$x_0(t) = \phi(t),$$

$$x_n(t)=\phi(t)+\int_0^t F(t, s, x_{n-1}(s), A_1 x_{n-1}(s), A_2 x_{n-1}(s))ds$$
$$+\int_0^t H(t, s, x_{n-1}(s), B_1 x_{n-1}(s), B_2 x_{n-1}(s))d\beta(s).$$

Further details follow by using the assumptions (A1)–(A2), (A4)–(A5), Schwarz inequality, stochastic integral isometry, the iterated procedure of Arnold [1, p. 109], Arnold [1, Theorem (5.1.1) (d), replacing the first exponent $k-1$ by k] and Lemma 2 and following an argument similar to that given in Itô [8, Theorem 3.1] with suitable modifications. The details are omitted.

Corollary. *Under the assumptions of Theorem* 1,

$$E[x^2(t)]\le\left(3\sup_{0\le s\le T} E[\phi^2(s)]+1\right)\exp[C_3(K, T)t]-1,$$

$$E[x^4(t)]\le\left(27\sup_{0\le s\le T} E[\phi^4(s)]+1\right)\exp[C_4(K, T)t]-1$$

with constants $C_3(K, T)>0$ and $C_4(K, T)>0$ depending on K and T.

Proof. Using standard results on stochastic integrals along the lines of Arnold [1], Berger and Mizel [2], Friedman [5], McKean [11] we get

$$E[1+x^2(t)]\le\left(3\sup_{0\le s\le T} E[\phi^2(s)]+1\right)+C_3(K, T)\int_0^t E[1+x^2(s)]ds.$$

Now an application of Gronwall's inequality yields

$$E[1+x^2(t)]\le\left(3\sup_{0\le s\le T} E[\phi^2(s)]+1\right)\exp[C_3(K, T)t]$$

with gives the first result.

Similarly, using $(a+b+c)^4\le 3^3(a^4+b^4+c^4)$, Theorem (5.1.1) (d) in [1] (see Arnold [1, p. 81, replacing the first exponent $k-1$ by k]), assumptions (A1) and (A4) and taking fourth powers of (1.4) and expectation, we get,

$$E[1+x^4(t)]\le\left(1+27\sup_{0\le s\le T} E[\phi^4(s)]\right)+C_4(K, T)\int_0^t E[1+x^4(s)]ds.$$

Applying Gronwall's inequality, we obtain the second estimate. This proves the corollary.

4. Stronger uniqueness and existence

In this section, first we shall establish a local uniqueness theorem which is useful in proving stronger version of the existence and uniqueness of Theorem 1. Consider the equation (1.4) along with the following equation

$$y(t)=\phi(t)+\int_0^t F^*(t, s, y(s), A_1^* y(s), A_2^* y(s))ds + \int_0^t H^*(t, s, y(s), B_1^* y(s), B_2^* y(s))d\beta(s), \qquad 0\le t\le T \tag{4.1}$$

where

$$A_1^* y(s)=\int_0^s f_1^*(s, \tau, y(\tau))d\tau, \qquad A_2^* y(s)=\int_0^s f_2^*(s, \tau, y(\tau))d\beta(\tau),$$

$$B_1^* y(s)=\int_0^s h_1^*(s, \tau, y(\tau))d\tau, \qquad B_2^* y(s)=\int_0^s h_2^*(s, \tau, y(\tau))d\beta(\tau),$$

$\phi(t)$ is a given $\mathscr{F}_t$-adapted continuous process, $f_1^*(t, s, y)$, $h_1^*(t, s, y)$ and $f_2^*(t, s, y)$, $h_2^*(t, s, y)$ are $\mathscr{F}_t$-measurable and $\mathscr{F}_s$-measurable respectively and continuous random functions defined for $0\le s\le t\le T$ and $-\infty<y<\infty$ and F^* and H^* are $\mathscr{F}_t$-adapted and $\mathscr{F}_s$-adapted respectively and continuous processes defined on $\{0\le s\le t\le T\}^2\times R^3$.

Theorem 2. *Suppose that the assumptions* (A2) *and* (A5) *hold. Let the assumptions* (A2) *and* (A5) *hold when* f_i, h_i, F *and* H *are replaced by* f_i^*, h_i^*, F^* *and* H^*, $i=1, 2$ *respectively and for some* $N>0$,

$$f_i(t, s, x)=f_i^*(t, s, x),\ h_i(t, s, x)=h_i^*(t, s, x), \qquad i=1, 2,$$
$$F(t, s, x, A_1x, A_2x)=F^*(t, s, x, A_1^*x, A_2^*x),$$
$$H(t, s, x, B_1x, B_2x)=H^*(t, s, x, B_1^*x, B_2^*x)$$

if $|x|\le N$, $0\le s\le t\le T$. *Assume that* $\phi(t)$ *is a* $\mathscr{F}_t$*-adapted continuous process with* $\sup_{0\le t\le T} E|\phi^4(t)|<\infty$. *Let* $x(t)$ *and* $y(t)$ *be the continuous solutions of the equations* (1.4) *and* (4.1) *with* $\sup_{0\le t\le T} E[x^2(t)]<\infty$ *and* $\sup_{0\le t\le T} E[y^2(t)]<\infty$ *respectively. Denote by* θ *the largest value of* $t\le T$ *for which* $\sup_{0\le s\le t}|x(s)|<N$ *and* θ^* *the largest value of* $t\le T$ *for which* $\sup_{0\le s\le t}|y(s)|<N$. *Then*

$$P(\theta=\theta^*)=1,$$

and

$$P\Big(\sup_{0\le s\le\theta}|x(s)-y(s)|=0\Big)=1.$$

Proof. Let

$$\Psi(t)=\begin{cases}1 & \text{if } \sup_{0\le s\le t}|x(s)|\le N,\\ 0 & \text{otherwise}\end{cases}$$

and

$$\Psi^*(t)=\begin{cases}1 & \text{if } \sup_{0\le s\le t}|y(s)|\le N\,,\\ 0 & \text{otherwise}.\end{cases}$$

Then both $\Psi(t)$ and $\Psi^*(t)$ are nonanticipating. We note that $\Psi(t)=1$ implies

$$f_i(t, s, x(s))=f_i^*(t, s, x(s))\,, \qquad h_i(t, s, x(s))=h_i^*(t, s, x(s))\,, \qquad i=1, 2\,,$$

$$F(t, s, x(s), A_1x(s), A_2x(s))=F^*(t, s, x(s), A_1^*x(s), A_2^*x(s))\,,$$

$$H(t, s, x(s)\,, B_1x(s), B_2x(s))=H^*(t, s, x(s), B_1^*x(s), B_2^*x(s)) \qquad \text{for} \quad s\le t\,.$$

By using the facts $a-c=(a-b)+(b-c)$, $|a+b|^2\le 2|a|^2+2|b|^2$ and Schwarz inequality, stochastic integral isometry, assumptions (A2) and (A5), equations (1.4) and (4.1) and $\Psi(t)=\Psi(t)\Psi(s)$, for $s\le t$, we have

$$E[\Psi(t)\,|\,x(t)-y(t)\,|^2]\le M(K, T)\int_0^t E[\Psi(s)\,|\,x(s)-y(s)\,|^2]\,ds$$

with a constant $M(K, T)>0$ depending on K and T.

By using Gronwall's inequality it follows that

$$E[\Psi(t)|\,x(t)-y(t)\,|^2]=0\,, \qquad t\in[0, T]\,.$$

Therefore by the continuity of $x(t)$ and $y(t)$, we get,

$$P\left(\sup_{0\le t\le T}[\Psi(t)\,|\,x(t)-y(t)\,|]=0\right)=1\,.$$

This implies that $x(t)$ and $y(t)$ coincide with probability one on the interval $[0, \theta]$. Thus it follows that

$$P(\theta\le\theta^*)=1\,.$$

By similar argument we find that $P(\theta\ge\theta^*)=1$, and the proof of the theorem is complete.

We now state and prove the following theorem which establishes the stronger version of our Theorem 1.

Theorem 3. *Let assumptions* (A1), (A4) *and* (A6) *hold. Suppose that for any* $N>0$, *there exists a constant* $K_N>0$ *such that for* $|x|\le N$, $|y|\le N$, $|x_i|\le N$, y_i, $z_i\in R$, $i=1, 2$, $0\le s\le t\le T$,

(A2′) $$|f_i(t, s, x)-f_i(t, s, y)|^2+|h_i(t, s, x)-h_i(t, s, y)|^2\le K_N^2|x-y|^2, \qquad i=1, 2, \quad \text{a.s.}\,,$$

(A5′) $$|F(t, s, x_1, y_1, z_1)-F(t, s, x_2, y_2, z_2)|^2+|H(t, s, x_1, y_1, z_1)-H(t, s, x_2, y_2, z_2)|^2\le K_N^2(|x_1-x_2|^2+|y_1-y_2|^2+|z_1-z_2|^2)\,, \qquad \text{a.s.}\,.$$

If $\phi(t)$ *is a* $\mathscr{F}_t$*-adapted continuous process with* $\sup_{0\le t\le T} E[\phi^4(t)]<\infty$, *then there exists a unique solution* $x(t)$ *of* (1.4) *satisfying* $\sup_{0\le t\le T} E[x^2(t)]<\infty$.

Remark. This theorem guarantees the existence of a unique solution of (1.4) even when the assumptions (A2) and (A5) are replaced by the weaker assumptions (A2′) and (A5′) respectively.

Proof. Let, for $y, z\in R$,

$$F_N(t, s, x, y, z)=\begin{cases} F(t, s, x, y, z) & \text{if } |x|\le N\,, \\ F(t, s, x, y, z)\left(2-\dfrac{|x|}{N}\right) & \text{if } N<|x|\le 2N\,, \\ 0 & \text{if } |x|>2N\,, \end{cases}$$

$$H_N(t, s, x, y, z)=\begin{cases} H(t, s, x, y, z) & \text{if } |x|\le N\,, \\ H(t, s, x, y, z)\left(2-\dfrac{|x|}{N}\right) & \text{if } N<|x|\le 2N\,, \\ 0 & \text{if } |x|>2N\,. \end{cases}$$

The functions F_N and H_N satisfy all the assumptions of Theorem 1 and hence Theorem 1 asserts the existence of a unique solution $x_N(t)$ satisfying $\sup_{0\le t\le T} E[x_N^2(t)]<\infty$ uniformly on N of the equation

$$x_N(t)=\phi(t)+\int_0^t F_N(t, s, x_N(s), A_1x_N(s), A_2x_N(s))ds + \int_0^t H_N(t, s, x_N(s), B_1x_N(s), B_2x_N(s))d\beta(s)\,. \tag{4.2}$$

From (4.2) and Lemma 3, we have,

$$P\left(\sup_{0\le t\le T}|x_N(t)|>N\right)\le P\left(\sup_{0\le t\le T}|\phi(t)|>\frac{N}{3}\right) +2\left(\frac{N^{-4}}{3}\right)C_2(K, T)\left[1+\sup_{0\le s\le T}E|x_N(s)|^4\right].$$

Since $\phi(t)$ is continuous with probability one and using corollary of Theorem 1, it follows that

$$\lim_{N\to\infty} P\left(\sup_{0\le t\le T}|x_N(t)|>N\right)=0\,. \tag{4.3}$$

Suppose θ_N be the largest value of t for which $\sup_{0\le s\le t}|x_N(s)|\le N$. For $N'>N$, Theorem 2 implies that $x_N(t)=x_{N'}(t)$ with probability one if $0\le t\le \theta_N$. Thus

$$P\left(\sup_{0\le s\le T}|x_N(t)-x_{N'}(t)|>0\right)\le P(\theta_N<T)=P\left(\sup_{0\le t\le T}|x_N(t)|>N\right).$$

From this and (4.3) it follows that the sequence $\{x_N(t)\}$ converges uniformly on $[0, T]$, with probability one to some limit $x(t)$ as $N\to\infty$. Furthermore, taking limits in (4.2), it follows that $x(t)$ is a solution of (1.4).

Let $x(t)$ and $y(t)$ denote two continuous solutions of (1.4) with $\sup_{0\le t\le T} E[x^2(t)]<\infty$ and $\sup_{0\le t\le T} E[y^2(t)]<\infty$ respectively. Define for $N>0$ and $t\in[0, T]$

$$\Psi(t)=\begin{cases}1 & \text{if } \sup_{0\le s\le t}|x(s)|\le N, \quad \sup_{0\le s\le t}|y(s)|\le N,\\ 0 & \text{otherwise}.\end{cases}$$

By following an argument similar to that given in Theorem 2, we find that

$$E[\Psi(t)|x(t)-y(t)|^2]\le M_N(K_N, T)\int_0^t E[\Psi(s)|x(s)-y(s)|^2]ds$$

with a constant $M_N(K_N, T)>0$ depending on K_N and T. Thus, it follows from Gronwall's inequality that

$$E[\Psi(t)|x(t)-y(t)|^2]=0.$$

This implies that

$$P[x(t)\ne y(t)]\le P\left[\sup_{0\le s\le T}|x(s)|>N\right]+P\left[\sup_{0\le s\le T}|y(s)|>N\right].$$

Noting that both $x(t)$ and $y(t)$ are continuous processes, we get,

$$P[x(t)\ne y(t)]=0, \qquad 0\le t\le T,$$

for $N\to\infty$, and consequently, we have

$$P[x(t)=y(t), \text{ for } 0\le t\le T]=1.$$

This proves the theorem.

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