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THE INFLUENCE OF SECOND COEFFICIENT ON SPIRAL-LIKE AND THE ROBERTSON FUNCTIONS

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1. Introduction

Let $P(\alpha, \beta, \lambda)$ denote the class of functions $p(z)=1+\sum_{n=1}^{\infty} p_n z^n$ which are analytic in the disc $E=\{z: |z|<1\}$ and satisfy the inequality

$$(1.1) \qquad \left|\frac{p(z)-1}{2\beta(p(z)-1+(1-\alpha)e^{-i\lambda}\cos\lambda)-(p(z)-1)}\right|<1$$

for some α, β, λ $(0\leq\alpha<1,\ 0<\beta\leq 1,\ -\pi/2<\lambda<\pi/2)$ and for all z in E. In [9], the following class has been introduced and studied in depth;

$$(1.2) \qquad S(\alpha, \beta, \lambda)=\left\{f\in N: \frac{zf'(z)}{f(z)}\varepsilon P(\alpha, \beta, \lambda),\ z\in E\right\},$$

where N denotes the class of analytic functions $f(z)$ with the normalization $f(0)=0=f'(0)-1$. A function $f\in S(\alpha, \beta, \lambda)$ is called a λ-spiral-like function of order α and type β. The class $S(\alpha, 1, \lambda)$ of λ-spiral-like functions of order α was introduced by Libera [6]. Furthermore, $S(0, 1, \lambda)$ is the class of the so called "spiral-like" functions defined by Špaček [13].

Let $p(z)=1+p_1z+p_2z^2+\cdots$ be in $P(\alpha, \beta, \lambda)$ and put $\theta=\exp\{-i\arg p_1\}$. Then $p(\theta z)=1+|p_1|z+\cdots\in P(\alpha, \beta, \lambda)$. Thus we see that there is no loss of generality in limiting our study to functions in $P(\alpha, \beta, \lambda)$ with non-negative real coefficients. It will be shown in Lemma 1 of this paper that $|p_1|\leq 2\beta(1-\alpha)\cos\lambda$, $(0\leq\alpha<1,\ 0<\beta\leq 1,\ -\pi/2<\lambda<\pi/2)$. From these observations, we define a subclass of $P(\alpha, \beta, \lambda)$, namely

$$(1.3) \qquad P_a(\alpha, \beta, \lambda)=\{p\in P(\alpha, \beta, \lambda): p'(0)=2a\beta(1-\alpha)\cos\lambda,\ 0\leq a\leq 1\}.$$

We now consider a subclass of univalent functions with fixed second coefficient generated from $P_a(\alpha, \beta, \lambda)$, viz.,

$$(1.4) \qquad S_a(\alpha, \beta, \lambda)=\left\{f(z)=z+(2a\beta(1-\alpha)\cos\lambda)z^2+\cdots: \frac{zf'(z)}{f(z)}\varepsilon P_a(\alpha, \beta, \lambda),\ z\in E\right\}.$$

Following Silverman and Telage [12], we define an another subclass of $P(\alpha, \beta, \lambda)$ as follows.

$$(1.5)\quad H_b(\alpha, \beta, \lambda) = \left\{ f(z) = z + a_2 z^2 + \cdots : \frac{zf'(z)}{f(z)} \varepsilon P(\alpha, \beta, \lambda), \right.$$
$$\left. |a_2| = 2b,\ 0 \le b \le \beta(1-\alpha)\cos\lambda,\ z \varepsilon E \right\}.$$

However, we observe that for $a = b/(\beta(1-\alpha)\cos\lambda)$, the results for the two classes, namely, $S_a(\alpha, \beta, \lambda)$ and $H_b(\alpha, \beta, \lambda)$ coincide.

We shall investigate how the second coefficient in the series expansion of the functions in the class $S_a(\alpha, \beta, \lambda)$ affects certain properties such as distortion, γ-spiral radius, and radius of starlikeness of these functions.

In [1], the author has defined a class, $C(\alpha, \beta, \lambda)$, of λ-Robertson functions of order α and type β. In fact, $f \in C(\alpha, \beta, \lambda)$ if and only if

$$(1.6)\qquad \left| \frac{zf''(z)/f'(z)}{2\beta(zf''(z)/f'(z) + (1-\alpha)e^{-i\lambda}\cos\lambda) - zf''(z)/f'(z)} \right| < 1$$

holds for some α, β, λ $(0 \le \alpha < 1,\ 0 < \beta \le 1,\ -\pi/2 < \lambda < \pi/2)$ and for all z in E. We observe that $C(0, 1, \lambda)$ and $C(0, (2\delta-1)/2\delta, \lambda)$ are the classes introduced and studied, respectively, by Robertson [11] and Kulshrestha [5].

It easily follows from definitions (1.2) and (1.6) that a function f is in $C(\alpha, \beta, \lambda)$ if and only if $zf'(z) \in S(\alpha, \beta, \lambda)$, $(z \in E)$. So analogous to our class $S_a(\alpha, \beta, \lambda)$, we shall study the class $C_a(\alpha, \beta, \lambda)$, where

$$(1.7)\qquad f \in C_a(\alpha, \beta, \lambda) \Longleftrightarrow zf'(z) \in S_a(\alpha, \beta, \lambda),\ (z \in E).$$

We observe that for $a = 1$, this class gives rise to the corresponding results obtained by the author in [1]; and for $a \ne 1$, the results are otherwise an improvement.

2. Growth estimates

In this section, we give two results. Our lemma has been used in the last section for generating certain subclasses of $P(\alpha, \beta, \lambda)$. The second result is the growth theorem for the class $P_a(\alpha, \beta, \lambda)$; and will be applied in the subsequent section.

Lemma 1. *If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, analytic in the open unit disk E, is in $P(\alpha, \beta, \lambda)$, then*

$$|p_n| \le 2\beta(1-\alpha)\cos\lambda$$

for all $n \ge 1$. The estimates are sharp.

Proof. Since $p \in P(\alpha, \beta, \lambda)$, the condition (1.1) coupled with an application of Schwarz's lemma implies

$$(2.1)\qquad p(z) = \frac{1 + ((2\beta-1) - 2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)\omega(z)}{1 + (2\beta-1)\omega(z)}$$

where ω is analytic in E and satisfies the conditions $\omega(0)=0$ and $|\omega(z)|<1$ for z in E. Now (2.1) may be written as

$$\left\{2\beta(1-\alpha)e^{-i\lambda}\cos\lambda+\sum_{k=1}^{\infty}(2\beta-1)p_kz^k\right\}\omega(z)=-\sum_{k=1}^{\infty}p_kz^k. \tag{2.2}$$

It is easy to see that in (2.2) each p_n on the right depends on $p_1, p_2, \cdots, p_{n-1}$ on the left. Thus (2.2) yields

$$\left\{2\beta(1-\alpha)e^{-i\lambda}\cos\lambda+\sum_{k=1}^{n-1}(2\beta-1)p_kz^k\right\}\omega(z)=-\sum_{k=1}^{n}p_kz^k-\sum_{k=n+1}^{\infty}t_kz^k,$$

where $\sum_{k=n+1}^{\infty}t_kz^k$ is absolutely and uniformly convergent in compacta on E. Using the fact that $|\omega(z)|<1$, squaring, and then integrating both sides we obtain

$$4\beta^2(1-\alpha)^2\cos^2\lambda+\sum_{k=1}^{n-1}(2\beta-1)^2|p_k|^2r^{2k}\geq\sum_{k=1}^{n}|p_k|^2r^k+\sum_{k=n+1}^{\infty}|t_k|^2r^{2k}.$$

Taking the limit as $r\to1$, we get

$$4\beta^2(1-\alpha)^2\cos^2\lambda+\sum_{k=1}^{n-1}(2\beta-1)^2|p_k|^2\geq|p_n|^2+\sum_{k=1}^{n-1}|p_k|^2,$$

that is

$$|p_n|^2+4\beta(1-\beta)\sum_{k=1}^{n-1}|p_k|^2\leq4\beta^2(1-\alpha)^2\cos^2\lambda.$$

Since $0<\beta\leq1$, this gives

$$|p_n|\leq2\beta(1-\alpha)\cos\lambda,\qquad(n\geq1)$$

which proves the lemma. The bounds are sharp for the functions

$$p_n(z)=\frac{1-(2\beta-1-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)z^n}{1-(2\beta-1)z^n}$$

for $n\geq1$ and $z\in E$.

Lemma 2 [3]. *If $\omega(z)=b_1z+\cdots$ is an analytic map of the unit disc into itself, then $|b_1|\leq1$ and*

$$|\omega(z)|\leq\frac{r(r+|b_1|)}{1+|b_1|r}$$

where $|z|=r$.

Theorem 1. *Let $p(z)=1+\sum_{n=1}^{\infty}p_nz^n$ be analytic in E. If $p\in P_a(\alpha,\beta,\lambda)$, then for $|z|=r<1$ and for all $\alpha,\beta,\lambda,\gamma,a$ $(0\leq\alpha<1,\ 0<\beta\leq1,\ -\pi/2<\lambda,\ \gamma<\pi/2,\ 0\leq a\leq1)$,*

$$\text{(2.3)}\quad \operatorname{Re}\{e^{i\gamma}p(z)\} \geq \frac{\begin{array}{c}(1+ar)^2\cos\gamma - 2\beta(1-\alpha)\cos\lambda\cdot r(r+a)(1+ar)\\ +(2\beta-1)\{2\beta(1-\alpha)\cos\lambda\cos(\gamma-\lambda)-(2\beta-1)\cos\gamma\}r^2(r+a)^2\end{array}}{(1+ar)^2-(2\beta-1)^2r^2(r+a)^2},$$

and

$$\text{(2.4)}\quad \operatorname{Re}\{e^{i\gamma}p(z)\} \leq \frac{\begin{array}{c}(1+ar)^2\cos\gamma + 2\beta(1-\alpha)\cos\lambda\cdot r(r+a)(1+ar)\\ +(2\beta-1)\{2\beta(1-\alpha)\cos\lambda\cos(\gamma-\lambda)-(2\beta-1)\cos\gamma\}r^2(r+a)^2\end{array}}{(1+ar)^2-(2\beta-1)^2r^2(r+a)^2}.$$

These bounds are sharp.

Proof. As in the proof of Lemma 1 we have

$$\text{(2.5)}\qquad p(z)=\frac{1+\{2\beta-1-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\}\omega(z)}{1+(2\beta-1)\omega(z)}$$

which, by direct computations, yeilds

$$\text{(2.6)}\qquad \omega(z)=\frac{1-p(z)}{2\beta(p(z)-1+(1-\alpha)e^{-i\lambda}\cos\lambda)-(p(z)-1)}=-ae^{i\lambda}z+\cdots.$$

Note that $\omega(z)$ is analytic map of Δ onto itself and $|-ae^{i\lambda}|=a\leq 1$. Then by Lemma 2, it follows that

$$\text{(2.7)}\qquad |\omega(z)|\leq\frac{r(r+a)}{1+ar},$$

where $|z|=r$. Letting $B(z)=e^{i\gamma}p(z)$, (2.5) may be written

$$\text{(2.8)}\qquad \omega(z)=\frac{e^{i\gamma}-B(z)}{(2\beta-1)B(z)-e^{i\gamma}(2\beta-1-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)}.$$

(2.7) and (2.8) together implies

$$\left|\frac{e^{i\gamma}-B(z)}{(2\beta-1)B(z)-e^{i\gamma}(2\beta-1-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)}\right|\leq\frac{r(r+a)}{1+ar}.$$

Thus it follows that $B(z)$ maps the disc $|z|\leq r$ onto a disc $|B(z)-D|<R$, where

$$D=e^{i\gamma}\left(1+\frac{(2\beta(2\beta-1)(1-\alpha)e^{-i\lambda}\cos\lambda)(r+a)^2r^2}{(1+ar)^2-(2\beta-1)^2(r+a)^2r^2}\right)$$

$$R=\frac{2\beta(1-\alpha)\cos\lambda\cdot r(r+a)(1+ar)}{(1+ar)^2-(2\beta-1)^2r^2(r+a)^2}.$$

This immediately leads to (2.3) and (2.4).

The equalities of Theorem 1 are obtained by the function

$$\text{(2.9)}\qquad p(z)=\frac{1+a(1+Ae^{it})z+Ae^{it}z^2}{1+a(1+(2\beta-1)e^{it})z+(2\beta-1)e^{it}z^2},$$

where $A=2\beta-1-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda$, and the values of t $(0\le t\le 2\pi)$ are properly chosen.

Setting $p(z)=zf'(z)/f(z)$ in Theorem, we obtain the growth estimates for the class $S_a(\alpha,\beta,\lambda)$. By taking the appropriate values of the parameters α, β we can get the corresponding results for several subclasses of λ-spiral-like functions introduced by many researchers. The growth theorem for the class $S_b(\alpha,1,\lambda)$ where $b=a(1-\alpha)\cos\lambda$ was obtained by Silverman and Telage [12]. Further, the growth theorem for $S_a(0,1,0)$ was found by Finkelstein [3].

3. The γ-spiral radius

Let S be the family of all normalized functions which are analytic and univalent in E. Following Silverman and Telage [12], if $f\in S$ and $|\gamma|<\pi/2$, then the γ-spiral radius of order δ $(0\le\delta<1)$ written $R(\gamma,\delta,f(z))$, is given by

$$R(\gamma,\delta,f(z))=\sup\left[r\colon \operatorname{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\}>\delta,\ |z|<r\right]; \tag{3.1}$$

and if $F\subset S$, then the γ-spiral radius of order δ of F, denoted $R(\gamma,\delta,F)$, is given by

$$R(\gamma,\delta,F)=\inf_{f\in F}R(\gamma,\delta,f(z))\,. \tag{3.2}$$

These definitions reduce to those of Libera [6] when $\delta=0$.

We now determine the γ-spiral radius of order δ of the class $S_a(\alpha,\beta,\lambda)$.

Theorem 2. *γ-spiral radius of order δ of the class $S_a(\alpha,\beta,\lambda)$ is the smallest positive root r_0 of the fourth degree equation*

$$Ar^4+Br^3+Cr^2+Dr+(\cos\gamma-\delta)=0\,, \tag{3.3}$$

where

$$A=(2\beta-1)\{2\beta(1-\alpha)\cos\lambda\cos(\gamma-\lambda)-(\cos\gamma-\delta)(2\beta-1)\}$$

$$B=-2a\{(2\beta-1)^2(\cos\gamma-\delta)+\beta(1-\alpha)(1-2(2\beta-1)\cos(\gamma-\lambda))\cos\lambda\}$$

$$C=4\beta(1-\beta)a^2(\cos\gamma-\delta)-2\beta(1-\alpha)\cos\gamma\{1+a^2-a^2(2\beta-1)\cos(\gamma-\lambda)\}\,,$$

$$D=2a\{\cos\gamma-\beta(1-\alpha)\cos\lambda-\delta\}\,.$$

The result is sharp.

Proof. Let $f\in S_a(\alpha,\beta,\lambda)$. Then

$$\frac{zf'(z)}{f(z)}=p(z)$$

for some $p\in P_a(\alpha,\beta,\lambda)$. Therefore by Theorem 1,

$$\operatorname{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\}>\delta$$

where the right side of (2.3) is $\geq\delta$. This is equivalent to

$$\begin{aligned}&(1+ar)^2\cos\gamma-2\beta(1-\alpha)\cos\lambda\cdot r(r+a)(1+ar)\\&\quad+(2\beta-1)\{2\beta(1-\alpha)\cos\lambda\cos(\gamma-\lambda)-(2\beta-1)\cos\gamma\}r^2(r+a)^2\\&\geq\delta\{(1+ar)^2-(2\beta-1)^2r^2(r+a)^2\}\end{aligned}$$

or

$$\begin{aligned}&(\cos\gamma-\delta)\{(1+ar)^2-(2\beta-1)^2r^2(r+a)^2\}\\&\qquad-2\beta(1-\alpha)\cos\lambda\cdot r(r+a)\{1+ar-(2\beta-1)\cos(\gamma-\lambda)\cdot r(r+a)\}\geq 0,\end{aligned}$$

which on simplification, and with the aid of (3.2) concludes the proof of the theorem. The extremal function is of the form (2.9).

Corollary 1. *γ-s.r. $S(\alpha,\beta,\lambda)$ is the smallest positive root r_0 of the quadratic equation*

$$(3.4)\qquad\begin{aligned}&(2\beta-1)\{2\beta(1-\alpha)\cos(\gamma-\lambda)\cos\lambda-(2\beta-1)\cos\gamma\}r^2\\&\qquad-2\beta(1-\alpha)\cos\lambda\cdot r+\cos\gamma=0.\end{aligned}$$

The result is sharp.

Proof. Set $\delta=0=a-1$ in (3.3). The least positive root of the equation is given by

$$\begin{aligned}&(2\beta-1)\{2\beta(1-\alpha)\cos\lambda\cos(\gamma-\lambda)-(2\beta-1)\cos\gamma\}r^4\\&\quad+2\{(2\beta-1)(2\beta(1-\alpha)\cos(\gamma-\lambda)\cos\lambda-(2\beta-1)\cos\gamma)-\beta(1-\alpha)\cos\lambda\}r^3\\&\quad+\{(2\beta-1)(2\beta(1-\alpha)\cos\lambda\cos(\gamma-\lambda)-(2\beta-1)\cos\gamma-4\beta(1-\alpha)\cos\lambda+\cos\gamma\}r^2\\&\quad+2\{\cos\gamma-\beta(1-\alpha)\cos\lambda\}r+\cos\gamma=0.\end{aligned}$$

Noting that $(1+r)^2$ is the common factor of the left side of the above equation we obtain, on simplification, the equation (3.4). The result is sharp for the function given by

$$(3.5)\qquad f(z)=\begin{cases}z/(1-(2\beta-1)e^{i\theta}z)^{2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\cdot(2\beta-1)^{-1}}, & \beta\neq\dfrac{1}{2}\\[2ex] z\exp((1-\alpha)\cos\lambda\cdot e^{i(\theta-\lambda)}z), & \beta=\dfrac{1}{2}\end{cases}$$

where

$$\theta=2\arctan\left\{\frac{1-(2\beta-1)r}{1+(2\beta-1)r}\cot\left(\frac{-\lambda}{2}\right)\right\}.$$

The result in the Corollary was also found in [9].

Corollary 2. γ-s.r. *of order* δ *of* $H_b(\alpha, \beta, \lambda)$ *is the smallest positive root of the equation*

$$(\cos\gamma-\delta)(u^2-(2\beta-1)^2v^2r^2)-2\beta(1-\alpha)\cos\lambda\cdot vr(u-(2\beta-1)\cos(\gamma-\lambda)\cdot vr)=0\,,$$

where

$$u=br+\beta(1-\alpha)\cos\lambda$$

$$v=b+\beta(1-\alpha)\cos\lambda\cdot r\,.$$

The result is sharp.

Proof. Letting $a=b/(\beta(1-\alpha)\cos\lambda)$ in Theorem 2, we find that the least positive root of the equation (3.3) is given by

$$\begin{aligned}&l^2(2\beta-1)\{2l\cos(\gamma-\lambda)-(2\beta-1)(\cos\gamma-\delta)\}r^4\\&+2bl\{l(2(2\beta-1)\cos(\gamma-\lambda)-1)-(2\beta-1)^2(\cos\gamma-\delta)\}r^3\\&+\{4b^2\beta(1-\beta)(\cos\gamma-\delta)-2l(b^2+l^2-l^2(2\beta-1)\cos(\gamma-\lambda))\}r^2\\&+2bl(\cos\gamma-l-\delta)r+l^2(\cos\gamma-\delta)=0\,,\end{aligned}$$

where

$$l=\beta(1-\alpha)\cos\lambda\,.$$

Writing

$$u=br+\beta(1-\alpha)\cos\lambda=br+l$$

$$v=b+\beta(1-\alpha)\cos\lambda\cdot r=b+lr$$

and rearranging the terms in the above equation we can easily obtain the result of this corollary.

Remark 1. Setting $\beta=1$, the above corollary gives the γ-s.r. of order δ of the class $H_b^\lambda(\alpha)\equiv H_b(\alpha, 1, \lambda)$.

Remark 2. When $\alpha=1/2$, $\beta=1$, $\lambda=0$ in Corollary 1 we note that (3.4) is linear, and we obtain the γ-s.r. of the class, K, of convex functions. Thus γ-s.r. $K=\cos\gamma$.

Remark 3. Different values of the parameters α, β and λ lead to γ-spiral radius of the classes studied by Libera [6], Makówka [8], and Kulshrestha [5].

4. Radius of starlikeness

It is evident that the 0-spiral radius of order δ of a subclass $F \subset S$ is the radius of starlikeness of order δ of F. Thus by fixing $\gamma = 0$ in Theorem 2, we immediately obtain the following

Theorem 3. *The radius of starlikeness of order δ of the class $S_a(\alpha, \beta, \lambda)$ is the smallest positive root of the fourth degree polynomial equation*

$$\begin{aligned}&(2\beta-1)\{2\beta(1-\alpha)\cos^2\lambda-(1-\delta)(2\beta-1)\}r^4\\&\quad-2a\{(1-\delta)(2\beta-1)^2+\beta(1-\alpha)\cos\lambda(1-2(2\beta-1)\cos\lambda)\}r^3\\&\quad+2\beta\{2a^2(1-\beta)(1-\delta)-(1-\alpha)\cos\lambda(1+a^2-a^2(2\beta-1)\cos\lambda)\}r^2\\&\quad+2a\{(1-\delta)-\beta(1-\alpha)\cos\lambda\}r+(1-\delta)=0\,.\end{aligned}$$

The result is sharp.

By fixing the parameters α, β, λ and a in Theorem 3 we obtain some interesting special cases.

Corollary 1. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic in E. If $f\in S(\alpha, \beta, \lambda)$, then f is univalent and starlike in*

$$|z|<\{\beta(1-\alpha)\cos\lambda+\sqrt{\beta^2(1-\alpha)^2\cos^2\lambda+(2\beta-1)^2-2\beta(1-\alpha)(2\beta-1)\cos^2\lambda}\}^{-1}\,.$$

The estimate for $|z|$ is sharp for the function (3.5).

Proof. Setting $\delta=0=a-1$ in Theorem 3, it follows that f is univalent and starlike for $|z|<r_0$, where r_0 is the smallest positive root of the equation

$$(2\beta-1)(2\beta(1-\alpha)\cos^2\lambda-(2\beta-1))r^2-2\beta(1-\alpha)\cos\lambda\cdot r+1=0\,. \tag{4.1}$$

Now the result easily follows from (4.1).

By taking $\alpha=0$ and $\beta=1$ in Corollary 1, we obtain the following result, which has been obtained independently and using different methods by Robertson [10], Libera [6], and Libera and Ziegler [7].

Corollary 2. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be a λ-spiral-like function; then f is starlike for*

$$|z|\le r_\lambda=\frac{1}{|\sin\lambda|+\cos\lambda}\,.$$

The estimate for r_λ is sharp for the function

$$f_0(z)=z/(1-e^{i\theta}z)^{2\cos\lambda\cdot e^{-i\lambda}},$$

where

$$\theta = 2\arctan\left\{\left((1-r)/(1+r)\right)\cot\left(\frac{-\lambda}{2}\right)\right\}.$$

Remark 1. We note that the maximum value of $|\sin\lambda| + \cos\lambda$ is $\sqrt{2}$ and occurs for $\lambda = \pm\pi/4$. Consequently, a λ-spiral-like function f is always univalent and starlike in $|z| < 1/\sqrt{2} = 0.707\cdots$.

Kulshrestha [5] studied the class $S^\lambda(\delta) \equiv S(0, (2\delta-1)/2\delta, \lambda)$. More precisely, a function $f \in N$ is said to be in the class $S^\lambda(\delta)$ if it satisfies the inequality

$$\left|\frac{e^{i\lambda}zf'(z)/f(z) - i\sin\lambda}{\cos\lambda} - \delta\right| < \delta,$$

for some δ, λ ($\delta > 1/2$, $-\pi/2 < \lambda < \pi/2$) and for all $z \in E$. Corollary 1 gives the following result for this class.

Corollary 3. *If f is in $S^\lambda(\delta)$ ($\delta > 1/2$), then f is starlike in*

$$|z| < 2\{(1+l)\cos\lambda + \sqrt{(1+l)^2\cos^2\lambda - 4l^2\,\mathrm{Re}(c)}\}^{-1}$$

where

$$c = (1+l)\cos\lambda\cdot e^{-i\lambda}/l - 1, \qquad l = 1 - 1/\delta.$$

The estimate is sharp for the function

$$f_0(z) = \begin{cases} z/(1 - le^{i\theta}z)^{((l+1)/l)\cos\lambda\cdot e^{-i\lambda}}, & l \neq 0 \\ z\exp\{\cos\lambda\cdot e^{i(\theta-\lambda)}z\}, & l = 0. \end{cases}$$

where θ is given by

$$\tan\theta/2 = \frac{1-lr}{1+lr}\cot\left(\frac{-\lambda}{2}\right).$$

Remark 2. Our results in the above corollary are improvements over the corresponding results obtained by Kulshrestha [5].

Remark 3. Different values of the parameters α, β in Corollary 1 give rise to the corresponding radius of starlikeness for the respective classes defined by Goel [4], Makӧwka [8], and many others.

We now set $H_b(0, 1, \lambda) \equiv H_b(\lambda)$. In the result that follows from Corollary 2 of Theorem 2 we relate starlike to spiral-like functions.

Corollary 4. *If $f \in H_b(0)$ and we set $D = \sec\gamma + |\tan\gamma|$ then*

$$\mathrm{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\} > 0 \qquad \text{for } |z| < r(b),$$

where

$$r(b)=\left[\frac{b(1-D)+\sqrt{b^2(1-D)^2+4D}}{2D}\right]$$

Furthermore $r(b)$ *is decreasing* $(0\leq b\leq 1)$ *with*

$$r(0)=\frac{1}{\sqrt{D}}$$

$$r(1)=\frac{1}{D}.$$

Remark 4. The above corollary was also found in [12].

5. The family $C_a(\alpha, \beta, \lambda)$

We can obtain results about the family $C_a(\alpha, \beta, \lambda)$ from those found for the class $S_a(\alpha, \beta, \lambda)$ by a simple application of (1.7) and thus observing that

$$\mathrm{Re}\left\{e^{i\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\delta \Longleftrightarrow \mathrm{Re}\left\{e^{i\gamma}\left(z\frac{g'(z)}{g(z)}\right)\right\}>\delta,$$

where $g(z)=zf'(z)$, $z\in E$. Furthermore, by fixing the parameters α, β, λ, γ and a in such results for the class $C_a(\alpha, \beta, \lambda)$ we can obtain several interesting special cases which coincide with the results found earlier by Libera and Ziegler [7], Chichra [2], Kulshrestha [5], and many others.

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