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ON A COVERING THEOREM OF WICKE AND WORRELL

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ABSTRACT. A generalization of Wicke and Worrell's theorem (Proc. Amer. Math. Soc. **55** (1976), 427–431) on covering property of topological spaces is provided.

The following theorem was first mentioned by Worrell and Wicke in [2] as Theorem (iv), but the proof was given twelve years later in [1].

Theorem 1. *Suppose that X is a countably compact topological space and $\mathscr{U}$ is the union of a countable collection $\{\mathscr{V}_n; n<\omega\}$ of collections of open subsects of X such that each $x\in X$ is in at least one element but not in more than countably many elements of some $\mathscr{V}_n$. Then some finite sub-collection of $\mathscr{U}$ covers X.*

The purpose of this note is to generalize this theorem.

We need some notations. If $\mathscr{U}$ is a collection of subsets of a topological space X, then define $\mathscr{D}(x, \mathscr{U})=\{U\in\mathscr{U}; x\in U\}$; $\mathrm{St}(x, \mathscr{U})=\bigcup_{U\in\mathscr{D}(x,\mathscr{U})} U$. $|\mathscr{D}(x, \mathscr{U})|$ denotes the cardinality of $\mathscr{D}(x, \mathscr{U})$.

Theorem 2. *Suppose that X is a countably compact topological space and $\mathscr{U}$ is an open cover. If $\mathscr{U}$ is the union of a countable collection $\{\mathscr{U}_n; n\leqslant\omega\}$ of collections of open subsets of X such that for each uncountable closed subset $S\subset X$, there is an $x\in S$ and an n satisfying $1\leqslant|\mathscr{D}(x, \mathscr{V}_n)|\leqslant\omega$, then $\mathscr{U}$ has a finite subcover..*

Proof. Assume the contrary. Since X is countably compact, $\mathscr{U}$ has no countable subcover. So X must be uncountable. Let $C=\{x\in X; 1\leqslant|\mathscr{D}(x, \mathscr{V}_n)|\leqslant\omega$ for some $n\}$. Then C is nonempty. Let $N(x)$ be the set of all positive integer n such that $1\leqslant|\mathscr{D}(x, \mathscr{V}_n)|\leqslant\omega$. Note that $\bigcup_{n\in N(x)}\mathrm{St}(x, \mathscr{V}_n)$ is a countable union of members of $\mathscr{U}$ for each $x\in X$. Let $R=X\setminus\bigcup_{x\in C}\bigcup_{n\in N(x)}\mathrm{St}(x, \mathscr{V}_n)$. Since R is a countable closed set, it is contained in a countable union U of members of $\mathscr{U}$. Now assume that there is a finite set $\{x_1, \cdots, x_k\}\subset C$ such that $C\setminus U\subset\bigcup_{1\leqslant i\leqslant k}\bigcup_{n\in N(x_i)}\mathrm{St}(x_i, \mathscr{V}_n)$. Let $W=U\cup\bigcup_{1\leqslant i\leqslant k}\bigcup_{n\in(x_i)}\mathrm{St}(x_i, \mathscr{V}_n)$. Then W is a countable union of members of $\mathscr{U}$ and $W\supset C$. Since $X\setminus W$ is countable, $\mathscr{U}$ has a countable subcover. This contradicts the first assumption of this proof. Hence there is no such a finite set, we can find a sequence $\{x_k\}_{k=1}^{\infty}$ in C such that $x_k\in(C\setminus U)\setminus\bigcup_{i\leqslant i\leqslant k}\bigcup_{n\in N(x_i)}\mathrm{St}(x_i, \mathscr{V}_n)$.

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Now we prove that the sequence $\{x_k\}_{k=1}^{\infty}$ has no ω-limit point. Let $x \in X$ be an arbitrary point. If $x \in U$, then U is an open neighborhood of x containing no points of $\{x_k\}_{k=1}^{\infty}$. If $x \in X \setminus U$, since $X \setminus U \subset X \setminus R = \bigcup_{x \in C} \bigcup_{n \in N(x)} \mathrm{St}(x, \mathscr{V}_n)$, there is an $\tilde{x} \in C$ such that $x \in \bigcup_{n \in N(\tilde{x})} \mathrm{St}(\tilde{x}, \mathscr{V}_n)$. Hence $x \in \mathrm{St}(\tilde{x}, \mathscr{V}_{n_0})$ for some n_0. Therefore $x \in V$ for an open set $V \in \mathscr{V}_{n_0}$. If $V \cap (\bigcup_{k=1}^{\infty} \{x_k\}) = \varnothing$, V is an open neighborhood of x containing no points of $\{x_k\}_{k=1}^{\infty}$. If $V \cap (\bigcup_{k=1}^{\infty} \{x_k\}) \neq \varnothing$, then $x_p \in V$ for some integer p and $V \subset \mathrm{St}(x_p, \mathscr{V}_{n_0})$. For $k > p$, since $x_k \in (C \setminus U) \setminus \bigcup_{1 \leqslant i \leqslant k} \bigcup_{n \in N(x_i)} \mathrm{St}(x_p, \mathscr{V}_n)$, we have $x_k \notin \mathrm{St}(x_p, \mathscr{V}_{n_0})$, hence $x_k \notin V$, V is an open neighborhood of x containing at most p points of the sequence $\{x_k\}_{k=1}^{\infty}$. Therefore $\{x_k\}_{k=1}^{\infty}$ has no ω-limit point, X is not countably compact, a contradiction.

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References

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[2] J. M. Worrell, Jr. and H. H. Wicke: *Characterizations of developable topological spaces*, Canad. J. Math., **17** (1965), 820–830.

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