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A NOTE ON THE LAW OF THE ITERATED LOGARITHM FOR MARTINGALES

By

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1. Introduction

In their paper [5], Lai and Wei obtained a general log log law for weighted sums of the form

$$\limsup_{n\to\infty}\left|\sum_{i=-\infty}^{\infty} b_{ni}\varepsilon_i\right| \Big/ \left(2\sum_{i=-\infty}^{\infty} b_{ni}^2 \log\log \sum_{i=-\infty}^{\infty} b_{ni}^2\right)^{1/2} = \sigma \text{ a.s.},$$

where the ε_i are independent random variables with zero means and a common variance σ^2, and $\{b_{ni}, n\geqq 1, -\infty<i<\infty\}$ is a double array of constants such that $\sum_{i=-\infty}^{\infty} b_{ni}^2<\infty$ for every n. They also applied this result to obtain laws of the iterated logarithm for partial sums of linear processes and for least squares estimates in regression models, and to improve the results of Tomkins [8, 9] concerning iterated logarithm behavior of the form $\sum_{i=1}^{n} f(i/n)\varepsilon_i$. It is our object here to extend their results to the martingale case.

Let $\{\varepsilon_n, \mathcal{F}_n, -\infty<n<\infty\}$ be a sequence of martingale differences such that

$$E(\varepsilon_n^2 \mid \mathcal{F}_{n-1})=\sigma^2>0 \quad \text{a.s. for all } n \tag{1.1}$$

and

$$E(|\varepsilon_n|^r \mid \mathcal{F}_{n-1})\leqq C \quad \text{a.s. for all } n, \text{ some } r>2 \text{ and some } C<\infty, \tag{1.2}$$

and let $\{a_{ni}, n\geqq 1, -\infty<i\leqq n\}$ be a double array of constants such that

$$\sum_{i=-\infty}^{n} a_{ni}^2<\infty \quad \text{for every } n. \tag{1.3}$$

Define

$$S_n=\sum_{i=-\infty}^{n} a_{ni}\varepsilon_i. \tag{1.4}$$

Conditions (1.1) and (1.3) guarantee almost sure convergence of the series in (1.4), and therefore S_n is well defined. Our main result is the following generalization of Theorem 1 of [5] (see the Note added in proof).

Theorem 1. *Let* $\{\varepsilon_n, \mathcal{F}_n, -\infty<n<\infty\}$ *be a sequence of martingale differences satisfying* (1.1) and (1.2), *and let* $\{a_{ni}, n\geqq 1, -\infty<i\leqq n\}$ *be a double array of constants satisfying* (1.3). *Define* S_n *as in* (1.4). *Assume that as* $n\to\infty$,

$$A_n=\sum_{i=-\infty}^{n} a_{ni}^2 \longrightarrow \infty \tag{1.5}$$

and

$$\sup_{i\leqq n} a_{ni}^2=o(A_n(\log A_n)^{-\rho}) \quad \text{for all } \rho>0. \tag{1.6}$$

(i) *If there exist constants* $c_i\geqq 0$ *and* $d>2/r$ *such that*

$$\sum_{i=-\infty}^{m}(a_{ni}-a_{mi})^2+\sum_{i=m+1}^{n} a_{ni}^2\leqq\Big(\sum_{i=m+1}^{n} c_i\Big)^d \quad \text{for } n>m\geqq m_0 \tag{1.7}$$

and

$$\Big(\sum_{i=m_0}^{n} c_i\Big)^d=O(A_n) \quad \text{as } n\to\infty, \tag{1.8}$$

then

$$\limsup_{n\to\infty}|S_n|/(2A_n \log\log A_n)^{1/2}\leqq\sigma \quad \text{a.s.} \tag{1.9}$$

(ii) *If*

$$\sum_{i=-\infty}^{m}(a_{ni}-a_{mi})^2+\sum_{i=m+1}^{n} a_{ni}^2\leqq g(n-m) \quad \text{for } n>m\geqq m_0, \tag{1.10}$$

where g *is a positive function on* $\{1, 2, \cdots\}$ *such that*

$$g(n)=O(A_n) \quad \text{as } n\to\infty, \tag{1.11}$$

$$\liminf_{n\to\infty} g(Kn)/g(n)>K^{2/r} \quad \text{for some integer } K\geqq 2 \tag{1.12}$$

and

$$\forall\gamma>0,\ \exists\delta<1 \text{ such that } \limsup_{n\to\infty}\{\max_{\delta n\leqq i\leqq n} g(i)/g(n)\}<1+\gamma, \tag{1.13}$$

then (1.9) *still holds.*

(iii) *Suppose that for every* $0<\gamma<\gamma_0$, *there exist integers* $1<n_1<n_2<\cdots$ *such that*

$$\limsup_{k\to\infty}\Big(\sum_{i\leqq n_{k-1}} a_{n_k,i}^2\Big)/A_{n_k}\leqq\gamma, \tag{1.14}$$

$$\limsup_{k\to\infty}(\log\log A_{n_k})/(\log k)\leqq 1+\gamma \tag{1.15}$$

and

$$\liminf_{k\to\infty}(\log\log A_{n_k})/(\log k)>0, \tag{1.16}$$

where n_k may depend on γ, then for every $-\sigma \leqq q \leqq \sigma$,

$$\liminf_{n\to\infty} |(2A_n \log\log A_n)^{-1/2} S_n - q| = 0 \quad \text{a.s.} \tag{1.17}$$

In the field of applications it will be often difficult to check condition (1.2), while the condition

$$\sup_n E|\varepsilon_n|^r < \infty \quad \text{for some} \quad r > 2 \tag{1.18}$$

is satisfied in many applications. For this reason, replacing the assumption (1.2) of Theorem 1 by the weaker assumption (1.18), we give the following iterated logarithm results.

Theorem 2. *Let $\{\varepsilon_n, \mathcal{F}_n, -\infty<n<\infty\}$ be a sequence of martingale differences satisfying* (1.1) *and* (1.18), *and let $\{a_{ni}, n\geqq 1, -\infty<i\leqq n\}$ be a double array of constants satisfying* (1.3), (1.5) *and* (1.6). *Define S_n as in* (1.4).

(i) *If there exist constants $c_i \geqq 0$ and $d>2/r$ such that* (1.7) *and* (1.8) *hold, then*

$$\limsup_{n\to\infty} |S_n|/(2\beta A_n \log\log A_n)^{1/2} \leqq \sigma \quad \text{a.s.,} \tag{1.19}$$

where $\beta = dr/(dr-2)$.

(ii) *If* (1.10), (1.11) *and* (1.13) *hold, and if*

$$\liminf_{n\to\infty} g(Kn)/g(n) \geqq K^{2\beta/r(\beta-1)} \quad \textit{for some } \beta>1 \textit{ and some integer } K\geqq 2, \tag{1.20}$$

then (1.19) *holds.*

Remarks. 1. Condition (1.13) is satisfied if either $g(n)$ is nondecreasing or $\max_{i\leq n} g(i) \sim g(n)$. Condition (1.12) (resp. (1.20)) is satisfied by $g(n)$ of the form $g(n)=n^\alpha L(n)$, where $\alpha>2/r$ (resp. $\alpha \geqq 2\beta/r(\beta-1)$) and $L(n)$ is a positive slowly varying function.

2. Define

$$f(m, n) = \sum_{i=-\infty}^{m} (a_{ni}-a_{mi})^2 + \sum_{i=m+1}^{n} a_{ni}^2 \quad \text{for} \quad n>m\geqq 1,$$

and $f(m, m)=0$ for $m\geqq 1$. If $f(m, n)$ satisfies (superadditivity)

$$f(m, k) + f(k, n) \leqq f(m, n) \quad \text{for} \quad n\geqq k\geqq m\geqq m_0,$$

then (1.7) and (1.8) hold with $d=1$, $c_{m_0}=0$ and $c_i = f(m_0, i) - f(m_0, i-1)$ for $i>m_0$.

3. For the particular case

$$a_{ni} = a_i \quad \text{if} \quad 1\leqq i\leqq n, \qquad a_{ni}=0 \quad \text{otherwise},$$

S_n reduces to the weighted sum $\sum_{i=1}^{n} a_i \varepsilon_i$. If, as $n\to\infty$

(1.21) (a) $A_n=\sum_{i=1}^{n} a_i^2 \longrightarrow \infty$, (b) $a_n^2=o(A_n(\log A_n)^{-\rho})$ for all $\rho>0$,

then the assumptions of Theorem 1(i) are satisfied with $d=1$ and $c_i=a_i^2$, and the assumptions of Theorem 1(iii) are satisfied by taking $n_k=\inf\{n>n_{k-1}: A_n\geqq L^k\}$, where $L>\gamma^{-1}$ (see [5]). The condition (1.21) (b) is equivalent to (1.6), and (1.21) (a)-(b) include $a_n=\pm n^\alpha$, $-1/2\leqq\alpha<\infty$, $a_n=\pm n^\alpha(\log n)^\beta$, $\alpha>-1/2$ or $\alpha=-1/2\leqq\beta<\infty$, etc. Tomkins [10] has obtained a log log law for weighted sums $\sum_{i=1}^{n} a_i\varepsilon_i$ of identically distributed martingale differences under slightly weaker conditions.

The above results provide a powerful tool for some applications. To illustrate we shall consider the stationary linear process

$$X_n=\sum_{i=0}^{\infty} b_i\varepsilon_{n-i}, \qquad \sum_{i=0}^{\infty} b_i^2<\infty, \tag{1.22}$$

where the $\{\varepsilon_n, \mathcal{F}_n, -\infty<n<\infty\}$ are martingale differences such that (1.1) holds, and $\mathcal{F}_n$ are σ-fields generated by ε_m, $m\leqq n$. This model is important in time series analysis, the martingale condition corresponding to the condition that the best linear predictor is the best predictor (both in the least squares sense; see Hannan and Heyde [3]).

Let for $n\geqq1$ and $i\leqq n$,

$$a_{ni}=\sum_{j=1}^{n} b_{j-i}, \qquad \text{where } b_i=0 \text{ if } i<0. \tag{1.23}$$

Then

$$S_n=\sum_{i=1}^{n} X_i=\sum_{i=-\infty}^{n} a_{ni}\varepsilon_i \tag{1.24}$$

and

$$g(n)=ES_n^2=\sigma^2 A_n, \qquad \text{where } A_n=\sum_{i=-\infty}^{n} a_{ni}^2. \tag{1.25}$$

We now apply Theorems 1 and 2 to obtain the following:

Corollary. *Let* $\{\varepsilon_n, \mathcal{F}_n, -\infty<n<\infty\}$ *be a sequence of martingale differences satisfying* (1.1), *and let* X_n *be a linear process defined by* (1.22). *Define* $\{a_{ni}\}$, S_n *and* $g(n)$ *by* (1.23), (1.24) *and* (1.25), *respectively.*

(i) *If* (1.2), (1.12) *and* (1.13) *hold, then*

$$\limsup_{n\to\infty}|S_n|/\{2g(n)\log\log g(n)\}^{1/2}\leqq1 \quad \text{a.s.}$$

(ii) *If* (1.13), (1.18) *and* (1.20) *hold, then*

$$\limsup_{n\to\infty}|S_n|/\{2\beta g(n)\log\log g(n)\}^{1/2}\leqq1 \quad \text{a.s.}$$

(iii) *If* (1.2) *holds, and if*

$$\sum_{i \le n \exp(-(\log n)^{\alpha})} a_{ni}^2 = o(g(n)) \qquad \textit{for all} \quad \alpha > 0 \tag{1.26}$$

and

$$\liminf_{n\to\infty} (\log\log g(n))/(\log\log n) > 0\,, \tag{1.27}$$

then for every $-1 \leqq q \leqq 1$,

$$\liminf_{n\to\infty} |\{2g(n)\log\log g(n)\}^{-1/2} S_n - q| = 0 \qquad \text{a.s.}$$

Conditions (1.12), (1.13) and (1.20) cover a wide range of correlation structures for the sequences $\{X_n\}$; see e.g., [4], [5] and [11]. Note that parts (i) and (ii) of the corollary were given by the author [11].

Proof. Note that

$$g(n-m) = \sigma^2 \left[\sum_{i=-\infty}^{m} (a_{ni} - a_{mi})^2 + \sum_{i=m+1}^{n} a_{ni}^2 \right] \qquad \text{for} \quad n > m \geqq 1\,.$$

Suppose that g satisfies (1.12) (or (1.20)) and (1.13). Then by Lemma 1 of Lai and Stout [4], $g(n) \to \infty$ as $n \to \infty$, and hence (1.5) holds. Note that (1.6) also holds (see [5], p. 327). We next suppose (1.26) and (1.27). Given $0 < \gamma < 1$, choose $1 < \delta < 1 + \gamma$ and define for $k = 1, 2, \cdots$

$$n_k = [\exp(k^{\delta})]\,.$$

Then (1.14), (1.15) and (1.16) hold (see [5], pp. 327 and 333). Hence Theorem 1 (ii)-(iii) and Theorem 2(ii) imply the corollary.

Theorem 1 (i), (iii) can be applied to extend a log log law for least squares estimates in regression models obtained by Lai and Wei [5] to the case that the disturbances form martingale differences. We can also apply Theorem 1 to extend Corollaries 1 and 2 of [5] to the martingale case. But we shall not enter into details.

2. Proofs of Theorem 1 (i)-(ii) and Theorem 2

The proofs of Theorem 1 (i)-(ii) and Theorem 2 closely follow that of Theorem 1 (i)-(ii) of [5]. Thus we only skech them. The following lemma was proved in [11].

Lemma 1. *Let* $\{\varepsilon_n, \mathcal{F}_n, -\infty < n < \infty\}$ *be a sequence of martingale differences such that* $E(\varepsilon_n^2 | \mathcal{F}_{n-1}) \leqq \sigma^2$ a.s. *for all* n, *and* (1.18) *holds. Let* $\{a_{ni}, n \geqq 1, -\infty < i \leqq n\}$ *be a double array of constants satisfying* (1.3), (1.5) *and* (1.6). *Let* $S_n = \sum_{i=-\infty}^{n} a_{ni}\varepsilon_i$. *Then for all* $\zeta > 1$ *and* $\theta > 0$, *as* $n \to \infty$

$$P[|S_n|>\zeta\sigma(2\theta A_n \log\log A_n)^{1/2}]=o(\exp(-\theta\log\log A_n)).$$

Proof of Theorem 1 (i)-(ii). Let $0<\delta<1$. By virtue of (1.2), we can cho ose $B>0$ such that

$$E[\varepsilon_i^2 I(|\varepsilon_i|>B)|\mathcal{F}_{i-1}]\leqq\delta^2\sigma^2 \qquad \text{a.s. for all } i.$$

Let

$$e_i=\varepsilon_i I(|\varepsilon_i|\leqq B)-E[\varepsilon_i I(|\varepsilon_i|\leqq B)|\mathcal{F}_{i-1}]$$

and

$$d_i=\varepsilon_i-e_i.$$

Then both $\{e_i, \mathcal{F}_i, -\infty<i<\infty\}$ and $\{d_i, \mathcal{F}_i, -\infty<i<\infty\}$ are sequences of martingale differences such that

$$E(e_i^2|\mathcal{F}_{i-1})\leqq E(\varepsilon_i^2|\mathcal{F}_{i-1})=\sigma^2 \qquad \text{a.s.}$$

and

$$E(d_i^2|\mathcal{F}_{i-1})\leqq E[\varepsilon_i^2 I(|\varepsilon_i|>B)|\mathcal{F}_{i-1}]\leqq\delta^2\sigma^2 \qquad \text{a.s.}$$

Putting e_i and d_i defined above in place of ε_i' and ε_i'', respectively, applying Lemma 1 stated above instead of Lemma 3 (ii) and making use of the Burkholder inequality [1] instead of the Marcinkiewicz-Zygmund inequality, then parts (i) and (ii) of Theorem 1 can be proved similarly to corresponding parts of Theorem 1 of [5].

Proof of Theorem 2. Let $\delta>0$. If (1.7) and (1.18) hold, then by the Burkholder inequality, we obtain that for $n>m\geqq m_0$

$$E|S_n-S_m|^r\leqq C_r(\sup_i E|\varepsilon_i|^r)\Big(\sum_{i=-\infty}^{m}(a_{ni}-a_{mi})^2+\sum_{i=m+1}^{n}a_{ni}^2\Big)^{r/2}$$

$$\leqq C_r(\sup_i E|\varepsilon_i|^r)\Big(\sum_{i=m+1}^{n}c_i\Big)^{\beta/(\beta-1)},$$

where $\beta=dr/(dr-2)$ and C_r is a positive constant depending only on r. Hence, applying Lemma 1 with $\theta=\beta+\delta$ and $\zeta=1+\delta$, and further applying Theorem 4 (i) of [5] (with $q=r$, $\lambda=\beta/(\beta-1)$ and $B_n=A_n$), we get

$$\limsup_{n\to\infty}|S_n|/(2A_n\log\log A_n)^{1/2}\leqq(1+\delta)(\beta+\delta)^{1/2}\sigma \qquad \text{a.s.}$$

Since δ is arbitrary, (1.19) follows as desired. Using Theorem 4 (ii) of [5], the proof of part (ii) of Theorem 2 follows similarly.

3. Proof of Theorem 1 (iii)

Define

(3.1) $$\varepsilon_{ni}'=\varepsilon_i I[|a_{ni}\varepsilon_i|\leqq A_n^{1/2}(\log\log A_n)^{-1}],$$

$$\varepsilon_{ni}=\varepsilon'_{ni}-E(\varepsilon'_{ni}|\mathcal{F}_{i-1}).$$

Then, $\{\varepsilon_{ni}, \mathcal{F}_i, -\infty<i\leqq n\}$ is a sequence of martingale differences such that

(3.2) $$E(\varepsilon_{ni}^2|\mathcal{F}_{i-1})\leqq\sigma^2 \quad \text{a.s. for all } i$$

and

(3.3) $$|a_{ni}\varepsilon_{ni}|\leqq 2A_n^{1/2}(\log\log A_n)^{-1} \quad \text{a.s. for all } i.$$

Let $I_k=\{n: n_{k-1}<n\leqq n_k\}$ and $B_k=\sum_{i\in I_k} a_{n_k,i}^2$. Then by (1.14).

(3.4) $$(I-\gamma+o(1))A_{n_k}\leqq B_k\leqq A_{n_k},\ \log\log B_k\sim\log\log A_{n_k}.$$

Further let $U_k=\sum_{i\in I_k} a_{n_k,i}\varepsilon_{n_k,i}$ and $Z_k=\sigma^{-1}B_k^{-1/2}U_k$. In the sequel, for simplicity of notation, we denote $a_{n_k,i}$ and $\varepsilon_{n_k,i}$ by a_{ki} and ε_{ki}, respectively. We now set down some lemmas under the assumptions of Theorem 1 (iii).

Lemma 2. *For all $\varepsilon>0$ and all large k,*

(3.5) $$E[I(Z_k>\varepsilon)|\mathcal{F}_{n_{k-1}}] \leqq\begin{cases}\exp(-\varepsilon/4c_k) & \text{a.s. } if\ \varepsilon c_k\geqq 1,\\ \exp[-(\varepsilon^2/2)(1-\varepsilon c_k/2)] & \text{a.s. } if\ \varepsilon c_k<1\end{cases}$$

where $c_k=2\sigma^{-1}(A_{n_k}/B_k)^{1/2}(\log\log A_{n_k})^{-1}$.

Proof. Assume $0<\lambda c_k\leqq 1$. Define

$$V_m^k=\exp(\lambda\sigma^{-1}B_k^{-1/2}\sum_{i=n_{k-1}+1}^{m} a_{ki}\varepsilon_{ki})\exp\left[-(\lambda^2/2)(1+\lambda c_k/2)B_k^{-1}\sum_{i=n_{k-1}+1}^{m} a_{ki}^2\right]$$

for $n_{k-1}<m\leqq n_k$, and $V_{n_{k-1}}^k=1$ a.s. It follows from (3.2) and (3.3) that

$$\begin{aligned}&E[\exp(\lambda\sigma^{-1}B_k^{-1/2}a_{ki}\varepsilon_{ki})|\mathcal{F}_{i-1}]\\ &\quad\leqq 1+(\lambda^2/2)(1+\lambda c_k/2)\sigma^{-2}B_k^{-1}a_{ki}^2E(\varepsilon_{ki}^2|\mathcal{F}_{i-1})\\ &\quad\leqq\exp[(\lambda^2/2)(1+\lambda c_k/2)B_k^{-1}a_{ki}^2] \quad \text{a.s.}\end{aligned}$$

Hence, $\{V_m^k, \mathcal{F}_m, n_{k-1}\leqq m\leqq n_k\}$ forms a supermartingale (cf. Stout [7], p. 299). Thus we get

$$E[\exp(\lambda Z_k)|\mathcal{F}_{n_{k-1}}]\leqq\exp[(\lambda^2/2)[1+\lambda c_k/2)] \quad \text{a.s.}$$

This implies that

(3.6) $$\begin{aligned}&E[I(Z_k>\varepsilon)|\mathcal{F}_{n_{k-1}}]\\ &\quad\leqq\exp(-\lambda\varepsilon)E[\exp(\lambda Z_k)|\mathcal{F}_{n_{k-1}}]\\ &\quad\leqq\exp(-\lambda\varepsilon)\exp[(\lambda^2/2)(1+\lambda c_k/2)] \quad \text{a.s.}\end{aligned}$$

Hence (3.5) follows from (3.6) by setting $\lambda=1/c_k$ if $\varepsilon c_k \geqq 1$, and $\lambda=\varepsilon$ if $\varepsilon c_k<1$.

Lemma 3. *Let $\alpha>0$ be given. Then, for all $\varepsilon>0$ and k sufficiently large,*

$$E[I(Z_k>\varepsilon)\,|\,\mathcal{F}_{n_{k-1}}] \geqq \exp[-(\varepsilon^2/2)(1+\alpha)] \quad \text{a.s.} \tag{3.7}$$

Proof. Since $E(\varepsilon_i\,|\,\mathcal{F}_{i-1})=0$ a.s., it follows from (1.1) and (1.2) that

$$\begin{aligned}\sigma^2-E(\varepsilon_{ni}^2\,|\,\mathcal{F}_{i-1}) &= E[(\varepsilon_i-\varepsilon'_{ni})^2\,|\,\mathcal{F}_{i-1}]+E^2(\varepsilon'_{ni}\,|\,\mathcal{F}_{i-1}) \\ &\leqq 2E(\varepsilon_i^2 I[\,|a_{ni}\varepsilon_i|>A_n^{1/2}(\log\log A_n)^{-1}]\,|\,\mathcal{F}_{i-1}) \\ &\leqq 2C\{(\sup_i a_{ni}^2)/A_n\}^{(r-2)/2}(\log\log A_n)^{r-2} \quad \text{a.s.}\end{aligned}$$

Hence by (1.6), for each $0<\delta<1$, there exists $N=N(\delta)$ such that

$$(1-\delta)\sigma^2<E(\varepsilon_{ni}^2\,|\,\mathcal{F}_{i-1}) \quad \text{a.s.} \tag{3.8}$$

for all $n>N$ and all $i\leqq n$.

Assume $0<\lambda c_k\leqq 1$, where c_k is defined in Lemma 2. Let $0<\delta<1$ and define

$$\begin{aligned}W_m^k &= \exp\Big(\lambda\sigma^{-1}B_k^{-1/2}\sum_{i=n_{k-1}+1}^{m} a_{ki}\varepsilon_{ki}\Big) \\ &\quad\times\exp\Big[-(\lambda^2/2)(1-\lambda c_k)(1-\delta)B_k^{-1}\sum_{i=n_{k-1}+1}^{m} a_{ki}^2\Big]\end{aligned}$$

for $n_{k-1}<m\leqq n_k$, and $W_{n_{k-1}}^k=1$ a.s. It follows from (3.3) and (3.8) that for all large k

$$\begin{aligned}&E[\exp(\lambda\sigma^{-1}B_k^{-1/2}a_{ki}\varepsilon_{ki})\,|\,\mathcal{F}_{i-1}] \\ &\quad\geqq 1+(\lambda^2/2)(1-\lambda c_k/2)\sigma^{-2}B_k^{-1}a_{ki}^2E(\varepsilon_{ki}^2\,|\,\mathcal{F}_{i-1}) \\ &\quad\geqq \exp[(\lambda^2/2)(1-\lambda c_k)(1-\delta)B_k^{-1}a_{ki}^2] \quad \text{a.s.}\end{aligned}$$

Hence, $\{W_m^k, \mathcal{F}_m, n_{k-1}\leqq m\leqq n_k\}$ forms a submartingale for each k sufficiently large. Thus we have for all large k

$$E[\exp(\lambda Z_k)\,|\,\mathcal{F}_{n_{k-1}}] \geqq \exp[(\lambda^2/2)(1-\lambda c_k)(1-\delta)] \quad \text{a.s.} \tag{3.9}$$

Since δ is arbitrary and $c_k\to 0$ as $k\to\infty$, the rest of the proof can be obtained by using (3.5) and (3.9), and arguing just as in the proof of Lemma 3 of Stout [6] (setting $\alpha_k=1$).

Lemma 4. *For all $\zeta>0$, $\xi>0$, $\theta\neq 0$ and all large k,*

$$\begin{aligned}&E\{I[(\theta-\zeta)\sigma\leqq(2B_k\log\log B_k)^{-1/2}U_k\leqq(\theta+\xi)\sigma]\,|\,\mathcal{F}_{n_{k-1}}\} \\ &\quad\geqq \exp(-\theta^2\log\log B_k) \quad \text{a.s.}\end{aligned} \tag{3.10}$$

Moreover, for all $\xi>\zeta>0$ and all large k,

(3.11)
$$E\{I[0\leqq(2B_k \log\log B_k)^{-1/2}U_k\leqq\xi\sigma]\mid\mathcal{F}_{n_{k-1}}\}$$
$$\geqq\exp(-\zeta^2 \log\log B_k) \quad \text{a.s.}$$

Proof. To prove (3.10), we only consider the case $\theta>0$. Since (3.5) and (3.7) continue to hold if we replace Z_k by $-Z_k$, the case $\theta<0$ follows similarly. Take $0<\zeta'<\zeta$ and $0<\xi'<\xi$ such that $\theta-\zeta'>0$. Applying Lemma 2 with $\varepsilon=(\theta+\xi)(2\log\log B_k)^{1/2}$, and noting that by (3.4), $\varepsilon c_k\to0$ as $k\to\infty$, we obtain that

(3.12)
$$E\{I[(2B_k \log\log B_k)^{-1/2}U_k>(\theta+\xi)\sigma]\mid\mathcal{F}_{n_{k-1}}\}$$
$$=E\{I[Z_k>(\theta+\xi)(2\log\log B_k)^{1/2}]\mid\mathcal{F}_{n_{k-1}}\}$$
$$\leqq\exp[-(\theta+\xi')^2\log\log B_k] \quad \text{a.s.}$$

for all large k. Take $\alpha>0$ such that $(\theta-\zeta')^2(1+\alpha)<\theta^2$. Then by Lemma 3,

(3.13)
$$E\{I[(2B_k \log\log B_k)^{-1/2}U_k>(\theta-\zeta')\sigma]\mid\mathcal{F}_{n_{k-1}}\}$$
$$=E\{I[Z_k>(\theta-\zeta')(2\log\log B_k)^{1/2}]\mid\mathcal{F}_{n_{k-1}}\}$$
$$\geqq\exp[-(\theta-\zeta')^2(1+\alpha)\log\log B_k]$$
$$\geqq2\exp(-\theta^2\log\log B_k) \quad \text{a.s.}$$

for all large k. From (3.12) and (3.13), (3.10) follows.

To prove (3.11), taking $\zeta<\tilde{\zeta}<\xi$, we obtain as in (3.12) that

(3.14)
$$E\{I[(2B_k \log\log B_k)^{-1/2}U_k>\xi\sigma]\mid\mathcal{F}_{n_{k-1}}\}$$
$$\leqq\exp(-\tilde{\zeta}^2\log\log B_k) \quad \text{a.s.}$$

for all large k. Take $0<\tau<\zeta$. Then as in (3.13),

(3.15)
$$E\{I[(2B_k \log\log B_k)^{-1/2}U_k>\tau\sigma]\mid\mathcal{F}_{n_{k-1}}\}$$
$$\geqq2\exp(-\zeta^2\log\log B_k) \quad \text{a.s.}$$

for all large k. From (3.14) and (3.15), (3.11) follows.

Proof of Theorem 1 (iii). We first note that

(3.16)
$$\sum_{k=1}^{\infty}P[|a_{ki}\varepsilon_i|\geqq A_{n_k}^{1/2}(\log\log A_{n_k})^{-1} \text{ for some } i]<\infty$$

(see [5]). By (1.16),

(3.17)
$$\log\log A_{n_k}\geqq d\log k \qquad \text{for all large } k \text{ and some } d>0.$$

Take $0<\gamma<d^2$. We now show that

(3.18) $P[|\sum_{i\notin I_k} a_{ki}\varepsilon_{ki}|\leqq\sigma\gamma^{1/4}(1+\gamma)(2A_{n_k}\log\log A_{n_k})^{1/2}$ for all large $k]=1$.

Let A, c and λ be positive constants such that

$$\sigma^2\sum_{i\notin I_k} a_{ki}^2\leqq A,\ |a_{ki}\varepsilon_{ki}|\leqq A^{1/2}c \text{ a.s. for all } i\notin I_k,\ \lambda c\leqq 1.$$

Define

$$T_k=\exp(\lambda A^{-1/2}\sum_{i\notin I_k} a_{ki}\varepsilon_{ki})\exp[-(\lambda^2/2)(1+\lambda c/2)A^{-1}\sigma^2\sum_{i\notin I_k} a_{ki}^2],$$

$$T_{j,l}=\exp\Big(\lambda A^{-1/2}\sum_{i=l}^{j} a_{ki}\varepsilon_{ki}\Big)\exp\Big[-(\lambda^2/2)(1+\lambda c/2)A^{-1}\sigma^2\sum_{i=l}^{j} a_{ki}^2\Big]$$

for $l\leqq j\leqq n_{k-1}$, and $T_{l-1,l}=1$ a.s. Then, as in the proof of Lemma 2, $\{T_{j,l}, \mathcal{F}_j, l-1\leqq j\leqq n_{k-1}\}$ forms a supermartingale, and hence $ET_{n_{k-1},l}\leqq 1$ for all $l\leqq n_{k-1}$. On the other hand, $T_{n_{k-1},l}\to T_k$ a.s. as $l\to-\infty$. Therefore, by the Fatou lemma,

$$ET_k\leqq\sup_{l\leq n_{k-1}} ET_{n_{k-1},l}\leqq 1,$$

which implies that

(3.19)
$$\begin{aligned}E[\exp(\lambda A^{-1/2}\sum_{i\notin I_k} a_{ki}\varepsilon_{ki})] &\leqq\exp[(\lambda^2/2)(1+\lambda c/2)A^{-1}\sigma^2\sum_{i\notin I_k} a_{ki}^2]\\ &\leqq\exp[(\lambda^2/2)(1+\lambda c/2)].\end{aligned}$$

By (1.14), $\sum_{i\notin I_k} a_{ki}^2\leqq\gamma(1+\gamma)^2A_{n_k}$ for all large k. Hence, putting $A=\sigma^2\gamma(1+\gamma)^2A_{n_k}$, $c=2\sigma^{-1}\gamma^{-1/2}(1+\gamma)^{-1}(\log\log A_{n_k})^{-1}$ and $\lambda=\gamma^{-1/4}(2\log\log A_{n_k})^{1/2}$ in the above argument, and noting that $\lambda c\to 0$ as $k\to\infty$, we obtain from (3.17) and (3.19) that

(3.20)
$$\begin{aligned}&P[\sum_{i\notin I_k} a_{ki}\varepsilon_{ki}>\sigma\gamma^{1/4}(1+\gamma)(2A_{n_k}\log\log A_{n_k})^{1/2}]\\ &\leqq\exp(-2\gamma^{-1/2}\log\log A_{n_k})\\ &\quad\times E\{\exp[\sigma^{-1}\gamma^{-3/4}(1+\gamma)^{-1}A_{n_k}^{-1/2}(2\log\log A_{n_k})^{1/2}\sum_{i\notin I_k} a_{ki}\varepsilon_{ki}]\}\\ &\leqq\exp[-\gamma^{-1/2}(1+o(1))\log\log A_{n_k}]\\ &\leqq\exp[-(\gamma^{-1/2}d+o(1))\log k]\end{aligned}$$

for all large k. Note that (3.19) (and therefore (3.20) as well) also holds with ε_{ki} replaced by $-\varepsilon_{ki}$. Thus we get

(3.21)
$$\begin{aligned}&P[|\sum_{i\notin I_k} a_{ki}\varepsilon_{ki}|>\sigma\gamma^{1/4}(1+\gamma)(2A_{n_k}\log\log A_{n_k})^{1/2}]\\ &\leqq 2\exp[-(\gamma^{-1/2}d+o(1))\log k]\end{aligned}$$

for all large k. Since $d>\gamma^{1/2}$, (3.18) follows from (3.21) and the Borel-Cantelli lemma.

we next show that all $\tau>0$

$$P\left[\left|\sum_{i=-\infty}^{n_k} a_{ki}E(\varepsilon'_{ki}\mid\mathcal{F}_{i-1})\right|\leqq\tau(A_{n_k}\log\log A_{n_k})^{1/2}\text{ for all large } k\right]=1. \tag{3.22}$$

By (1.6), for all $\tau>0$ and $\theta>0$,

$$\begin{aligned}P\left[\left|\sum_{i=-\infty}^{n_k} a_{ki}E(\varepsilon'_{ki}\mid\mathcal{F}_{i-1})\right|>\tau(A_{n_k}\log\log A_{n_k})^{1/2}\right] \\ \leqq\tau^{-1}(A_{n_k}\log\log A_{n_k})^{-1/2}E\left|\sum_{i=-\infty}^{n_k} a_{ki}E(\varepsilon'_{ki}\mid\mathcal{F}_{i-1})\right| \\ \leqq\tau^{-1}(A_{n_k}\log\log A_{n_k})^{-1/2}\sum_{i=-\infty}^{n_k}E\{|a_{ki}\varepsilon_i|I[|a_{ki}\varepsilon_i|>A_{n_k}^{1/2}(\log\log A_{n_k})^{-1}]\} \\ \leqq\tau^{-1}(\sup_i E|\varepsilon_i|^r)\{(\sup_i a_{ki}^2)/A_{n_k}\}^{(r-2)/2}(\log\log A_{n_k})^{r-3/2} \\ =o(\exp(-\theta\log\log A_{n_k}).\end{aligned} \tag{3.23}$$

Hence (3.22) follows from (3.17), (3.23) and the Borel-Cantelli lemma.
By (1.15) and (3.4),

$$\text{og}\log B_k\leqq(1+\gamma)^2\log k\qquad\text{for all large } k. \tag{3.24}$$

In view of (3.24), we can apply Lemma 4 and a conditional version of the Borel-Cantelli lemma (see Doob [2], p. 323) to obtain that for every $-1\leqq\theta\leqq1$ and $\eta>0$,

$$P[|(2B_k\log\log B_k)^{-1/2}U_k-(1+\gamma)^{-1}\theta\sigma|\leqq\eta\text{ i.o.}]=1. \tag{3.25}$$

Assume $0\leqq\theta\leqq1$ (the case $-1\leqq\theta<0$ follows similarly). By (3.4) and (3.24), for all large k,

$$A_{n_k}\log\log A_{n_k}\geqq B_k\log\log B_k\geqq(1-\gamma)^2A_{n_k}\log\log A_{n_k}. \tag{3.26}$$

By (3.25) and (3.26),

$$\begin{aligned}P[(1-\gamma')\{(1+\gamma)^{-1}\theta\sigma-\eta\}\leqq(2A_{n_k}\log\log A_{n_k})^{-1/2}U_k \\ \leqq(1+\gamma)^{-1}\theta\sigma+\eta\text{ i.o.}]=1,\end{aligned} \tag{3.27}$$

where $\gamma'=\gamma$ or 0 according as $(1+\gamma)^{-1}\theta\sigma-\eta>0$ or $\leqq0$. Consequently, from (3.16), (3.18), (3.22) and (3.27) it follows that

$$\begin{aligned}P[(1-\gamma')\{(1+\gamma)^{-1}\theta\sigma-\eta\}-\sigma\gamma^{1/4}(1+\gamma)-2^{-1/2}\tau \\ \leqq(2A_n\log\log A_n)^{-1/2}S_n \\ \leqq(1+\gamma)^{-1}\theta\sigma+\eta+\sigma\gamma^{1/4}(1+\gamma)+2^{-1/2}\tau\text{ i.o.}]=1.\end{aligned} \tag{3.28}$$

Since γ, η and τ are arbitrary, (1.17) follows from (3.28).

Note added in proof. In this paper, we made a restriction to the one-sided case ($a_{ni}=0$, $i>n$), but it is not essential. It is easy to see that similar methods can be applied to obtain results for the two-sided case. Theorem 1(i)-(ii) and Theorem 2 continue to hold for the two-sided case if (1.3)-(1.7) and (1.10) are replaced by the corresponding ones of [5]. For Theorem 1(iii), however, the assumption (1.14) must be replaced by (1.13) of [5] in which the subsets I_k of integers satisfy $\max\{n: n\in I_k\}<\min\{n: n\in I_{k+1}\}$ for $k\geqq 1$. In [5], only the disjointness of I_k's was required. Theorem 1 of [5] then holds for martingale differences satisfying (1.1) and (1.2) under the additional assumption on I_k stated above.

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