YOKOHAMA MATHEMATICAL
JOURNAL VOL. 33, 1985

ASYMPTOTIC NORMALITY OF A RECURSIVE STOCHASTIC ALGORITHM WITH OBSERVATIONS SATISFYING SOME ABSOLUTE REGULARITY CONDITION

By

KEN-ICHI YOSHIHARA and HIROSHI TAKAHATA

(Received May 2, 1985)

1. Introduction.

Let $\{\xi_n, -\infty<n<\infty\}$ be a zero mean strictly stationary process defined on a probability space $(\Omega, \mathcal{B}, P)$. Define the process $\{y_n\}$ by

$$y_n=\theta+\xi_n \tag{1.1}$$

where θ is an unknown parameter. In real time problems, as an estimate of θ, it is preferable to use "recursive M-estimator" of a stochastic approximation type. The reason is explained in [2]. When $\xi_n's$ are not necessarily independent, the asymptotic properties are less known. Holst [2] studied the properties of a simple recursive estimator of θ in the case when the sequence $\{\xi_n\}$ is m-dependent and conjectured that his results will also hold under some condition of a mixing type instead of the condition on m-dependence.

In this paper, we show that Holst's conjecture is valid when $\{\xi_n\}$ is absolutely regular, i.e., $\{\xi_n\}$ satisfies the condition

$$\beta(n)=E\{\sup_{B\in\mathcal{F}_n^\infty}|P(A|\mathcal{F}_{-\infty}^0)-P(A)|\}\longrightarrow 0 \qquad (n\to\infty) \tag{1.2}$$

where $\mathcal{F}_a^b$ denotes the σ-algebra generated by $\xi_a, \cdots, \xi_b$ $(a\leqq b)$.

2. Results.

Let $\{y_n\}$ be the process defined by (1.1). Let $\{x_n\}$ be the recursive estimator of θ defined by

$$\begin{cases} x_{n+1}=x_n+(n+1)^{-1}\psi(y_{n+1}-x_n) & (n\geqq 1) \\ x_0 \text{ is arbitrary.} \end{cases} \tag{2.1}$$

Let

$$f(x)=E\psi(\xi_k-x). \tag{2.2}$$

We consider the following assumptions:

A1. ψ is bounded, i.e., $|\psi(x)| \leqq K$ for all x where K is a positive constant.

A2. ψ is nonincreasing and has at most finitely many discontinuity points.

A3. $f(x) = -\gamma x + \delta(x)$ where $\gamma > 1/2$ and $\delta(x) = o(x)$ as $x \to 0$.

A4. f satisfies a Lipschitz condition, i.e.,

(2.3) $$|f(x_1) - f(x_2)| \leqq M_0 |x_1 - x_2|.$$

A5. $\{\xi_n\}$ is a strictly stationary, absolutely regular sequence of random variables such that $E\xi_0 = 0$ and $\beta(n) = O(e^{-\lambda n})$ for some $\lambda > 0$.

Now, for any m and for $n(\geqq m+1)$ let

(2.4) $$V_n = \psi(\xi_n - x_{n-1}) - E\{\psi(\xi_n - x_{n-1}) \mid \mathcal{F}_1^{n-m-1}\}$$

and

(2.5) $$Z_n = E\{\psi(\xi_n - x_{n-1}) \mid \mathcal{F}_1^{n-m-1}\} - f(x_{n-m-1}).$$

Then, there exists a constant L such that

$$|V_n| \leqq L \quad \text{a.s.}$$

if A1 is satisfied.

Theorem 1. *If $f(x) = 0$ only for $x = 0$ and if* A1, A2, A4 *and* A5 *hold, then for any $T(>0)$, $\varepsilon(>0)$ and n sufficiently large*

(2.6) $$P(|x_n - \theta| \geqq \varepsilon) \leqq \exp\left[-an^{L^{-1}\min(L, -f(T), f(-T)) - \rho}\right]$$

where ρ is an arbitrary positive number such that

$$0 < \rho < L^{-1} \min(L, -f(T), f(-T)).$$

Remark 1. From Theorem 1 it is easily follows that $x_n \to \theta$ with probability one if conditions of Theorem 1 are satisfied.

Theorem 2. *Assume that* A1-A5 *hold. Then*

(2.7) $$E|x_n - \theta|^{2p} = O(n^{-p'})$$

for all $p'(\theta < p' < p)$.

Theorem 3. *Assume that* A1-A2 *hold. Then*

(2.8) $$\sqrt{n}(x_n - \theta) \longrightarrow N(0, (2\gamma - 1)^{-1}\sigma^2)$$

where γ is the one defined in A3 *and*

(2.9) $$\sigma^2 = r(0) + 2\sum_{j=1}^{\infty} r(j) > 0$$

with

$$r(j)=E\psi(\xi_0)\psi(\xi_j). \tag{2.10}$$

Remark 2. It is known that if A1 and A5 hold, then the series in the right hand side of (2.9) converges absolutely.

In proofs, c, with or without subscript, will be used as a positive constant whose value is not always same, $[s]$ denotes the largest integer m such that $m\leqq s$ and $I(s)$ denotes the indicator of the set A.

3. Auxiliary lemmas.

In this and following sections, we assume that $\{\xi_n\}$ is absolutely regular. The next lemma is well known.

Lemma A. *If X is $\mathfrak{F}_{-\infty}^{k}$-measurable and $|X|\leqq c_1$ and if Y is $\mathfrak{F}_{k+n}^{\infty}$-measurable and $|Y|\leqq c_2$, then*

$$|EXY-EX\,EY|\leqq 4c_1c_2\beta(n). \tag{3.1}$$

Lemma 3.1. *Let $h(x, y)$ be a Borel measurable function such that $|h(x, y)|\leqq K_0$ for all x and y. Let X be an $\mathfrak{F}_1^k$-measurable random variable and let Y be an $\mathfrak{F}_{k+m}^{n}$-measurable random variable. Further, let $H(x)=Eh(x, Y)$. Then*

$$E|E\{h(X, Y)|\mathfrak{F}_1^k\}-H(X)|\leqq 2K_0\beta(m). \tag{3.2}$$

Proof. Let Q and R be probability distributions of X and Y, respectively. Let $\bar{P}$ be the joint distribution (X, Y) and $P(y|z)$ a regular conditional probability distribution of Y given $X=x$. Then

$$\text{R.H.S of (3.2)}=\int\left|\int h(x, y)P(dy|x)-\int h(x, y)R(dy)\right|Q(dx)$$

$$\leqq K_0\iint|P(dy|x)-R(dy)|Q(dx)$$

$$=K_0\,\mathrm{Var}\,[\bar{P}-Q\times R]$$

where $\mathrm{Var}\,[\bar{P}-Q\times R]$ denotes the total variation of $\bar{P}-Q\times R$. But, in [4] it was proved that

$$\mathrm{Var}\,[\bar{P}-Q\times R]=2\beta(m).$$

Hence, we have (3.2)

In proofs of Theorems, without loss of generality, we assume that $\theta=0$ and so instead of (2.1) we consider

$$\begin{cases} x_{n+1}=x_n+(n+1)^{-1}\psi(\xi_{n+1}-x_n), \\ x_0 \text{ is arbitrary.} \end{cases} \tag{3.3}$$

To prove Theorems, above, proof of theorems in [1] are examined and, where needed, slightly changed, but the ideas are those of [1].

Now, for $j\geqq m+1$ and $1\leqq i\leqq m$, let

$$\eta_{j,i}=E\{\phi(\xi_j-x_{j-1})\,|\,\mathfrak{F}_1^{j-i}\}-E\{\phi(\xi_j-x_{j-1})\,|\,\mathfrak{F}_1^{j-i-1}\} \tag{3.4}$$

Then, by (2.4) and (3.4)

$$\sum_{j=k}^{n} j^{-1}V_j=\sum_{i=1}^{m}\left\{\sum_{j=k}^{n} j^{-1}\eta_{j,i}\right\} \tag{3.5}$$

and each summand in the right hand side of (3.5) is a sum of martingale differences. So, by the technique in [3] we have the following lemmas which correspond to the ones in [1].

Lemma 3.2. *Let* $m(\geqq 1)$ *and* $a(\geqq 1)$ *be arbitrary. Let* $k\geqq m+2$. *Assume* $0\leqq C\leqq(m+1)La\log a$. *If* A1 *and* A5 *hold, then*

$$P\Big(\sum_{j=k}^{n} j^{-1}V_j\geqq C\Big)\leqq(m+1)\exp\left\{-\frac{C^2(k-m-2)}{2aL^2(m+1)^2}\right\}, \tag{3.6}$$

Proof. The proof is similar to that of Lemma 1 in [3] and so is omitted

Next, let a_0 be a positive constant such that $a_0\lambda>3$ where λ is the one in A5.

Lemma 3.3. *Let* n *be sufficiently large and* $m=[a_0\log n]$. *Suppose that* $0\leqq C\leqq 8L\log 2$, A1 *and* A5 *hold.*

(i) *If* D *is a positive constant, then for any* k $(m+2\leqq k\leqq a_1n)$

$$P\Big(-\sum_{j=k}^{n} j^{-1}V\leqq -C\sum_{j=k}^{n} j^{-1}+D\Big)\leqq\exp(-a_2n^{C/L-\rho}) \tag{3.7}$$

where a_1 *and* a_2 *are positive constants and* $\rho(0<\rho<C/L)$ *is arbitrary.*

(ii) *If* $D=O(\log\log n)$, *then* (3.7) *hold for all* k, $(m+2\leqq k\leqq(\log n)^3)$.

Proof. We use the technique in [3]. (3.7) is obvious if $C>L$.

Firstly, we consider case (i). Suppose that $0\leqq C\leqq L$ and put

$$k_n=[e^{-4}e^{-(D/L)}k_1{}^{-(C/L)}n^{C/L}].$$

If we put

$$a_1=\exp\left\{-\Big(4+\frac{D}{C}\Big)\frac{L}{C}\right\},$$

then it is clear that $m+2\leqq k\leqq a_1n$ implies $m+2\leqq k\leqq k_n$.

By the definition of k_n and A1 we have that for any $t(0\leqq t\leqq L^{-1}j\log 2)$

$$P\Big(\sum_{j=k}^{n} j^{-1}V_j\leqq -C\sum_{j=k}^{n} j^{-1}+D\Big)$$

$$\leqq P\Big(\sum_{j=k}^{k_n-1} j^{-1}V_j\leqq -C\sum_{j=k}^{n} j^{-1}+D+2L\Big)+P\Big(\sum_{j=k_n}^{n} j^{-1}V_j\leqq -2L\Big)$$

$$=P\Big(-\sum_{i=1}^{m}\sum_{j=k_n}^{n} j^{-1}\eta_{j,i}\geqq 2L\Big)$$

$$\leqq \sum_{i=1}^{m} P\Big(\sum_{j=k_n}^{n} j^{-1}\eta_{j,i}\geqq (2L)/(m+1)\Big)$$

$$\leqq (m+1)e^{-2tL/(m+1)}\exp\Big(t^2L^2\sum_{j=k_n}^{n} j^{-2}\Big).$$

Putting

$$t=\Big\{(m+1)L\sum_{j=k_n}^{n} j^{-2}\Big\}^{-1}$$

and noting $m=O(\log n)$, we have (3.7).

The proof of (ii) is carried out analogously and so omitted.

For any $m(\geqq 1)$, let

$$h_n=\sum_{k=n-m}^{n} k^{-1}\psi(\xi_k-x_{k-1}) \tag{3.8}$$

Lemma 3.4. *Let* $m=[a_0\log a]$. *Then, there exists a positive constant* b *such*

$$|Z_n|\leqq bn^{-1}\log n \qquad \text{a.s.} \tag{3.9}$$

where Z_n *is the one defined by* (2.5).

Proof. Firstly, we show that for any positive constant C

$$\begin{aligned}|E\{\psi(\xi_n-x_{n-m-1}+cn^{-1}\log n)\,|\,\mathcal{F}_1^{n-m-1}\}\\ -f(x_{n-m-1}-cn^{-1}\log n)|\leqq n^{-1} \qquad \text{a.s.}\end{aligned} \tag{3.10}$$

Since by Lemma 3.1, A1 and A5

$$\begin{aligned}P(|E\{\psi(\xi_n-x_{n-m-1}+cn^{-1}\log n)\,|\,\mathcal{F}_1^{n-m-1}\}\\ -f(x_{n-m-1}-cn^{-1}\log n)|>n^{-1})\\ \leqq nE|E\{\psi(\xi_n-x_{n-m-1}+cn^{-1}\log n)\,|\,\mathcal{F}_1^{n-m-1}\}\\ -f(x_{n-m-1}-cn^{-1}\log n)|\\ \leqq 2Mn\beta(m)\leqq 2Mn^{-2},\end{aligned}$$

so by Borel-Cantelli's lemma we have (3.10).

Now, by (3.1) and (3.8)

$$x_n=x_{n-m}+h_n \tag{3.11}$$

with

$$|h_n|\leqq Km(n-m-1)^{-1}\leqq cn^{-1}\log n \qquad \text{a.s.}$$

since ψ is bounded. Hence, using A2, A5, A4 and (3.17) we have

$$\begin{aligned}&E\{\psi(\xi_n-x_{n-1})\,|\,\mathcal{F}_1^{n-m-1}\}\\&\quad=E\{\psi(\xi_n-x_{n-m-1}-h_{n-1})\,|\,\mathcal{F}_1^{n-m-1}\}\\&\quad\leqq E\{\psi(\xi_n-x_{n-m-1}+cn^{-1}\log n)\,|\,\mathcal{F}_1^{n-m-1}\}\\&\quad\leqq f(x_{n-m-1}-cn^{-1}\log n)+n^{-1}\\&\quad\leqq f(x_{n-m-1})+cn^{-1}\log n \qquad \text{a.s.}\end{aligned}$$

and similarly we have

$$E\{\psi(\xi_n-x_{n-1})\,|\,\mathcal{F}_1^{n-m-1}\}\geqq f(x_{n-m-1})-cn^{-1}\log n \qquad \text{a.s.}$$

Thus, therproof is completed.

4. Proofs of Theorems 1 and 2.

Proof of Theorem 1. In what follows, a_0 is the one in the preceding section. Let $m=[a_0\log n]$. Let $\delta(0<\delta<1)$ be a number such that $[n\delta]\geqq m+1$ and define the events

$$B=\left\{\min_{0\leq j\leq n-[\delta n]-1}x_{[\delta n]-m+j}\geqq\frac{\varepsilon}{2},\ x_n\geqq\varepsilon\right\}$$

$$A_{m+1}=\left\{\min_{1\leq j\leq n-m-1}x_j\geqq\frac{\varepsilon}{2},\ x_n\geqq\varepsilon\right\}$$

$$A_k=\left\{x_{k-m-1}<\frac{\varepsilon}{2},\ \min_{k-m\leq j\leq n-m-1}x_j\geqq\frac{\varepsilon}{2},\ x_n\geqq\varepsilon\right\}$$
$$(k=m+2,\ m+3,\ \cdots,\ n-1)$$

$$A_n=\left\{x_{n-m-1}<\frac{\varepsilon}{2},\ x_n\geqq\varepsilon\right\}.$$

By (2.4) and (2.5) we can write x_n as

$$\begin{aligned}x_n&=x_{n-1}+n^{-1}\psi(\xi_n-x_{n-1})\\&=x_{n-1}+n^{-1}V_n+n^{-1}Z_n+n^{-1}f(x_{n-m-1})\\&=x_k+\sum_{j=1}^{n}j^{-1}V_j+\sum_{j=k+1}^{n}j^{-1}Z_j+\sum_{j=k+1}^{n}j^{-1}f(x_{j-m+1})\\&\quad(m+1\leqq k\leqq n-1).\end{aligned}\tag{4.1}$$

Putting $k=[\delta n]$ and using the fact that f is nonincreasing we have

$$P(x_{[\delta n]}\leqq 2T,\ B)$$
$$\leqq P\Big(x_{[\delta n]}\leqq 2T,\ \min_{[\delta n]-m\leq j\leq n-m-1}x_j\geqq\frac{\varepsilon}{2},$$

$$x_{[\delta n]}+\sum_{j=[\delta n]+1}^{n} j^{-1}V_j+\sum_{j=[\delta n]+1}^{n} j^{-1}Z_j+\sum_{j=[\delta n+1}^{n} j^{-1}f(x_{j-m-1})\leqq\varepsilon\Big)$$

$$\leqq P\Big(-\sum_{j=[\delta n]+1} j^{-1}V_j\leqq 2T-\varepsilon+f\Big(\frac{\varepsilon}{2}\Big)\sum_{j=[\delta n]+1}^{n} j^{-1}+\sum_{j=[\delta n]+1}^{n} j^{-1}Z_j\Big).$$

Since by Lemma 3.4

$$\Big|\sum_{j=k}^{n} j^{-1}Z_j\Big|\leqq ck^{-1}$$

and $f(\varepsilon/2)<0$, we can choose δ so small that

$$2T+f\Big(\frac{\varepsilon}{2}\Big)\sum_{j=[\delta n]+1}^{n} j^{-1}+\sum_{j=[\delta n]+1}^{n} j^{-1}Z_j<-1$$

for all n sufficiently large. Hence, using Lemma 3.2

$$P(x_{[\delta n]}\leqq 2T,\ B)\leqq P\Big(-\sum_{j=[\delta n]+1}^{n} j^{-1}V_j<-1\Big)$$

(4.2)

$$\leqq c_1\exp(-c_2 n).$$

Proceeding as above and taking into account of the fact that $f(x_{j-m-1})<0$ for j $(k+1\leqq j\leqq n,\ [\delta n]+1\leqq k\leqq n-1)$, we obtain

$$P(x_n\geqq\varepsilon,\ B)\leqq\sum_{k=[\delta n]+1}^{n} P(A_k)$$

$$\leqq\sum_{k=[\delta n]+1}^{n} P\Big(x_{k-m-1}<\frac{\varepsilon}{2},\ -\sum_{j=k+1}^{n} j^{-1}V_j\leqq x_k-\varepsilon+\sum_{j=k+1}^{n} j^{-1}Z_j\Big).$$

We note that if $x_{k-m-1}<\varepsilon/2$ and k is sufficiently large, then

$$x_k=x_{k-m-1}+\sum_{j=k-m}^{k} j^{-1}\psi(\xi_j-x_{j-1})$$

$$\leqq x_{k-m-1}+K(m+1)(k-m)^{-1}<\frac{3\varepsilon}{4}$$

and

$$\sum_{j=k+1}^{n} j^{-1}Z_j<\frac{\varepsilon}{8}\qquad\text{a.s.}$$

Hence, by Lemma 3.2

$$P(x_n\geqq\varepsilon,\ B)\leqq\sum_{k=[\delta n]+1}^{n} P(A_k)$$

(4.3)

$$\leqq\sum_{k=[\delta n]+1}^{n} P\Big(-\sum_{j=k+1}^{n} j^{-1}V_j\leqq-\frac{\varepsilon}{8}\Big)$$

$$\leqq \sum_{k=[\delta n]+1}^{n} c_1 \exp(-c_2 k)$$

for all n sufficiently large.

Finally, to estimate $P(x_{[\delta n]}>2T)$, we put

$$A'_{m+1}=\{\min_{1\leq j\leq [\delta n]-m-1} x_j \geqq T,\ x_{[\delta n]}>2T\},$$

$$A'_m=\{\min_{k\leq j\leq [\delta n]-m-1} x_{j-m-1} \geqq T,\ x_{[\delta n]}>2T\}$$

$$(k=m+2,\ m+3,\ \cdots,\ [\delta n]-1),$$

and

$$A'_{[\delta n]}=\{x_{[\delta n]-m-1}<T,\ x_{[\delta n]}>2T\}.$$

For n sufficiently large, let $k_1=[b(\log n)^2]$ such that for $k\geqq k_1$

$$x_k \leqq x_{k-m-1}+K(m+1)(k-m)^{-1} \leqq x_{k-m-1}+3T/4$$

where T is an arbitrarily given positive number. Then for $k(m+2\leqq k\leqq k_1)$

$$|x_k| \leqq |x_0|+K\sum_{i=1}^{k_1} j^{-1} \leqq c \log\log n$$

and so by Lemma 3.4

$$\left| x_k-2T+\sum_{j=k+1}^{[\delta n]} j^{-1}Z_j \right|$$

$$\leqq |x_k|+2T+b\sum_{j=k+1}^{[\delta n]} j^{-2}\log j$$

$$\leqq c_2 \log\log n \qquad \text{a.s.}$$

where $0<\delta<1$ and c_2 is some positive constant. Hence, by Lemma 3.3 (ii)

$$P(A'_k) \leqq P\Big(x_{k-m-1}<T,\ -\sum_{j=k+1}^{[\delta n]} j^{-1}V_j \leqq f(T)\sum_{j=k+1}^{[\delta n]} j^{-1}+x_k-2T+\sum_{j=k+1}^{[\delta n]} j^{-1}Z_j\Big)$$

$$\leqq P\Big(-\sum_{j=k+1}^{[\delta n]} j^{-1}V_j \leqq f(T)\sum_{j=k+1}^{[\delta n]} j^{-1}+c_2\log\log n\Big)$$

$$\leqq c_3 \exp(-c_4 n^{-f(T)/L-\rho})$$

for any ρ $(0<\rho<-f(T)/L)$ and k $(m+2\leqq k<k_1)$. We note that

$$P(A'_{m+1}) \leqq c_5 \exp(-c_6 n^{-f(T)/L-\rho}).$$

For any $k\,(k_1\leqq k\leqq [a_1\delta n]$, a_1 being some constant obtained from Lemma 3.3 (i)).

$$P(A'_k) \leqq P\Big(-\sum_{j=k+1}^{[\delta n]} j^{-1}V_j=f(T)\sum_{j=k+1}^{[\delta n]} j^{-1}-\frac{T}{4}+\sum_{j=k+1}^{[\delta n]} j^{-1}Z_j\Big)$$

$$\leqq c_7 \exp(-c_8 n^{-f(t)/L})$$

using Lemma 3.3 (i) and Lemma 3.4.

Finally, if $k \geqq [a_1 \delta n]$ we have

$$\sum_{j=k+1}^{[\delta n]} j^{-1} Z_j \leqq c n^{-1} \log n \qquad \text{a.s.}$$

and so for all n sufficiently large

$$\sum_{j=k+1}^{[\delta n]} j^{-1} Z_j < \frac{T}{8} \qquad \text{a.s.}$$

Hence, by Lemma 3.3 (i)

$$P(A_k') \leqq P\Big(-\sum_{j=k+1}^{[\delta n]} j^{-1} V_j \leqq -\frac{T}{8}\Big) \leqq c_1 \exp(-c_2 n).$$

Further

$$P(A'_{[\delta n]})=0$$

for n sufficiently large. Consequently, we have

$$P(x_{[\delta n]} > 2T) \leqq \exp(-c_1 n^{-f(T)/L-\rho}) + \exp(-c_2 n) \tag{4.4}$$

for all n sufficiently large.

Thus, it follows from (4.2)-(4.4) that

$$\begin{aligned} P(x_n \geqq \varepsilon) &\leqq P(x_{[\delta n]} > 2T) + P(x_{[\delta n]} \leqq 2T,\ B) + P(x_n \geqq \varepsilon,\ \bar{B}) \\ &\leqq \exp(-c_1 n^{-f(T)/L-\rho}) + \exp(-c_2 n) \end{aligned} \tag{4.5}$$

for all n sufficiently large.

Similarly, we have

$$P(x_n \leqq -\varepsilon) \leqq \exp(-c_3 n^{f(-T)/L-\rho}) + \exp(-c_4 n). \tag{4.6}$$

Thus, the proof is completed.

Proof of Theorem 2. As before, let $m=[a_0 \log n]$. We use Holst's method in [1]. The theorem is proved by induction. We note that

$$\begin{aligned} x_n^2 = x_{n-1}^2 &+ n^{-2}\psi(\xi_n - x_{n-1}) + 2n^{-1} x_{n-1} \\ &+ 2n^{-1} x_{n-1}\{\psi(\xi_n - x_{n-1}) - f(x_{n-1})\} + 2n^{-1} x_{n-1} f(x_{n-1}) \end{aligned} \tag{4.7}$$

By (3.11)

$$x_{n-1} = x_{n-m-1} + h_{n-1}$$

where

$$|h_{n-1}| \leqq c n^{-1} \log n \qquad \text{a.s.}$$

as in the proof of Lemma 3.4, and by (2.4) and (2.5)

$$x_{n-1}\{\psi(\xi_n - x_{n-1}) - f(x_{n-1})\}$$

$$(4.8)\qquad =x_{n-2m}V_n+x_{n-2m}Z_n+x_{n-2m}\{f(x_{n-2m})-f(x_{n-1})\}$$
$$+h_{n-1}\{\psi(\xi_n-x_{n-1})-f(x_{n-1})\}.$$

We remark that $EV_n=0$, x_{n-2m} is $\mathcal{F}_1^{n-2m}$-measurable and V_n is $\mathcal{F}_{n-m}^n$-measurable, so by Lemma A and the fact that $|x_n|\leqq c\log n$ a.s.

$$|Ex_{n-2m}V_n|\leqq|Ex_{n-2m}||EV_n|+c\rho(m)\log n\leqq cn^{-3}.$$

We note that Lemma 3.4

$$|Z_n|\leqq cn^{-1}\log n \qquad \text{a.s.}$$

and by A1

$$|h_{n-1}\{\psi(\xi_n-x_{n-1})-f(x_{n-1}))\}|\leqq cn^{-1}\log n \qquad \text{a.s.}$$

Furthermore, by A4

$$|f(x_{n-m-1})-f(x_{n-1})|\leqq M|x_{n-m-1}-x_{n-1}|$$
$$\leqq M|h_{n-1}|\leqq cn^{-1}\log n \qquad \text{a.s.}$$

we have

$$Ex_n^2\leqq Ex_{n-1}^2+2n^{-1}E(x_{n-1}f(x_{n-1}))$$
$$+cn^{-2}\log nE|x_{n-m-1}|+cn^{-2}\log n.$$

By A3 there are constants $\varepsilon>0$ and γ' such that $xf(x)\leqq-\gamma'x^2$ for $|x|<\varepsilon$ and $\gamma>\gamma'>1/2$. Since $xf(x)\leqq 0$, so

$$E(x_{n-1}f(x_{n-1}))\leqq E(x_{n-1}f(x_{n-1})I(|x_{n-1}|<\varepsilon))$$
$$\leqq-\gamma'E(x_{n-1}^2I(|x_{n-1}|<\varepsilon))$$
$$=-\gamma'E(x_{n-1}^2)+\gamma'E(x_{n-1}^2I(|x_{n-1}|\geqq\varepsilon)).$$

As for all n sufficiently large

$$|x_n|\leqq c\log n \qquad \text{a.s.}$$

so by Theorem 1

$$E(x_{n-1}^2I(|x_{n-1}|\geqq\varepsilon))=O(n^{-1}).$$

On the other hand, by Theorem 1

$$E|x_{n-m-1}|\leqq\varepsilon.$$

Hence,

$$(4.9)\qquad Ex_n^2\leqq(1-2\gamma'n^{-1})Ex_{n-1}^2+cn^{-2}\log n.$$

Using Lemma 3.4 in [1] we obtain from (4.8) that

$$(4.10)\qquad Ex_n^2=O(n^{-\min(2\gamma',1-\tau)})=O(n^{-1+\tau})$$

where τ is an arbitrary small positive number. From (4.10) it follows that for any small positive number τ

$$E|x_n|\leqq\{Ex_n^2\}^{1/2}=O(n^{-1/2-\tau}). \tag{4.11}$$

Now, let p be an integer such that $p\geqq 2$. By the Taylor expansion

$$\begin{aligned}|x_n|^{2p}&=|x_{n-1}+n^{-1}\psi(\xi_n-x_{n-1})|^{2p}\\&\leqq|x_{n-1}|^{2p}+2pn^{-1}\psi(\xi_n-x_{n-1})|x_{n-1}|^{2p-2}x_{n-1}\\&\quad+c_1n^{-2}|x^{2p-2}+c_2n^{-2p}(\log n)^{2p-2}\qquad\text{a.s.}\end{aligned}$$

Furthermore

$$|x_{n-1}|^{2p-2}=|x_{n-m-1}+h_{n-1}|^{2p-2}=|x_{n-m-1}|^{2p-2}+B_n$$

where

$$B_n\leqq cn^{-1}\log n|x_{n-m-1}|^{2p-3}+c_2n^{-2p+2}(\log n)^{2p-2}\qquad\text{a.s.}$$

Using similar caliculations as in (4.8), we get

$$\begin{aligned}&x_{n-1}|x_{n-1}|^{2p-2}(\psi(\xi_n-x_{n-1})-f(x_{n-1}))\\&\quad=(x_{n-m-1}+h_{n-1})(|x_{n-m-1}|^{2p-2}+B_n)(V_n+Z_n+f(x_{n-m-1})-f(x_{n-1}))\\&\quad\leqq x_{n-m-1}|x_{n-m-1}|^{2p-2}V_n+c_1n^{-1}\{|x_{n-m-1}|^{2p-1}+|x_{n-m-1}|^{2p-2}\}\\&\qquad+c_2n^{-2p+2}(\log n)^{2p}|x_{n-m-1}|\\&\qquad+c_3n^{-2}|x_{n-m-1}|^{2p-3}+c_4n^{-2p+1}(\log n)^{2p}\qquad\text{a.s.}\end{aligned}$$

We remark that for all n sufficiently large

$$\begin{aligned}&E(x_{n-m-1}|x_{n-m-1}|^{2p-2}V_n)\\&\quad\leqq E|x_{n-m-1}|^{2p-1}|EV_n|+c\,(\log n)^{2p-1}\beta(m)\\&\quad\leqq cn^{-3}.\end{aligned}$$

Hence for an arbitrary small positive number τ

$$\begin{aligned}E|x_n|^{2p}&\leqq E|x_{n-1}|^{2p}+2pn^{-1}E\{x_{n-1}|x_{n-1}|^{2p-2}f(x_{n-1})\}\\&\quad+c_1n^{-2}\{E|x_{n-m-1}|^{2p-1}+E|x_{n-m-1}|^{2p-2}\}\\&\quad+c_2n^{-2p+2}(\log n)^{2p}E|x_{n-m-1}|\\&\quad+c_3n^{-3}|x_{n-m-1}|^{2p-3}+c_4n^{-2p+1}(\log n)^{2p}\\&\leqq(1-2p\gamma'n^{-1})E|x_{n-1}|^{2p}+cn^{-p+\tau}\end{aligned}$$

since from Theorem 1 we obtain as above

$$E(x_{n-1}|x_{n-1}|^{2p-2}f(x_{n-1}))\leqq-\gamma'E|x_{n-1}|^{2p}+cn^{-p+\tau}$$

and for $2s \leqq 2p-1$

$$E|x|^{2s}=O(n^{-s+\tau})$$

for all arbitary small positive number τ. Hence, by Lemma 3.4 in [1]

$$E|x_n|^{2p}=O(n^{-\min(2p\gamma',\, p-\tau)})=O(n^{-p+\tau})$$

for all $\tau(>0)$ sufficiently small. By induction Theorem 2 is obtained for all even number $2p$.

Now, for $\beta(0<\beta<p)$

$$E|x_n|^{2\beta} \leqq (E|x_n|^{2p})^{(\beta/p)}=O(n^{-\beta-\tau})$$

for all $\tau(>0)$ sufficiently small.

Thus, the theorem holds for all values of $p'(0<p'<p)$.

5. Proof of Theorem 3.

Let $m=[h_0 \log n]$, as before. Let N be a number such that $\gamma(N+1)^{-1}<1$. we consider some sequences of positive number as in [1]. Define

$$\beta_{n,k}=\begin{cases} \prod_{j=k+1}^{n}(1-\gamma j^{-1}), & k=N,\ N+1. \cdots, n-1 \\ 1, & k=n, \end{cases} \tag{5.1}$$

$$b_n=\left\{\sum_{k=N}^{n}\gamma^2\beta_{n,k}^2 k^{-2}\right\}^{-(1/2)} \tag{5.2}$$

and

$$\alpha_{n,k}=b_n\gamma\beta_{n,k}k^{-1}, \qquad k=N,\ N+1,\ \cdots,\ n. \tag{5.3}$$

It is shown in [1] that

$$\alpha_{n,k} \leqq c_1 n^{(1/2)-\gamma}k^{-1+\gamma}. \tag{5.4}$$

Following lemmas were proved in [1].

Lemma B. *We have*

$$C_k k^{\gamma}n^{-\gamma} \leqq \beta_{n,k} \leqq D_k k^{\gamma}n^{-\gamma}, \qquad k=N,\ N+1,\ \cdots,\ n$$

for some C_k and D_k which do not depend on n and which fulfill $0<C_k \leqq 1 \leqq D_k<\infty$ *and*

$$\lim_{k\to\infty} C_k=\lim_{k\to\infty} D_k=1.$$

Lemma C. *For* $\gamma>1/2$

$$\lim_{n\to\infty}\gamma^2 b_n^2 n^{-1}=\lim_{n\to\infty} n\sum_{k=N}^{n}\beta_{n,k}^2 k^{-2}=(2\gamma-1)^{-1}.$$

Lemma D. *Let $\gamma > 1/2$ and let $\{W_k\}$ be a sequence of real numbers converging to W. If $k_0 > N$ is a fixed positive integer, then*

$$\lim_{n\to\infty} \sum_{k=k_0}^{\infty} \alpha_{n,k}^2 W_k = W.$$

Hence, to prove Theorem 3, it is enough to show

$$\gamma b_n x_n \xrightarrow{D} N(0, \sigma^2) \tag{5.5}$$

since by Lemma C

$$\gamma b_n n^{-(1/2)} \longrightarrow (2\gamma - 1)^{1/2} \qquad \text{as} \quad n \to \infty.$$

Let n be so large that $m+2 \geqq N$. Let

$$T_n = \gamma(x_{n-1} - x_{n-m-1}) + Z_n$$

and

$$\delta_n = \delta(x_{n-m-1}).$$

Then, using (2.4), (2.5) and A3 we can write x_n as

$$\begin{aligned} x_n &= (1 - \gamma n^{-1}) x_{n-1} + n^{-1} V_n + n^{-1} \delta_n + n^{-1} T_n \\ &= \beta_{n,m+2} x_{m+2} + \sum_{k=m+3}^{n} k^{-1} \beta_{n,k} + \sum_{k=m+3}^{n} k^{-1} \beta_{n,k} \delta_k + \sum_{k=m+3}^{n} k^{-1} \beta_{n,k} T_k. \end{aligned} \tag{5.6}$$

Hence, to prove (5.5), it is enough to show that the following four relations hold:

$$b_n \beta_{n,m+2} x_{m+2} \xrightarrow{P} 0, \tag{5.7}$$

$$b_n \sum_{k=m+3}^{n} k^{-1} \beta_{n,k} \beta_k \xrightarrow{P} 0, \tag{5.8}$$

$$b_n \sum_{k=m+3}^{n} k^{-1} \beta_{n,k} T_k \xrightarrow{P} 0, \tag{5.9}$$

$$S_n = b_n \gamma \sum_{k=m+3}^{n} k^{-1} \beta_{n,k} V_k = \sum_{k=m+3}^{n} \alpha_{n,k} V_k \xrightarrow{D} N(0, \sigma^2), \tag{5.10}$$

Since by (5.3) and (5.4)

$$\begin{aligned} b_n \beta_{n,m+2} &\leqq c m n^{(1/2)-\gamma} m^{-1+\gamma} \\ &\leqq c n^{(1/2)-\gamma} (\log n)^{\gamma} \longrightarrow 0 \qquad (n \to \infty), \end{aligned}$$

if $\gamma > 1/2$ and by Theorem 1 $E|x_{m+2}|$ is bounded, so (5.7) holds.

Next, since by Theorem 2

$$E x_n^2 = O(n^{-1+2\tau})$$

for all $\tau (>0)$ sufficiently small and

$$x_n \longrightarrow 0 \qquad \text{a.s.}$$

so (5.8) holds. In fact, this conclusion is obtained by [1] but we obtain it here for completeness.

Let $t>0$. Since $\delta(x)=o(x)$, for $t>0$ we can find $\varepsilon>0$ with the property that

$$|\delta(x)|\leqq t^2|x|.$$

Since $x_n \to 0$ a.s., so we can choose N_1 so that

$$P(|x_j|\leqq \varepsilon,\ j\geqq N_1)>1-t.$$

Then, for all $\tau(>0)$ sufficiently small

$$\begin{aligned}
&P\left\{\left|b_n \sum_{k=m+3}^{n} k^{-1}\beta_{n,k}\delta_k\right|>t\right\}\\
&\quad\leqq t+P\left\{\left|b_n \sum_{k=m+3}^{n} k^{-1}\beta_{n,k}\delta_k\right|>t : \max_{j\geq N_3}|x_j|\leqq \varepsilon\right\}\\
&\quad\leqq t+P\left\{t^2 b_n \sum_{k=N_3}^{n} k^{-1}\beta_{n,k}|x_k|>t\right\}\\
&\quad\leqq t+tE\left\{b_n \sum_{k=N_3}^{n} k^{-1}\beta_{n,k}|x_k|\right\}\\
&\quad\leqq t+ctb_n \sum_{k=N_3}^{n} k^{-1}\beta_{n,k}k^{-(1/2)+\tau}\\
&\quad\leqq ct.
\end{aligned}$$

As $b_n\beta_{n,k}\to 0$ for any fixed k, so we have (5.8).

Thirdly, by Lemma 3.4

$$|T_k|\leqq \gamma|h_{k-1}|+|Z_k|\leqq ck^{-1}\log n.$$

Hence, using (5.4)

$$\begin{aligned}
E\left(b_n \sum_{k=m+3}^{n} k^{-1}\beta_{n,k}T_k\right)^2 &\leqq c\left(n^{(1/2)-\gamma} \sum_{k=m+3}^{n} k^{\gamma-2}\right)^2(\log n)^2\\
&\leqq \begin{cases} cn^{-1}(\log n)^2 & \text{if } \gamma>1\\ cn^{-1}(\log n)^4 & \text{if } \gamma=1\\ cn^{1-2\gamma}(\log n)^2 & \text{if } 1/2<\gamma<1. \end{cases}
\end{aligned}$$

and so

$$E\left(b_n \sum_{k=m+3}^{n} k^{-1}\beta_{n,k}T_k\right)^2 \longrightarrow 0 \qquad (n\to\infty),$$

which implies (5.7).

Finally, we prove (5.8). Let q be a positive number such that $1/2<q<1$ if $\gamma>1$ and $(2\gamma)^{-1}<q<1$ if $1/2<\gamma\leqq 1$. Let

$$k_n(i)=\begin{cases} m+2, & i=1, \\ i[n^{1-q}], & i=2, 3, \cdots, r_n-1 \\ n, & i=r_n \end{cases} \tag{5.11}$$

where $r_n=[n^q]$. Then $\{k_n(i)\}$ has the following properties:

$$k_n(i+1)-k_n(i)>2m+1 \qquad (i=1, \cdots, r_n-1) \tag{5.12}$$

$$\lim_{n\to\infty} n^{1-2\gamma} \sum_{j=1}^{r_n} k_n(i)^{2\gamma-2}=0 \tag{5.13}$$

$$\lim_{n\to\infty} n^{(1/2)-\gamma} \max_{1\leq i\leq r_n^{-1}} \{(k_n(i+1))^{\gamma}-(k_n(i))^{\gamma}\}=0 \tag{5.14}$$

and if $W_k\to W$, then

$$\lim_{n\to\infty} \sum_{i=1}^{r_n-1} \sum_{k=k_n(i)+m+1}^{k_n(i+1)-j} \alpha_{n,k}\alpha_{n,k+j}W_k=W \qquad (j=0, 1, \cdots, m) \tag{5.15}$$

(cf. Lemma 5.5 in [1]).

For $i(i=1, 2, \cdots, r_n-1)$, put

$$x_{n,i}=\sum_{j=k_n(i)+m+1}^{k_n(i+1)} \alpha_{n,j}V_j \tag{5.16}$$

$$b_{n,j}^2=\sum_{j=k_n(i)+m+1}^{k_n(i+1)} \alpha_{n,k}^2 r(0) + 2\sum_{p=1}^{m} \sum_{k=k_n(i)+m+1}^{k_n(i+1)-p} \alpha_{n,k}\alpha_{n,k+p}r(p) \tag{5.17}$$

and

$$\mathcal{F}_{n,i}=\mathcal{F}_1^{k_n(i+1)}. \tag{5.18}$$

Since $x_{n,i}$ is $\mathcal{F}_{n\ i}$-measurable and

$$E\{x_{n\ i-1}\}=0$$

so $\{x_{n,i}\}$ is a martingale difference array. Now, rewriting S_n as

$$S_n=\sum_{i=1}^{r_n-1} x_{n,i}+\sum_{i=1}^{r_n-1} \sum_{j=1}^{m} \alpha_{n,k_n(i)+j} V_{k_n(i)+j},$$

we show that

$$\sum_{i=1}^{r_n-1} x_{n,i} \longrightarrow N(0, \sigma^2) \tag{5.19}$$

and

$$\sum_{i=1}^{r_n-1}\sum_{j=1}^{m}\alpha_{n,k_n(i)+j}V_{k_n(i)+j}\xrightarrow{P}0. \tag{5.20}$$

To prove (5.19), it is enough to show that

$$I_1=E\left[\sum_{i=1}^{r_n-1}\sum_{j=1}^{m}\alpha_{n,k_n(i)+j}V_{k(i)+j}\right]^2\longrightarrow 0 \tag{5.21}$$

as $n\to\infty$. We note that since $V_i\in\mathcal{F}_{i-m}^{i}$, $EV_i=0$ and $|V_i|\leqq L$ a.s., so

$$|EV_iV_j|\leqq 2L\beta(j-i-m) \quad \text{if} \quad j-i-m>0. \tag{5.22}$$

Hence, it follows from Lemma A, A5 and (5.4) that as $n\to\infty$

$$\begin{aligned} I_1&=\sum_{i,j=1}^{r_n-1}\sum_{k,j=1}^{m}\alpha_{n,k_n(i)+k}\alpha_{n,k_n(j)+l}EV_{k_n(i)+k}V_{k_n(j)+l} \\ &=\sum_{i=1}^{r_n-1}\sum_{k,l=1}^{m}\alpha_{n,k_n(i)+k}\alpha_{n,k_n(i)+l}EV_{k_n(i)+k}V_{k_n(i)+l} \\ &\quad+2\sum_{1\leq i<j\leq r_n-1}\sum_{k,l=1}^{m}\alpha_{n,k_n(i)+k}\alpha_{n,k_n(j)+l}EV_{k_n(i)+k}V_{k_n(j)+l} \\ &\leqq L\sum_{i=1}^{r_n-1}\sum_{k,l=1}^{m}\alpha_{n,k_n(i)+k}\alpha_{n,k_n(i)+l} \\ &\quad+c\sum_{1\leq i<j\leq r_n-1}\sum_{k,l=1}^{m}\alpha_{n,k_n(i)+k}\alpha_{n,k_n(j)+l}\beta(k_n(j)-k_n(i)-2m)\longrightarrow 0, \end{aligned} \tag{5.23}$$

which implies (5.21).
To prove (5.20), we need the following lemma.

Lemma 5.1. *If* A1 *and* A5 *hold, then for any* $\varepsilon>0$ *there exists an integer* m_0 *such that*

$$|\sigma^2-\sigma_{m_0}^2|<\varepsilon \tag{5.24}$$

and

$$\left|\sum_{i=1}^{r_n-1}b_{n,i}^2-\sigma_{m_0}^2\right|\leqq c_1\varepsilon+o(1) \tag{5.25}$$

as $n\to\infty$, *that is,*

$$\lim_{n\to\infty}\sum_{i=1}^{r_n-1}b_{n,i}^2=\sigma^2 \tag{5.26}$$

where

$$\sigma_{m_0}^2=r(0)+2\sum_{i=1}^{m_0}r(i).$$

Proof. We note that by Lemma A1 and A5

$$\left|\sum_{i=k}^{\infty} r(i)\right| \leqq c \sum_{i=k}^{\infty} \beta(i) \leqq ce^{-\lambda k}.$$

Hence, for any $\varepsilon>0$ we can choose m_0 satisfying (5.24)

Let ε and m_0 be fixed. Let j be an arbitrary integer such that $0<j<m_0$. By (5.4) and (5.11)

$$\begin{aligned}
&\sum_{i=1}^{r_n-1} \sum_{k=k_n(i)+m_0+1}^{k_n(i)+m+1} \alpha_{n,k}\alpha_{n,k+j}r(j) \\
&\leqq c\sum_{i=1}^{r_n-1} n^{1-2\gamma}\{k_n(i)\}^{-2+2\gamma}m \\
&\leqq cn^{1-2\gamma}(n^{1-q})^{-2+2\gamma}\log n \sum_{i=1}^{r_n} i^{-2+2\gamma} \\
&\leqq cn^{1-2\gamma}n^{(1-q)(-2+2\gamma)}(n^q)^{-1+2\gamma}\log n \\
&=O(n^{-1+q}\log n)
\end{aligned}$$

and

$$\lim_{n\to\infty}\sum_{i=1}^{r_n-1} \sum_{k=k_n(i)+m_0+1}^{k_n(i+1)} \alpha_{n,k}\alpha_{n,k+j}r(j)=r(j).$$

Hence

$$\sum_{i=1}^{r_n-1} \sum_{k=k_n(i)+m+1}^{k_n(i+1)} \alpha_{n,k}\alpha_{n,k+j}r(j) \longrightarrow r(j) \qquad (j=0,1,\cdots,m_0). \tag{5.27}$$

Further, by (5.4), (5.11), and the definition of m_0

$$\begin{aligned}
&\sum_{i=1}^{r_n-1} \sum_{p=m_0+1}^{m} \sum_{k=k_n(i)+m+1}^{k_n(i+1)-p} \alpha_{n,k}\alpha_{n,k+p}|r(p)| \\
&\leqq c\sum_{i=1}^{r_n-1} n^{1-2\gamma}\{k_n(i)\}^{-2+2\gamma}\{k_n(i+1)-k_n(i)\}\sum_{p=m_0+1}^{m}\beta(p) \\
&\leqq c\varepsilon n^{1-2\gamma}(n^{1-q})^{-1+2\gamma}\sum_{i=1}^{r_n-1} i^{-2+2\gamma} \\
&\leqq c\varepsilon n^{q-2\gamma q}(r_n)^{-1+2\gamma} \\
&\leqq c\varepsilon n^{q-2\gamma q}(n^q)^{-1+2\gamma} \\
&\leqq c\varepsilon .
\end{aligned} \tag{5.28}$$

Now, (5.25) follows from (5.27) and (5.28), and the proof is completed.

Since $\{x_{n,i}\}$ is a martingale difference array with respect to $\mathcal{F}_{n,i}$, so, in order to prove (5.19), by Lemma 5.1 it suffices to show that

$$\lim_{n\to\infty}\sum_{i=1}^{r_n-1} E|E\{x_{n,i}^2|\mathcal{F}_{n,i-1}\}-b_{n,i}^2|=0 \tag{5.29}$$

and for any $\varepsilon>0$.

$$\lim_{n\to\infty}\sum_{i=1}^{r_n-1} E\{x_{n,i}^2 I(|x_{n,i}|>\varepsilon)\}=0. \tag{5.30}$$

Firstly, we prove (5.30). We note that

$$\begin{aligned}\max_{1\le i\le r_n-1}|x_{n,i}| &\leqq \max_{1\le i\le r_n-1}\left|\sum_{j=k_n(i)+m+1}^{k_n(i+1)}\alpha_{n,j}V_j\right| \\ &\leqq cn^{(1/2)-\gamma}\max_{1\le i\le r_n-1}\sum_{j=k_n(i)+m+1}^{k_n(i+1)} j^{\gamma-1} \\ &\leqq cn^{(1/2)-\gamma}\max_{1\le i\le r_n-1}\{(k_n(i+1))^\gamma-(k_n(i))^\gamma\}.\end{aligned}$$

Hence, (5.30) follows from (5.14).

Now, we proceed to prove (5.29). For i $(1\leqq i\leqq r_n-1)$ let

$$I_i=E|E\{x_{n,i}^2|\mathfrak{F}_{n,i-1}\}-b_{n,i}^2|.$$

Then

$$\begin{aligned} I_i &\leqq \sum_{k=k_n(i)+m+1}^{k_n(i+1)}\alpha_{n,k}^2 E|E\{V_k^2|\mathfrak{F}_{n,i}\}-r(0)| \\ &+2\sum_{k=k_n(i)+m+1}^{k_n(i+1)-m-1}\sum_{p=1}^{m}\alpha_{n,k}\alpha_{n,k+p}E|E\{V_kV_{k+p}|\mathfrak{F}_{n,i-1}\}-r(p)| \\ &+2\sum_{k=k_n(i)-m-1}^{k_n(i+1)-m-1}\sum_{p=m+1}^{k_n(i+1)-k}\alpha_{n,k}\alpha_{n,k+p}E|E\{V_kV_{k+p}|\mathfrak{F}_{n,i-1}\}| \\ &=J_{1i}+J_{2i}+J_{3i},\qquad \text{(say)}.\end{aligned} \tag{5.31}$$

By Lemma 3.1 and (5.22)

$$\begin{aligned} |J_{3i}| &\leqq 2\sum_{k=k_n(i)+m+1}^{k_n(i+1)-m-1}\sum_{p=m+1}^{k_n(i+1)-k}\alpha_{n,k}\alpha_{n,k+p}[|EV_kV_{k+p}|+c\beta(k-k_n(i)-m)] \\ &\leqq \sum_{k=k_n(i)+m+1}^{k_n(i+1)-m-1}\sum_{p=m+1}^{k_n(i+1)-k}\alpha_{n,k}\alpha_{n,k+p}\{\beta(p-m)+\beta(k-k_n(i)-m)\} \\ &\leqq c\{n^{(1/2)-\gamma}(k_n(i))^{-1+\gamma}\}^2\{k_n(i+1)-k_n(i)\} \\ &\leqq cn^{-\gamma+q-\gamma q}i^{-2+2\gamma}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{i=1}^{r_n-1}|J_{3i}| &\leqq cn^{-\gamma+q-\gamma q}(r_n)^{-1+2\gamma} \\ &\leqq cn^{-\gamma(1-q)}\longrightarrow 0\end{aligned} \tag{5.32}$$

as $n\to\infty$.

Let

$$U_k=\psi(\xi_k-x_{k-1})-f(x_{k-m-1}).$$

Then

$$V_k=U_k-Z_k$$

with Z_k is the one defined by (2.5). Hence, we have

$$E(V_k V_{k+p}|\mathcal{F}_{n,i-1})=E(U_k U_{k+p}|\mathcal{F}_{n,i-1})+R_{k,p}$$

for $k=k_n(i)+m+1, \cdots, k_n(i+1)-p$ and $p=0, 1, \cdots, m$ where

$$|R_{k,p}|\leqq ck^{-1}\log n \qquad \text{a.s.}$$

since U_k is bounded and by Lemma 3.4

$$|Z_k|\leqq ck^{-1}\log n \qquad \text{a.s.}$$

Now, let

$$\psi_k=\psi(\xi_k),$$

$$H_k=\psi(\xi_k-x_{k-1})-\psi(\xi_k)$$

and

$$f_k=f(x_{k-m-1}).$$

Then

$$U_k U_{k+p}=(\psi_k+H_k-f_k)(\psi_{k+p}+H_{k+p}-f_{k+p})$$

$$=\psi_k\psi_{k+p}+G_{k,p}, \qquad \text{(say)}.$$

Since

$$|E\{\psi_k\psi_{k+p}|\mathcal{F}_{n,i-1}\}-r(p)|\leqq c\beta(k-k_n(i))$$

for $k=k_n(i)+m+1, \cdots, k_n(i+1)-p$ *and* $p=0, 1, \cdots, m$, so we have

$$|J_{1i}|\leqq \sum_{k=k_n(i)+m+1}^{k_n(i+1)} \alpha^2_{n,k}[c\beta(k-k_n(i)-m)$$

(5.33)
$$+E|E\{G_{k,0}|\mathcal{F}_{n,i-1}\}|+ck^{-1}_m]$$

and

(5.34)
$$|J_{2i}|\leqq 2\sum_{p=1}^{m}\sum_{k=k_n(i)+m+1}^{k_n(i+1)-p} \alpha_{n,k}\alpha_{n,k+p}[c\beta(k-k_n(i)-m)$$

$$+E|E\{G_{k,p}|\mathcal{F}_{n,i-1}\}|+ck^{-1}m].$$

Now, we use the same method of estimation in [1]. Let $\eta_{k+p}=(k+p)^{-(1/3)}$ $(0\leqq p\leqq m)$. Using A1, A2, Chebyshev's inequality, the definition of f and A4 we have

$$E(\psi_{k+p'}H_{k+p}|\mathcal{F}_{n,i-1})$$

$$=E(\psi_{k+p}H_{k+p}[I\{0\leqq x_{k+p-1}\leqq\eta_{k+p}\}+I\{-\eta_{k+p}\leqq x_{k+p-1}<0\}$$

$$+I\{|x_{k+p-1}|>\eta_{k+p}\}]|\mathcal{F}_{n,i-1})$$

$$-c[E\{\psi(\xi_{k+p'}-\eta_{k+p})|\mathcal{F}_{n,i-1}\}+E\{\psi(\xi_{k+p'}\eta_{k+p}|\mathcal{F}_{n,i-1}\}$$
$$+P(|x_{k+p-1}|>\eta_{k+p})]$$
$$\leqq c[-f(\eta_{k+p})+f(-\eta_{k+p})+\eta_{k+p}^{-2}E\{x_{k+p-1}^2|\mathcal{F}_{n,i-1}\}]$$
$$\leqq c[\eta_{k+p}+\eta_{k+p}^{-2}E\{x_{k+p-1}^2|\mathcal{F}_{n,i-1}\}]$$

$0\leqq p,\ p'\leqq m$. With the same method we can prove

$$E\{\psi_{k+p'}H_{k+p}|\mathcal{F}_{n,i-1}\}\geqq -c[\eta_{k+p}+\eta_{k+p}^{-2}E\{x_{k+p-1}^2|\mathcal{F}_{n,i-1}\}].$$

The remaining terms of $G_{k,p}$ can be estimated analogously. Hence, for $p(0\leqq p\leqq 1)$ we have

$$E|E\{G_{k,p}|\mathcal{F}_{n,i-1}\}|\leqq c[\eta_k+\eta_{k+p}^{-2}E(x_{k+p-1}^2)+\eta_{k+p}^{-2}E(x_{k+p-m-1}^2)]\leqq ck^{-(1/3)}$$

and so from (5.33) and (5.34)

$$|J_{1i}|\leqq c\sum_{k=k_n(i)+m}^{k_n(i+1)}\alpha_{n,k}^2k^{-(1/3)}$$

and

$$|J_{2i}|\leqq c\sum_{k=k_n(i)+m}^{k_n(i+1)}\alpha_{n,k}\alpha_{n,k+p}k^{-(1/3)}.$$

Thus, from (5.33) and (5.34)

$$\max\Big(\sum_{i=1}^{r_n-1}|J_{1i}|,\ \sum_{i=1}^{r_n-1}|J_{2i}|\Big)\longrightarrow 0\qquad (n\to\infty). \tag{5.35}$$

Now, it follows from (3.32) and (3.35) that

$$\sum_{i=1}^{r_n-1}I_i\longrightarrow 0\qquad (n\to\infty)$$

which implies that (5.29) holds. So, the proof of Theorem 3 is completed.

References

[1] Holst, U. (1980) : *Convergence of a recursive stochastic algorithm with m-dependent observations.* Scand. J. Statist. 7, 207-215.

[2] Holst, U. (1981). *Convergence of a recursive stochastic algorithm with m-dependent observations.* Correction note. Scand. J. Statist. 8, 124.

[3] Komolós, J. and Révész, P. (1972) : *On the rate of convergence of the Robbins-Monro method.* Z. Wahrscheinlichkeitstheorie verw. Gebiete 25, 39-47.

[4] Rozanov, Yu. A. and Volkonski, V.A. (1961). *Some limit theorems for random functions II.* Theory Probab. Appl. 6, 186-198.

[5] Sacks, J. (1958) : *Asymptotic distribution of stochastic approximation procedures.* Ann. Math. Statist. 29, 373-405.

Department of Mathematics
Faculty of Engineering
Yokohama National University
156, Tokiwadai, Hodogaya, Yokohama,
240 JAPAN
and
Department of Mathematics,
Tokyo Gakugei University
4-1-1 Nukuikita-Machi, Koganei, Tokyo,
184 JAPAN