YOKOHAMA MATHEMATICAL
JOURNAL VOL. 33, 1985

CENTRAL LIMIT THEOREMS FOR STATIONARY MIXING SEQUENCES

By

KEN-ICHI YOSHIHARA

(Received April 18, 1985)

I. Introduction.

Let $\{X_k, -\infty<k<\infty\}$ be a strictly stationary mixing sequences of real-valued random variables defined on a probability space $(\Omega, \mathcal{A}, P)$. Thus, the sequence $\{X_k\}$ satisfies either the ϕ-mixing condition

(1.1) $$\phi(n)=\sup_{A\in\mathcal{M}_{-\infty}^{k}, B\in\mathcal{M}_{k+n}^{\infty}}\left|\frac{P(AB)-P(A)P(B)}{P(A)}\right|\downarrow 0$$

or the strong mixing condition

(1.2) $$\alpha(n)=\sup_{A\in\mathcal{M}_{-\infty}^{k}, B\in\mathcal{M}_{k+n}^{\infty}}|P(AB)-P(A)P(B)|\downarrow 0,$$

where $\mathcal{M}_a^b$ denote the σ-algebra generated by $X_a, \cdots, X_b (a\leqq b)$. Assume $EX_1=0$ and $EX_1^2<\infty$. For each $n(\geqq 1)$ define $S_n=\sum_{j=1}^{n}X_j$ and $s_n=\{ES_n^2\}^{1/2}$. We shall say that the sequence $\{X_k\}$ satisfies the central limit theorem if

(1.3) $$\lim_{n\to\infty}P\left(\frac{1}{s_n}S_n<x\right)=(2\pi)^{-1/2}\int_{-\infty}^{x}e^{-t^2/2}dt=\Phi(x).$$

In [3], several sufficient conditions under which the sequence $\{X_k\}$ satisfies the central limit theorem were obtained but the problem of whether the following conjecture holds remained unsolved:

Conjecture A. If a strictly stationary sequence $\{X_j\}$ is ϕ-mixing and satisfies

(1.4) $$EX_j=0, \qquad EX_j^2<\infty, \qquad \lim_{n\to\infty}V\left(\sum_{j=1}^{n}X_j\right)=\infty,$$

then it satisfies the central limit theorem.

The main object of this paper is to show if, for a ϕ-mixing sequence $\{X_j\}$ satisfying (1.4), $\{S_n^2/s_n^2\}$ is uniformly integrable, then $\{X_j\}$ satisfies the central limit theorem (Theorem 2).

2. The central limit theorem for strong mixing sequences.

Firstly, we show a lemma.

Lemma. *Let a strictly stationary sequence $\{X_j\}$ satisfy the strong mixing condition with mixing coefficient $\alpha(n)$. Suppose $EX_j=0$ and $EX_j^2<\infty$. Suppose further that $\{S_n^2/s_n^2\}$ is uniformly integrable. Then, the characteristic function $\phi_n(t)$ of S_n/s_n may be written as*

$$\phi_n(t)=1-\frac{t^2}{2}+t^2R_n(t) \tag{2.1}$$

where $R_n(t)\to 0$ uniformly in n as $t\to 0$.

Proof. By Taylor's expansion, noting $EX_1=0$, we can rewrite $\phi_n(t)$ as

$$\phi_n(t)=1-\frac{t^2}{2}-\frac{t^2}{2}E\left[\frac{S_n^2}{s_n^2}(e^{ih(S_n/s_n)}-1)\right] \tag{2.2}$$

for some $h\,(0<|h|<|t|)$. Since by assumption $\{S_n^2/s_n^2\}$ is uniformly integrable, so for an arbitrary positive ε, we can choose a number $T=T(\varepsilon)$ so large that

$$E\left[\frac{S_n^2}{s_n^2}I\left(\left\{\frac{S_n^2}{s_n^2}>T\right\}\right)\right]<\varepsilon \qquad \text{for all } n \tag{2.3}$$

where $I(A)$ denotes the indicator of the set A. Then

$$\left|E\left[\frac{S_n^2}{s_n^2}(e^{ih(S_n/s_n)}-1)\right]\right|\leqq TE\left[|e^{ih(S_n/s_n)}-1|I\left(\left\{\frac{S_n^2}{s_n^2}\leqq T\right\}\right)\right] \tag{2.4}$$
$$+2E\left[\frac{S_n^2}{s_n^2}I\left(\left\{\frac{S_n^2}{s_n^2}>T\right\}\right)\right]$$

Since the first term in the righthand side of the above inequality tends uniformly in n to zero as $t\to 0$, so the conclusion follows from (2.2), (2.3) and (2.4).

Let G be the class of all slowly varying functions which satisfy the following conditions:

If $g\in G$ then there exists an integer-valued function $\gamma_g=\gamma_g(n)$ such that $\gamma_g(n)\to\infty$ and $\gamma_g=o(n)$ as $n\to\infty$ and

$$\lim_{n\to\infty}\frac{g(r)}{g(n)}=1 \tag{2.5}$$

holds for all integer-valued function $r=r(n)$ satisfying $r(n)\geqq\gamma_g(n)$ and $r=o(n)$.

In what follows, we shall agree to denote by $[x]$ the integer part of x.

Theorem 1. *Let the strictly stationary sequence $\{X_j\}$ satisfy the strong mixing condition with mixing coefficient $\alpha(n)$. Suppose $EX_1=0$ and $EX_1^2<\infty$. Suppose further that $\{S_n^2/s_n^2\}$ is uniformly integrable. Then in order that the sequence $\{X_j\}$ satisfies the central limit theorem it is necessary and sufficient that*

$$s_n^2=ES_n^2=ng(n)\to\infty \qquad (n\to\infty) \tag{2.6}$$

Proof. Firstly, we show that if (2.6) holds for some $g\in G$, then $\{X_j\}$ satisfies the central limit theorem. As in the proof of Theorem 18.4.1 in [3] we put

$$\lambda(n)=\max\left\{(\alpha([n^{1/4}]))^{1/3},\ \frac{1}{\log n}\right\}$$

$$p=\max\left\{\left[\frac{n\alpha([n^{1/4}])}{\lambda(n)}\right],\ \left[\frac{n^{3/4}}{\lambda(n)}\right],\ \gamma_g(n)\right\}, \tag{2.7}$$

$$q=[n^{1/4}],\qquad k=\left[\frac{n}{p+q}\right].$$

Then, it is obvious that

(i) $p\to\infty,\ q\to\infty,\ k\to\infty,\ p=o(n),\ q=o(p)$ as $n\to\infty$,

(ii) $\dfrac{n^{1-\beta}q^{1+\beta}}{p^2}=O(n^{-(1+3\beta)/4})=o(1)$ if $\beta>0$,

(iii) $k\alpha(q)\leqq\dfrac{n}{q}\alpha(q)\leqq\alpha([n^{1/4}])\dfrac{\lambda(n)}{\alpha([n^{1/4}])}=0$ as $n\to\infty$,

Let

$$Z_n=\frac{S_n}{s_n},\ Z'_n=\sum_{i=0}^{k-1}\xi_i\ \text{ and }\ Z''_n=\frac{1}{s_n}\sum_{i=0}^{k}\eta_i \tag{2.8}$$

where

$$\xi_i=\sum_{j=i(p+q)+1}^{i(p+q)+p}X_j,\qquad i=0,1,\cdots,k-1$$

$$\eta_i=\sum_{j=i(p+q)+p+1}^{(i+1)(p+q)}X_j,\qquad i=0,1,\cdots,k-1,$$

$$\eta_k=\sum_{j=k(p+q)+1}^{n}X_j.$$

By the method used in the proof of Theorem 18.4.1 in Ibragimov and Linnik (1971), we can prove that for any $\varepsilon>0$

$$P(|Z''_n|>\varepsilon)\to 0\quad\text{as}\quad n\to\infty. \tag{2.9}$$

So, by Lemma 18.4.1 in [3] the limit distribution of Z_n coincides with the limit distribution of Z'_n. But, by (iii) we can prove that

$$Ee^{itZ'_n}-(\varphi_n(t))^k\to 0\qquad(n\to\infty) \tag{2.10}$$

where $\varphi_n(t)$ is the characteristic function of $s_n^{-1}\xi_1$. (See, the proof of (18.4.2) in [3]). We note here that as $E\xi_0=0$, so by Lemma

$$\varphi_n(t)=E\{\exp(its_n^{-1}\xi_1)\}=1-\frac{t^2}{2}\frac{E\xi_1^2}{s_n^2}+\left(t^2\frac{s_p^2}{s_n^2}\right)R_n\left(t\frac{s_p}{s_n}\right) \tag{2.11}$$

Where

$$\frac{kE\xi_1^2}{s_n^2}=\frac{kpg(p)}{ng(n)}\to 1 \quad \text{as} \quad n\to\infty,$$

since by assumption $g\in G$ and by (2.7) $p(n)\geqq\gamma_g(n)$ and $kp/n\to 1$ as $n\to\infty$. Hence, by (2.11)

$$(\varphi_n(t))^k\to e^{-t^2/2} \quad \text{as} \quad n\to\infty. \tag{2.12}$$

Now, it follows from (2.11) and (2.14) that $\{X_j\}$ satisfies the central limit theorem.

Conversely, we show that if $\{X_j\}$ satisfies the central theorem then (2.6) holds for some $g\in G$. By Theorem 18.1.1 in [3]

$$s_n^2=nh(n) \tag{2.13}$$

where $h(n)$ is a slowly varying function of n. As in (2.7) we put

$$\begin{aligned} p_0&=\max\left\{\left[\frac{n\alpha([n^{1/4}])}{\lambda(n)}\right], \left[\frac{n^{3/4}}{\lambda(n)}\right]\right\}, \\ q_0&=[n^{1/4}], \qquad k_0=\left[\frac{n}{p_0+q_0}\right]. \end{aligned} \tag{2.14}$$

where $\lambda(n)$ is the same one as in (2.7). Then, by the above method we can prove that

$$P(B_n^{-1}S_n<x)\longrightarrow\Phi(x) \tag{2.15}$$

where

$$B_n=k_0^2s_{p_0}^2=k_0p_0h(p_0). \tag{2.16}$$

Hence, by (2.13)

$$\lim_{n\to\infty}\frac{h(p_0)}{h(n)}=\lim_{n\to\infty}\frac{n}{k_0p_0}\lim_{n\to\infty}\frac{k_0p_0h(p_0)}{nh(n)}=\lim_{n\to\infty}\frac{B_n}{s_n^2}=1, \tag{2.17}$$

since by (2.14)

$$\lim_{n\to\infty}\frac{n}{k_0p_0}=1$$

and by (2.15) and (1.3)

$$\lim_{n\to\infty}\frac{B_n}{s_n^2}=1.$$

Analogously, we can prove that for any p such that $p=o(n)$ and $p(n)\geqq p_0(n)$ (with $k=[n/(p+q_0)]$)

(2.18) $$\lim_{n\to\infty} \frac{h(p)}{h(n)}=1,$$

which implies $h\in G$ (with $\gamma_h(n)=p_0(n)$). It follows from (2.13) and (2.6) holds for the function $h(n)$. Thus, the proof is completed.

Remark. Let $\{X_j\}$ be a strictly stationary strong mixing sequence satisfying the following conditions:

(a) $EX_j=0$, $EX_j^2<\infty$ and $s_n^2\to\infty$ $(n\to\infty)$.

(b) $\{S_n^2/s_n^2\}$ is uniformly integrable,

(c) $EX_1X_n\geqq cn^{-1}$ for all n sufficiently large where c is positive constant.

Then, it follows from Theorem 1 that the sequence $\{X_j\}$ does [not satisfy the central limit theorem, since, in this case, for the function defined by $s_n^2=nh(n)$ the relation

$$h(n)\geqq K\log n \qquad (n\to\infty)$$

holds and so $h\notin G$.

3. The central limit theorem for a general ϕ-mixing sequence.

In this section, we consider a general ϕ-mixing sequence.

Theorem 2. *Suppose that a strictly stationary sequence $\{X_j\}$ is ϕ-mixing and satisfies $EX_1=0$ and $EX_j^2<\infty$. Assume that $\{S_n^2/s_n^2\}$ is uniformly integrable.*

Then, in order that the sequence $\{X_j\}$ satisfies the central limit theorem it is necessary and sufficient that

(3.1) $$\lim_{n\to\infty} s_n^2=ES_n^2=\infty$$

holds.

Proof. It is enough to prove that if $s_n^2\to\infty$, then $\{X_j\}$ satisfies the central limit theorem. We note that if $s_n^2\to\infty$, then by Theorem 18.2.3 in [3]

(3.2) $$s_n^2=nh(n)$$

with $h(n)$ being a slowly varying function of n. So, by Theorem 1, it suffices to prove that $h(n)\in G$.

As in (2.3), we put

(3.3) $$\begin{aligned}&\lambda(n)=\max\{(\phi([n^{1/4}]))^{1/3},\ 1/\log n\}\\ &p(n)=\max\{[n(([n^{1/4}])/\lambda(n))^{1/2}],\ [n^{3/4}/\lambda(n)]\}\\ &q(n)=[n^{1/4}], \qquad k=[n/(p+q)].\end{aligned}$$

Then, it is obvious that

$$p\to\infty,\ q\to\infty,\ q=o(p),\ p=o(n),\ k\to\infty$$

(3.4) $$\frac{n^{1-\beta}q^{1+\beta}}{p^2}=O(n^{(1+8\beta)/12})=o(1)\ \text{for some}\ \beta>0,$$

$$k^2\phi(n)\leqq(n/p)^2\phi(q)\leqq\lambda(n)\longrightarrow 0.$$

Using the above system $\{p, q, k\}$ define $\{\xi_0, \cdots, \xi_{k-1}\}$ and $\{\eta_0, \cdots, \eta_{k-1}, \eta_k\}$ as in the proof of Theorem 1.

We note that by the Schwarz inequality

$$E\left\{\sum_{i=0}^{k-2}(\xi_i+\eta_i)\right\}^2-2\left[E\left|\sum_{i=0}^{k-2}(\xi_i+\eta_i)\right|^2E|\xi_{k-1}+\eta_{k-1}+\eta_k|^2\right]^{1/2}$$

$$\leqq ES_n^2\leqq E\left\{\sum_{i=0}^{k-2}(\xi_i+\eta_i)\right\}^2+\left[E\left|\sum_{i=0}^{k-2}(\xi_i+\eta_i)\right|^2E|\xi_{k-1}+\eta_{k-1}+\eta_k|^2\right]^{1/2}$$

$$+E|\xi_{k-1}+\eta_{k-1}+\eta_k|^2,$$

i.e.,

(3.5) $$\begin{aligned}&s^2{}_{(k-1)(p+q)}-2s_{(k-1)(p+q)}s_{n-(k-1)(p+q)}\leqq s_n^2\\&\leqq s^2{}_{(k-1)(p+q)}+2s_{(k-1)(p+q)}s_{n-(k-1)(p+q)}+s^2_{n-(k-1)(p+q)}.\end{aligned}$$

On the other hand, as by Lemma 18.2.3 in [3]

$$\sup_{0\leq j\leq m}\frac{h(m+j)}{h(m)}\leqq 4\qquad(m\to\infty),$$

so by (3.2), (3.3) and the fact $q=o(p)$

$$\lim_{n\to\infty}\frac{s^2_{n-(k-1)(p+q)}}{ks_p^2}=\lim_{n\to\infty}\frac{n-(k-1)(p+q)}{kp}\,\frac{h(p+q)}{h(p)}\,\frac{h(n-(k-1)(p+q))}{h(p+q)}$$

$$\leqq 16\lim_{n\to\infty}\frac{2(p+q)}{kp}=0.$$

Hence, in order to prove

(3.6) $$\lim_{n\to\infty}\frac{h(n)}{h(p)}=\lim_{n\to\infty}\frac{s_n^2}{ks_p^2}=1,$$

it is enough to show

(3.7) $$\lim_{n\to\infty}\frac{s^2_{(k-1)(p+q)}}{ks_p^2}=1.$$

Now, we note that

(3.8) $$s^2_{(k-1)(p+q)}=E\Big(\sum_{i=0}^{k-2}\xi_i\Big)+2E\Big(\sum_{i=0}^{k-2}\xi_i\Big)\Big(\sum_{j=0}^{k-2}\eta_j\Big)+E\Big(\sum_{j=0}^{k-2}\eta_j\Big)^2.$$

By (3.4) and the well-known inequality (Theorem 17.2.3) in [3])

$$E\Big(\sum_{i=0}^{k-2}\eta_i\Big)^2=\sum_{i=0}^{k-2}E\eta_i^2+2\sum_{0\leq i<j\leq k-2}E\eta_i\eta_j \tag{3.9}$$

$$\leqq ks_q^2+2ks_q^2\sum_{l=1}^{k-2}\phi^{1/2}(lp)$$

$$\leqq ks_q^2[1+2\{k^2\phi(q)\}^{1/2}]$$

$$=ks_q^2(1+o(1))$$

and similarly

$$E\Big(\sum_{i=0}^{k-2}\xi_i\Big)^2=ks_p^2(1+o(1)). \tag{3.10}$$

Hence, (3.6) follows from (3.7)-(3.10).

Let $r=r(n)$ be any integer-valued function such that $r=o(n)$ and $r(n)\geqq p(n)$. Then, by the above method we can show that

$$\lim_{n\to\infty}\frac{h(n)}{h(r)}=1. \tag{3.11}$$

Thus, $h\in G$ (with $\gamma_h(n)=p(n)$) and the proof is completed.

Reference

[1] Billingsley, P.: *Convergence of probability measures*, New York: Wiley 1968.

[2] Bradley, R.C., Jr.: *A remark on the central limit question for dependent random variables.* J. Appl. Prob. 17 94-101 (1980).

[3] Ibragimov, J.A. and Linnik Yu. V.: *Independent and stationary sequences of random variables*, Groningen: Wolters-Noordhoff 1971.

Department of Mathematics
Faculty of Engineering
Yokohama National University
156 Tokiwadai, Hodogaya
Yokohama, Japan