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SOJOURNS OF MULTIDIMENSIONAL GAUSSIAN PROCESSES WITH DEPENDENT COMPONENTS

By

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1. Introduction and a result

Let $\{X(t)=(X_1(t), X_2(t), \cdots, X_p(t))', t\geqq 0\}$ be a measurable separable p-dimensional stationary Gaussian process, where x' is the transposed vector of x. This paper deals with the limiting distribution of sojourn time by $X(t)$ in the fixed sphere, under some assumptions on the dependence among the components of p-dimensional process $X(t)$. In [4], the author studied the case $p=2$ and Berman [1] treated the p-dimensional stationary Gaussian process with independent components in the case of the expanding or shrinking spheres. The result in this paper is the direct extension of Theorem 1 (II) in [4] to the case of general dimension and the idea of the proof is the same.

Assumptions and notation are the following. Suppose that

$$EX(0)=0,$$

$$R(t)\equiv EX(0)X(t)'=(R_{ij}(t))_{1\leq i,j\leq p},$$

$$R_{ij}(t)=\begin{cases} r(t), & \text{if } i=j, \\ \rho(t), & \text{if } i\neq j, \end{cases}$$

$r(t)$ and $\rho(t)$ are continuous,

$$r(0)=1, \quad \rho(0)=\rho_0 \qquad (0\leqq\rho_0<1),$$

$$r(t)\sim t^{-\alpha}L(t) \quad \text{as} \quad t\to\infty, \tag{1.1}$$

$$\rho(t)\sim\rho_\infty t^{-\alpha}L(t) \quad \text{as} \quad t\to\infty, \tag{1.2}$$

where $0<\alpha<1$, $0\leqq\rho_\infty<1$ and L is a slowly varying function at infinity. In [4], we assumed $\rho_\infty=0$. However, it is not necessary to assume it by the same reasoning as in [7]. We shall also treat the functional limit theorem.

Define

$$M(t)=\int_0^t I[X(s)\in D]ds, \quad t>0,$$

where $I[\cdot]$ is the indicator function and

(1.3) $$D=\left\{(x_1, \cdots, x_p)' \,\Big|\, \sum_{j=1}^{p} x_j^2 \leqq 1\right\}.$$

Our result is the following.

Theorem 1. *Let $Z(\tau)$, $\tau\in[0, \infty)$, be the Rosenblatt process with the representation*

$$Z(\tau)=\sqrt{(1-2\alpha)(2-2\alpha)}\,\Gamma\left(\frac{1+\alpha}{2}\right)\left\{\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma(\alpha)\right\}^{-1}$$

$$\int_{-\infty}^{\infty} dB(u_1)\int_{-\infty}^{u_1} dB(u_2)\int_0^{\tau} \{(s-u_1)(s-u_2)\}^{-(\alpha+1)/2} I[u_1<s]I[u_2<s]ds,$$

$B(\cdot)$ being a standard Brownian motion, and let $Z_j(\tau)$, $\tau\in[0, \infty)$, $j=1\cdots, p$, are independent copies of $Z(\tau)$, $\tau\in[0, \infty)$. Then, under the above conditions, as $t\to\infty$

$$\Delta_t(\tau)\equiv\{\mathrm{Var}\, M(t)\}^{-1/2}\{M(t\tau)-EM(t\tau)\}$$

converges weakly in the space $C[0, \infty)$ to

(1.4) $$\Delta(\tau)\equiv\left\{c_1^2\left(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\right)^2+\sum_{j=2}^{p} c_j^2\left(\frac{1-\rho_\infty}{1-\rho_0}\right)^2\right\}^{-1/2}$$

$$\left\{c_1\left(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\right)Z_1(\tau)+\sum_{j=2}^{p} c_j\left(\frac{1-\rho_\infty}{1-\rho_0}\right)Z_j(\tau)\right\},$$

where $\{c_j, j=1, \cdots, p\}$ are non-zero constants determined later in the proof.

2. Proof

Define by $\mathfrak{F}$ the class of real-valued functions of p variables square-integrable with respect to $\prod_{j=1}^{p}\phi(x_j)$, $\phi(x)$ being the standard normal density, and let $H_n(x)$, $n=0, 1, 2, \cdots$ denote the n-th normalized Hermite polynomial defined by $H_n(x)=(-1)^n\phi(x)^{-1}(d^n/dx^n)\phi(x)$. Then $f\in\mathfrak{F}$ has the expansion

(2.1) $$f(x_1, \cdots, x_p)=\sum_{n=0}^{\infty}\ \sum_{\sum_{j=1}^{p} n_j=n} c(n_1, \cdots, n_p)\prod_{j=1}^{p} H_{n_j}(x_j)$$

in the mean square sense, where

$$c(n_1, \cdots, n_p)=\left\{\prod_{j=1}^{p} n_j!\right\}^{-1}\int_{R^p} f(x_1, \cdots, x_p)\prod_{j=1}^{p}\{H_{n_j}(x_j)\phi(x_j)dx_j\},$$

$$n_j\geqq 0,\ \sum_{j=1}^{p} n_j=n.$$

The validity of (2.1) is well-known for $p=1, 2$ (cf. [3]) and it can be shown for general p by the same reasoning as for $p=1$, if we use the following lemma (a special case of Lemma 3.2 in [5]).

Lemma 1. *Let* $(\xi_1, \cdots, \xi_{2p})$ *be* $2p$*-dimensional Gaussian random variables satisfying*

$$E\xi_j=0,\ E\xi_j^2=1, \qquad 1\leqq j\leqq 2p,$$

$$E\xi_j\xi_{j+p}=r_j, \qquad 1\leqq j\leqq p,$$

$$E\xi_j\xi_k=0, \qquad \text{if} \quad (j, k)\notin\{(j, j), (j+p, j+p), (j, j+p);\ 1\leqq j\leqq p\}.$$

Then

$$E\Big[\prod_{j=1}^{p} H_{n_j}(\xi_j)H_{m_j}(\xi_{j+p})\Big]=\prod_{j=1}^{p}\delta_{n_j m_j} n!r_j^{n_j},$$

where $\delta_{nm}=1$ *or* 0 *according as* $n=m$ *or* $n\neq m$.

Let

$$T=(t_{ij})_{1\leq i, j\leq p}, \tag{2.2}$$

where

$$t_{1j}=\frac{1}{\sqrt{p(1+(p-1)\rho_0)}}, \qquad 1\leqq j\leqq p,$$

$$t_{ij}=\frac{1}{\sqrt{i(i-1)(1-\rho_0)}}, \qquad 2\leqq i\leqq p, \qquad 1\leqq j<i,$$

$$t_{ii}=-\frac{i-1}{\sqrt{i(i-1)(1-\rho_0)}}, \qquad 2\leqq i\leqq p,$$

$$t_{ij}=0, \qquad 2\leqq i\leqq p-1, \qquad i<j\leqq p,$$

and define $Y(t)=(Y_1(t), \cdots, Y_p(t))'$ by $Y(t)=TX(t)$. Then

$$\tilde{R}(t)\equiv EY(0)Y(t)'=(\tilde{R}_{ij}(t))_{1\leq i, j\leq p},$$

where

$$\tilde{R}_{ij}(t)=\begin{cases}\dfrac{r(t)+(p-1)\rho(t)}{1+(p-1)\rho_0}, & \text{if} \quad i=j=1,\\ \dfrac{r(t)-\rho(t)}{1-\rho_0}, & \text{if} \quad i=j=2, 3, \cdots, p,\\ 0, & \text{if} \quad i\neq j.\end{cases} \tag{2.3}$$

Therefore, $\{Y_j(t)\}$, $1\leqq j\leqq p$, are independent Gaussian processes

We then have

$$M(t)=\int_0^t I[X(s)\in D]ds=\int_0^t I[Y(s)\in\tilde{D}]ds,$$

where

$$\tilde{D}=\{(y_1, \cdots, y_p)'\,|\,T^{-1}(y_1, \cdots, y_p)'\in D\},$$

and by (2.1)

$$M(t)=\sum_{n=0}^{\infty}\sum_{\Sigma_{j=1}^{p}n_j=n} c(n_1, \cdots, n_p)\int_0^t \prod_{j=1}^{p} H_{n_j}(Y_j(s))ds,$$

where

$$c(n_1, \cdots, n_p)=\left\{\prod_{j=1}^{p} n_j!\right\}^{-1}\int_{\tilde{D}} \prod_{j=1}^{p}\{H_{n_j}(x_j)\phi(x_j)dx_j\}. \tag{2.4}$$

For proving the theorem, the following proposition is essential. For an $f\in\mathcal{F}$, define m by

$$m=\min\{n\geqq 1 \mid \text{there exists } c(n_1, \cdots, n_p)\neq 0 \text{ with } \textstyle\sum_{j=1}^{p} n_j=n \text{ in the expansion (2.1)}\}.$$

We call this m the Hermite rank of $f\in\mathcal{F}$, following the definition due to Taqqu [6].

Proposition 1. *Suppose that the Hermite rank of f is m and $r(t)\sim t^{-\alpha}L(t)$, $\rho(t)\sim\rho_\infty t^{-\alpha}L(t)$ for some α with $0<\alpha<1/m$. Write*

$$K_t(\tau)=\int_0^{t\tau} f(Y_1(s), \cdots, Y_p(s))ds$$

and

$$I_t(\tau)=\sum_{\Sigma_{j=1}^{p}m_j=m} c(m_1, \cdots, m_p)\int_0^{t\tau}\prod_{j=1}^{p} H_{m_j}(Y_j(s))ds.$$

Then as $t\to\infty$

$$\{\operatorname{Var} K_t(1)\}^{-1/2}\{K_t(\tau)-EK_t(\tau)\}$$

is asymptotically equal to

$$\{\operatorname{Var} I_t(1)\}^{-1/2}I_t(\tau)$$

in the sense of the finite dimensional distributions.

Proof. If we could show that for any $a_j\in\boldsymbol{R}$, $\tau_j\in[0, \infty)$, $j=1, \cdots, h$,

$$\lim_{t\to\infty} E\left|\sum_{j=1}^{h} a_j[\{\operatorname{Var} K_t(1)\}^{-1/2}\{K_t(\tau_j)-EK_t(\tau_j)\} - \{\operatorname{Var} I_t(1)\}^{-1/2}I_t(\tau_j)]\right|^2=0$$

then the proposition follows. For that, it is enough to show that

$$\lim_{t\to\infty} E|\{\operatorname{Var} K_t(1)\}^{-1/2}\{K_t(\tau)-EK_t(\tau)\} - \{\operatorname{Var} I_t(1)\}^{-1/2}I_t(\tau)|^2=0. \tag{2.5}$$

Note that

$$EK_t(\tau)=c(0, \cdots, 0)t\tau,$$

and

$$K_t(\tau)-EK_t(\tau)=I_t(\tau)+R_t(\tau),$$

where

$$R_t(\tau)=\sum_{n=m+1}^{\infty}\sum_{\Sigma_{j=1}^{p}n_j=n} c(n_1,\cdots,n_p)\int_0^{t\tau}\prod_{j=1}^{p}H_{n_j}(Y_j(s))ds.$$

We have

$$E\{K_t(\tau)-EK_t(\tau)\}^2=EI_t(\tau)^2+ER_t(\tau)^2+2EI_t(\tau)R_t(\tau),$$

however by Lemma 1

$$\begin{aligned}EI_t(\tau)R_t(\tau)=&\sum_{n=m+1}^{\infty}\sum_{\Sigma_{j=1}^{p}m_j=m}\sum_{\Sigma_{j=1}^{p}n_j=n} c(m_1,\cdots,m_p)c(n_1,\cdots,n_p)\\ &\times\int_0^{t\tau}\int_0^{t\tau}E\Big[\prod_{j=1}^{p}H_{m_j}(Y_j(u))H_{n_j}(Y_j(v))\Big]du\,dv\\ =&0.\end{aligned}$$

Hence

$$\text{Var }K_t(\tau)=\text{Var }I_t(\tau)+\text{Var }R_t(\tau). \tag{2.6}$$

We have, by Lemma 1, (2.3), (1.1) and (1.2),

$$\begin{aligned}\text{Var }I_t(\tau)=&\sum_{\Sigma_{j=1}^{p}m_j=m}\sum_{\Sigma_{j=1}^{p}k_j=m} c(m_1,\cdots,m_p)c(k_1,\cdots,k_p)\\ &\times\int_0^{t\tau}\int_0^{t\tau}E\Big\{\prod_{j=1}^{p}H_{m_j}(Y_j(u))H_{k_j}(Y_j(v))\Big\}du\,dv\\ =&\sum_{\Sigma_{j=1}^{p}m_j=m} c(m_1,\cdots,m_p)^2\int_0^{t\tau}\int_0^{t\tau}m_1!\\ &\times\Big(\frac{r(u-v)+(p-1)\rho(u-v)}{1+(p-1)\rho_0}\Big)^{m_1}\prod_{j=2}^{p}m_j!\Big(\frac{r(u-v)-\rho(u-v)}{1-\rho_0}\Big)^{m_j}du\,dv\\ =&2\sum_{\Sigma_{j=1}^{p}m_j=m} c(m_1,\cdots,m_p)^2\int_0^{t\tau}(t\tau-s)m_1!\\ &\times\Big(\frac{r(s)+(p-1)\rho(s)}{1+(p-1)\rho_0}\Big)^{m_1}\prod_{j=2}^{p}m_j!\Big(\frac{r(s)-\rho(s)}{1-\rho_0}\Big)^{m_j}ds\\ \sim&2\sum_{\Sigma_{j=1}^{p}m_j=m} c(m_1,\cdots,m_p)^2\int_0^{t\tau}(t\tau-s)m_1!\\ &\times\Big(\frac{s^{-\alpha}L(s)+(p-1)\rho_\infty s^{-\alpha}L(s)}{1+(p-1)\rho_0}\Big)^{m_1}\prod_{j=2}^{p}m_j!\Big(\frac{s^{-\alpha}L(s)-\rho_\infty s^{-\alpha}L(s)}{1-\rho_0}\Big)^{m_j}ds\\ =&2\sum_{\Sigma_{j=1}^{p}m_j=m} c(m_1,\cdots,m_p)^2 m_1!\Big(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\Big)^{m_1}\end{aligned}$$

$$\times \prod_{j=2}^{p} m_j!\Big(\frac{1-\rho_\infty}{1-\rho_0}\Big)^{m_j}\int_0^{t\tau}(t\tau-s)s^{-m\alpha}L(s)^m ds$$
$$\sim C(t\tau)^{2-m\alpha}L(t\tau)^m,$$

where

$$C=\frac{2}{(1-m\alpha)(2-m\alpha)}\sum_{\Sigma_{j=1}^p m_j=m} c(m_1, \cdots, m_p)^2 m_1!\Big(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\Big)^{m_1}\prod_{j=2}^{p} m_j!\Big(\frac{1-\rho_\infty}{1-\rho_0}\Big)^{m_j}.$$

On the other hand,

$$\mathrm{Var}\, R_t(\tau)=2\sum_{n=m+1}^{\infty}\sum_{\Sigma_{j=1}^p n_j=n} c(n_1, \cdots, n_p)^2\int_0^{t\tau}(t\tau-s)$$
$$\times n_1!\Big(\frac{r(s)+(p-1)\rho(s)}{1+(p-1)\rho_0}\Big)^{n_1}\prod_{j=2}^{p} n_j!\Big(\frac{r(s)-\rho(s)}{1-\rho_0}\Big)^{n_j}ds.$$

For any $n_1, \cdots, n_p$ with $\Sigma_{j=1}^p n_j \geqq m+1$, there exist $q_1, \cdots, q_p$ depending on $n_1, \cdots, n_p$ such that $q_j \leqq n_j$, $\Sigma_{j=1}^p q_j=m+1$. Then

$$\mathrm{Var}\, R_t(\tau)\leqq 2\sum_{n=m+1}^{\infty}\sum_{\Sigma_{j=1}^p n_j=n} c(n_1, \cdots, n_p)^2\int_0^{t\tau}(t\tau-s)$$
$$\times n_1!\Big(\frac{r(s)+(p-1)\rho(s)}{1+(p-1)\rho_0}\Big)^{q_1}\prod_{j=2}^{p} n_j!\Big(\frac{r(s)-\rho(s)}{1-\rho_0}\Big)^{q_j}ds$$
$$\sim 2\sum_{n=m+1}^{\infty}\sum_{\Sigma_{j=1}^p n_j=n} c(n_1, \cdots, n_p)^2 n_1!\Big(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\Big)^{q_1}$$
$$\times \prod_{j=2}^{p} n_j!\Big(\frac{1-\rho_\infty}{1-\rho_0}\Big)^{q_j}\int_0^{t\tau}(t\tau-s)s^{-(m+1)\alpha}L(s)^{m+1}ds.$$

Note that the Parseval identity gives us

$$\sum_{n=0}^{\infty}\sum_{\Sigma_{j=1}^p n_j=n} c(n_1, \cdots, n_p)^2\prod_{j=1}^{p} n_j!=\int_{R^p} f(x_1, \cdots, x_p)^2\prod_{j=1}^{p}(\phi(x_j)dx_j)<\infty.$$

Hence for large t,

$$\mathrm{Var}\, R_t(\tau)=o(\mathrm{Var}\, I_t(\tau)), \tag{2.7}$$

which together with (2.6) gives us

$$\mathrm{Var}\, K_t(\tau)\sim \mathrm{Var}\, I_t(\tau).$$

Thus the left hand side of (2.5) turns out to be

$$\lim_{t\to\infty}\{\mathrm{Var}\, I_t(1)\}^{-1}E|K_t(\tau)-EK_t(\tau)-I_t(\tau)|^2$$
$$=\lim_{t\to\infty}\{\mathrm{Var}\, I_t(1)\}^{-1}ER_t(\tau)^2=0$$

by (2.7). The proof of the proposition is thus complete.

The limiting distribution in the theorem is determined by the following lemma.

Lemma 2. *Suppose that* (1.1) *and* (1.2) *are satisfied with* $0<\alpha<1/2$ *and let*

$$J_t(\tau)=\int_0^{t\tau}\sum_{j=1}^{p} c_j H_2(Y_j(s))ds, \qquad \tau\in[0, \infty).$$

Then as $t\to\infty$

$$\{\mathrm{Var}\, J_t(1)\}^{-1/2} J_t(\tau)$$

converges weakly in the space $C[0, \infty)$ *to* $\Delta(\tau)$ *defined in* (1.4).

Proof. Let

$$J_t^j(\tau)=\int_0^{t\tau} H_2(Y_j(s))ds, \qquad j=1, \cdots, p, \qquad \tau\in[0, \infty).$$

Since $\{Y_j(t)\}$, $j=1, \cdots, p$, are independent,

$$\mathrm{Var}\, J_t(1)=\sum_{j=1}^{p} c_j^2\, \mathrm{Var}\, J_t^j(1).$$

We have

$$\begin{aligned}\mathrm{Var}\, J_t^1(1)&=\int_0^t\int_0^t EH_2(Y_1(u))H_2(Y_1(v))du\,dv\\ &=2\int_0^t\int_0^t\left(\frac{r(u-v)+(p-1)\rho(u-v)}{1+(p-1)\rho_0}\right)^2 du\,dv\\ &=4\int_0^t (t-s)\left(\frac{r(s)+(p-1)\rho(s)}{1+(p-1)\rho_0}\right)^2 ds\\ &\sim\frac{4}{(1-2\alpha)(2-2\alpha)}\left(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\right)^2 t^{2-2\alpha}L(t)^2\end{aligned}$$

and for $j=2, \cdots, p$

$$\begin{aligned}\mathrm{Var}\, J_t^j(1)&=\int_0^t\int_0^t EH_2(Y_j(u))H_2(Y_j(v))du\,dv\\ &=2\int_0^t\int_0^t\left(\frac{r(u-v)-\rho(u-v)}{1-\rho_0}\right)^2 du\,dv\\ &=4\int_0^t (t-s)\left(\frac{r(s)-\rho(s)}{1-\rho_0}\right)^2 ds\\ &\sim\frac{4}{(1-2\alpha)(2-2\alpha)}\left(\frac{1-\rho_\infty}{1-\rho_0}\right)^2 t^{2-2\alpha}L(t)^2.\end{aligned}$$

Hence

$$\{\mathrm{Var}\, J_t(1)\}^{-1/2} J_t(\tau)\sim\left\{c_1^2\left(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\right)^2+\sum_{j=2}^{p} c_j^2\left(\frac{1-\rho_\infty}{1-\rho_0}\right)^2\right\}^{-1/2}$$

$$\times\left\{c_1\left(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\right)\{\operatorname{Var} J_l^1(1)\}^{-1/2}J_l^1(\tau)\right.$$

$$\left.+\sum_{j=2}^{p}c_j\left(\frac{1-\rho_\infty}{1-\rho_0}\right)\{\operatorname{Var} J_l^j(1)\}^{-1/2}J_l^j(\tau)\right\}.$$

By the non-central limit theorem ([2], [6]), under our conditions,

$$\{\operatorname{Var} J_l^j(1)\}^{-1/2}J_l^j(\tau)\overset{W}{\Longrightarrow} Z_j(\tau), \qquad \tau\in[0, \infty)$$

in the sense of weak convergence in the space $C[0, \infty)$. Since $\{J_l^j(\tau), \tau\in[0, \infty)\}$ are independent, the proof of the lemma is complete.

Now, to prove the theorem, we have to calculate (2.4) for

$$\tilde{D}=\{(y_1, \cdots, y_p)' \mid T^{-1}(y_1, \cdots, y_p)'\in D\},$$

where T and D are defined in (2.2) and (1.3), respectively. Since

$$T^{-1}=(s_{ij})_{1\leq i,j\leq p},$$

where

$$s_{i1}=\frac{\sqrt{1+(p-1)\rho_0}}{\sqrt{p}}, \qquad 1\leqq i\leqq p,$$

$$s_{ij}=\frac{\sqrt{1-\rho_0}}{\sqrt{i(i-1)}}, \qquad 2\leqq j\leqq p, \qquad 1\leqq i\leqq j,$$

$$s_{ii}=-\frac{(i-1)\sqrt{1-\rho_0}}{\sqrt{i(i-1)}}, \qquad 2\leqq i\leqq p,$$

$$s_{ij}=0, \qquad 2\leqq j\leqq p-1, \qquad j<i\leqq p,$$

we have

$$\tilde{D}=\left\{(y_1, \cdots, y_p)' \,\middle|\, (1+(p-1)\rho_0)y_1^2+(1-\rho_0)\sum_{j=2}^{p}y_j^2\leqq 1\right\}.$$

Obviously,

$$c(n_1, \cdots, n_p)=0$$

for any $n_1, \cdots, n_p$ with $n_1+\cdots+n_p=1$, since $\tilde{D}$ is symmetric with respect to each axis. Also,

$$c(n_1, \cdots, n_p)=0$$

for any $n_1, \cdots, n_p$ with $n_1+\cdots+n_p=2$ such that $n_j\leqq 1$ for all j.

Note that

$$\int_{-a}^{a}H_2(y)\phi(y)dy=-2a\phi(a),$$

and put

$$\alpha\equiv\alpha(y_2, \cdots, y_p)\equiv\left\{\frac{1-(1-\rho_0)\sum_{j=2}^{p} y_j^2}{1+(p-1)\rho_0}\right\}^{1/2}.$$

Then the standard calculation gives us

$$c(2, 0, \cdots 0)=\frac{1}{2}\int\underset{D}{\cdots}\int H_2(y_1)\phi(y_1)\phi(y_2)\cdots\phi(y_p)dy_1dy_2\cdots dy_p$$

$$=-\int\underset{A}{\cdots}\int\alpha\phi(\alpha)\phi(y_2)\cdots\phi(y_p)dy_2\cdots dy_p<0,$$

where

$$A=\left\{(y_2, \cdots, y_p)'\,\middle|\,\sum_{j=2}^{p} y_j^2\leqq(1-\rho_0)^{-1}\right\}.$$

Further put

$$\beta\equiv\beta(y_1, y_3, y_4, \cdots, y_p)$$
$$\equiv\left\{(1-\rho_0)^{-1}[1-(1+(p-1)\rho_0)y_1^2]-\sum_{j=3}^{p} y_j^2\right\}^{1/2},$$

then we have

$$c(0, 2, 0, 0, \cdots, 0)=-\int\underset{B}{\cdots}\int\beta\phi(\beta)\phi(y_1)\phi(y_3)\phi(y_4)\cdots\phi(y_p)dy_1dy_3dy_4\cdots dy_p<0$$

where

$$B=\left\{(y_1, y_3, y_4, \cdots, y_p)'\,\middle|\,(1-\rho_0)^{-1}[1-(1+(p-1)\rho_0)y_1^2]\geqq\sum_{j=3}^{p} y_j^2\right\},$$

and

$$c(0, 2, 0, \cdots, 0)=c(0, 0, 2, 0, \cdots, 0)=\cdots=c(0, 0, \cdots, 0, 2).$$

Therefore the Hermite rank of $f(y_1, \cdots, y_p)=I[(y_1, \cdots, y_p)'\in\tilde{D}]$ is 2. If we determine $\{c_j\}$ in the theorem by $c_j=c(n_1, \cdots, n_p)$, where $n_j=2, n_i=0$ for $i\neq j$, then Proposition 1 and Lemma 2 prove the finite dimensional convergence of $\Delta_t(\tau)$ to $\Delta(\tau)$.

It remains to prove the tightness of $\{\Delta_t(\tau), \tau\in[0, \infty)\}$. We have, for $0<\tau_1<\tau_2$

$$E|\Delta_t(\tau_2)-\Delta_t(\tau_1)|^2=\{\mathrm{Var}\, M(t)\}^{-1}E\left|\int_{t\tau_1}^{t\tau_2}(I[X(s)\in D]-EI[X(s)\in D])ds\right|^2. \tag{2.8}$$

Since the Hermite rank of $f(x_1, \cdots, x_p)\equiv I[(x_1, \cdots, x_p)'\in D]$ is 2 in our case, as in the proof of Proposition 1, we have

$$\mathrm{Var}\, M(t)\sim Ct^{2-2\alpha}L(t)^2 \tag{2.9}$$

and

$$E\left|\int_{t\tau_1}^{t\tau_2}(I[X(s)\in D]-EI[X(s)\in D])ds\right|^2 \tag{2.10}$$

$$\leqq \text{const.} \times \int_0^{t(\tau_2-\tau_1)} (t(\tau_2-\tau_1)-s)s^{-2\alpha}L(s)^2 ds$$

$$= \text{const.} \times \{t(\tau_2-\tau_1)\}^{2-2\alpha} L(t(\tau_2-\tau_1))^2.$$

It follows from (2.8)-(2.10) that

$$E|\Delta_t(\tau_2)-\Delta_t(\tau_1)|^2 \leqq \text{const.} \times (\tau_2-\tau_1)^{1+\gamma},$$

where $\gamma=1-2\alpha>0$, and hence the tightness of $\{\Delta_t(\tau)\}$ follows from the well-known criterion.

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