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ON ELEMENTARY DEFORMATIONS OF MAPS OF SURFACES INTO 3-MANIFOLDS I

By

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§ 1. Introduction.

S. Smale [S] classified the set of immersions of S^2 into $E^n(2<n)$ by regular homotopy. M. Hirsch [Hr] generalized Smale's result as follows: The regular homotopy classes of a C^∞ manifold M^m to a C^∞ manifold $N^n(m>n)$ are in one-to one correspondence with homotopy classes of bundle maps of tangent bundle $T(M^m)$ to $T(N^n)$. This Smale-Hirsch theorem is a theorem about global moves of homotopy. To attack the Poincaré Conjecture, W. Haken [Hk] examined local moves between immersions of a 2-sphere into a homotopy 3-ball N^3. What he found is that ∂N^3 can be deformed into a 3-ball in N^3 by four types of local deformations, elementary deformations, such that these four types of deformations take place in a special order. It is natural to consider the following question: how many kinds of elementary deformations are needed, for two given immersions which are regularly homotopic, to convert one immersion to the other? Or more generally, how many kinds of elementary deformations are needed to convert one nice map to another nice map? A nice map is a piecewise linear map of a surface into a 3-manifold whose singularities consist of a finite number of double curves, triple points and branch points.

In this paper we shall consider deformations of a homotopy between two nice maps of a surface in a 3-manifold into a finite sequence of elementary deformations. Elementary deformations are basic local moves of nice maps.

This is the first of two papers devoted to the study of deformations of a homotopy between two nice maps of a surface to a 3-manifold into a finite sequence of elementary deformations. In the second paper [HN], we shall prove the following:

Theorem. *Jf two maps f and g are nice maps of a closed surface into the interior of a 3-manifold, and if they are homotopic, then one deforms to the other by a finite sequence of six kinds of deformations and their inverses which we call elementary deformations.*

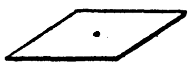
In this note we shall define special deformations, which are compositions of the elementary deformations, and prove a special case (Disk-trade lemma) of the above. Pushing-disk lemma (Lemma 2.3) is a key lemma of Theorem. The proof of Theorem is parallel to the one of Pushing-disk lemma.

We work in the piecewise linear category.

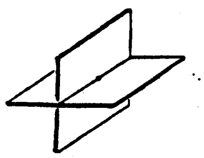
The interior, closure, and boundary of $(\cdots)$ are denoted by $\mathrm{Int}(\cdots)$, $\mathrm{Cl}(\cdots)$, and $\partial(\cdots)$ respectively. The unit interval and the closed interval $[-1, 1]$ will be denoted by I and J respectively.

We use the definitions and notations in [R] and [Z] without notice.

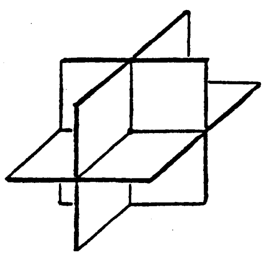
For a map $f: F^2 \to M^3$ of a closed surface to a 3-manifold, the *singularities* of f is the closure of the set of those points $x \in f(F^2)$ for which $f^{-1}(x)$ consists of more than one point. A point $x \in f(F^2)$ is called a *regular* point, if it does not belong to the sungularities of f (see [P]). The map f is said to be *nice*, provided that the singularities of f consist of *double curves* in which two sheets pierce each other, *triple points* in which three sheets pierce each other, and *branch points* from each of which exactly one double curve originates, and that at each point of F^2 whose image is not a branch point, the map f is locally homeomorphic. Neighborhoods of the singularities of f are shown in Figure 1.1. We consider that neither triple points nor branch poihts belong to double curves. Hence the image of f is the disjoint union of the set of regular points, the set



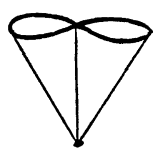
A neighborhood of a regular point.



A neighbord of a double point which is homeomorphic to the set $(J\times 0 \quad 0\times J)\times I$.



A neighborhood of a triple point which is homeomorphic to the set $J\times J\times 0 \quad J\times 0\times J \quad 0\times J\times J$.



A neighborhood of a branch point which is a cone over figure 8.

Fig. 1.1

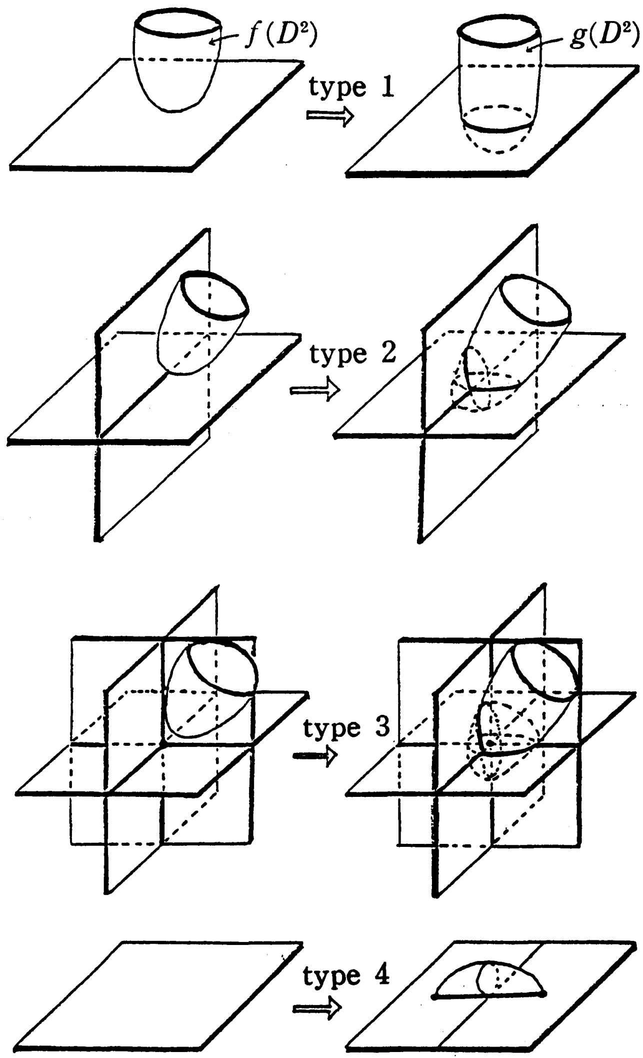


Figure 1.2

of double curves, the set of triple points, and the set of branch points.

We define as follows two kinds of modification on the set of nice maps of a surface F^2 to a manifold M^3. Let $f: F^2 \to M^3$ be a nice map. Modification Γ of f to a new nice map $g: F^2 \to M^3$ is called an *h-move*, if there exist a disk D^2 in F^2 and a ball B^3 in M^3 such that the maps f and g coincide outside the disk D^2 and such that the images $f(D^2)$ and $g(D^2)$ lie in the interior of the ball B^3. The pair (D^2, B^3) is called a *support pair* of the h-move Γ. An h-move Γ of f to g is called an *elementary deformation* of type i ($i=0, 1, 2, 3, 4, 5, 6$), if one of the following conditions holds for a support pair (D^2, B^3) of the h-move Γ (see Figure 1.2 and Figure 1.3.):

Type 0. There exists an orientation preserving homeomorphism of B^3 to

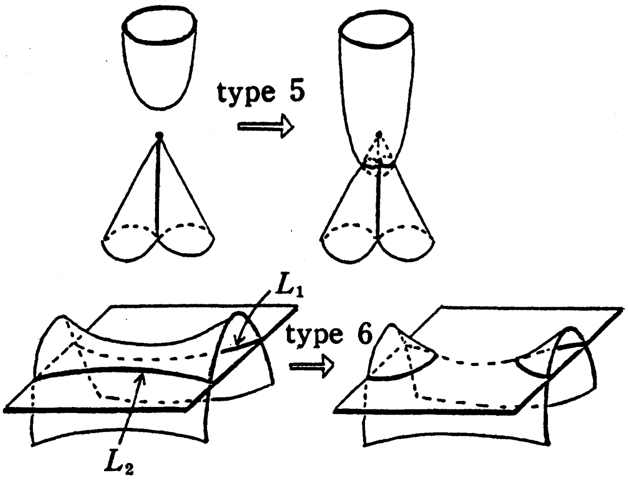


Fig. 1.3

itself that is the identity on the boundary of B^3 and that sends $f(F^2)\cap B^3$ to $g(F^2)\cap B^3$.

Type 1. The intersection $B^3\cap f(F^2)$ consists of two disjoint disk. The union $f(D^2)\cup g(D^2)$ bounds a 3-ball in B^3 whose interior intersects $f(F^2)$ by an open disk.

Type 2. The intersection $B^3\cap f(B^2)$ consists of three disks, two of which pierce each other by a double curve and the other one contains $f(D^2)$ without intersecting the two disks. The union $f(D^2)\cup g(D^2)$ bounds a 3-ball in B^3 which intersects each of the three disks by a disk, one of these is $f(D^2)$ and the other two disks pierce each other by a double curve.

Type 3. The intersection $B^3\cap f(F^2)$ consists of four disks, three of which pierce each other and the other one contains $f(D^2)$ without intersecting the three disks. The union $f(D^2)\cup g(D^2)$ bounds a 3-ball in B^3 which intersects each of the four disks by a disk, one of these four disks is $f(D^2)$ and the other three disks pierce each other.

Type 4. The intersection $B^3\cap f(F^2)$ consists of a disk. The intersection $B^3\cap g(F^2)$ contains exactly two branch points and one double curve which originates from one of the branch points and terminates the other branch point.

Type 5. The intersection $B^3\cap f(F^2)$ consists of a disk containing $f(D^2)$ and a singular disk containing exactly one branch point and a double curve which originates from the branch point. The union $f(D^2)\cup g(D^2)$ bounds a ball in B^3 which intersects $f(F^2)$ by the disk $f(D^2)$ and a singular disk branch point and a double curve.

Type 6. The intersection $B^3\cap f(F^2)$ consists of two disks which intersect each other by two double curves, say L_1 and L_2. The same situation holds for the intersection $B^3\cap g(F^2)$. But each of the double curves in $B^3\cap g(F^2)$ originates from an end point of L_1 and terminates an end point of L_2.

Each inverse deformation of an elementary deformation of Type i is called an elementary deformation of Type i^{-1}, which we also call an *elementary deformation*. We ignore elementary deformations of Type 0 in the proofs. Namely we do not mention where and when we use elementary deformations of Type 0 in each proof.

Our deformations of Type 1, 2, 6 correspond to Haken's deformations of Type 1*a*, 1*b*, 2 respectively. Haken's deformation of Type 3 is the composition of his deformations of Type 1*b* and 2. To deform nice maps, deformations of Type 3, 4, 5 are definitely needed.

We use the sign /// to indicate the end of proofs.

§ 2. Special deformations.

In this section we investigate elementary deformations and develop elementary techniques.

Let $f_1, f_2 : F^2 \to M^3$ be nice maps, one of which is obtained from the other

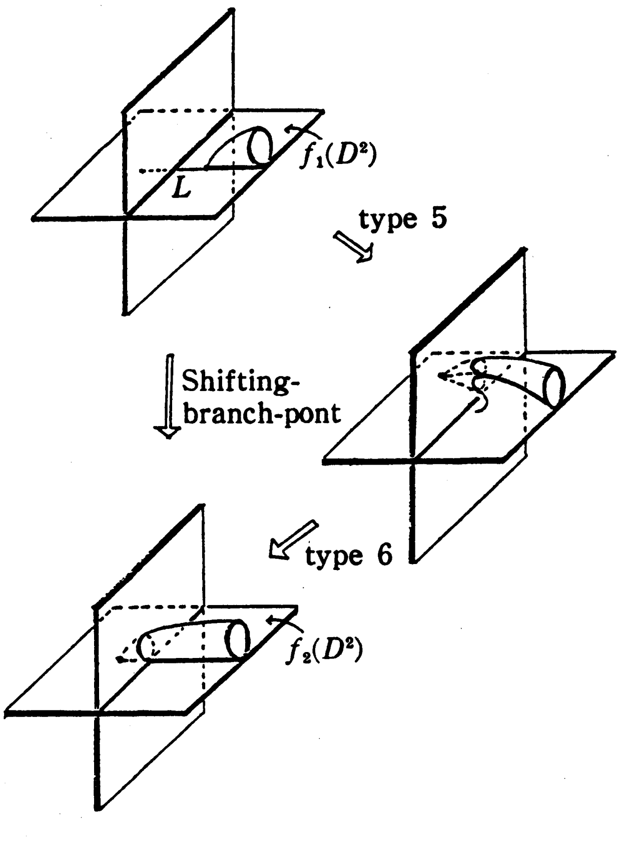


Fig. 2.1

by a single h-move with a support pair (D^2, B^3). We say that f_1 deforms to f_2 by *shifting-branch-point* (along a curve L), if their images in B^3 are like those shown in Figure 2.1.

Lemma 2.1. (Shifting-branch-point) *Let $f_1, f_2: F^2 \to M^3$ be nice maps. If one of them deforms to the other by shifting-branch-point, then there exists a finite sequence of elementary deformations which deforms one to the other.*

Proof. First apply an elementary deformation of Type 5, and then an elementary deformation of Type 6. The detail will be omitted (see Figure 2.1).///

Let $f_1, f_2: F^2 \to M^3$ be nice maps, one of which is obtained from the other by a single h-move with a support pair (D^2, B^3). We say that f_1 deforms to f_2 by *squeezing-off*, if their images in B^3 are like those shown in Figure 2.2.

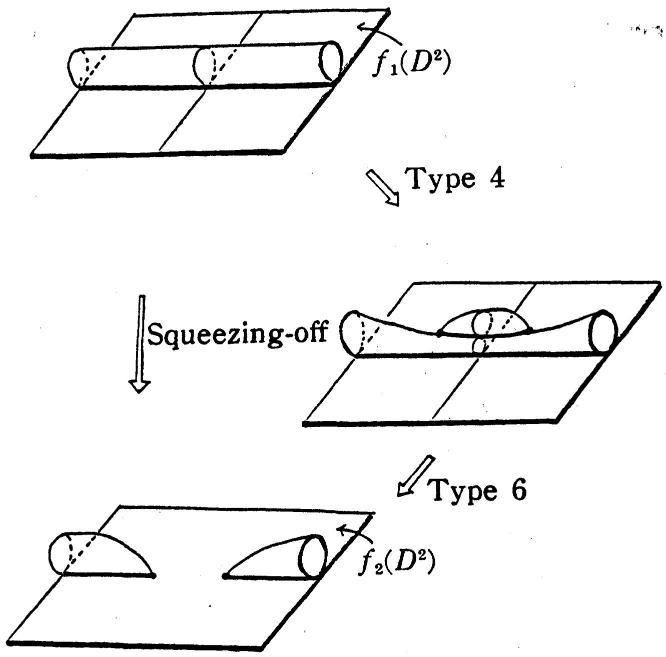


Fig. 2.2

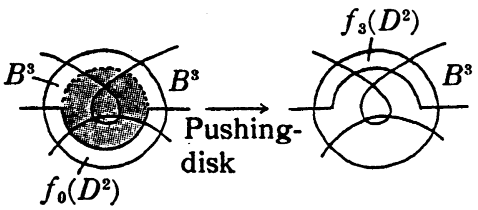
Lemma 2.2. (Squeezing-off) *Let $f_1, f_2: F^2 \to M^3$ be nice maps. If one of them deforms to the other by squeezing-off, then there exists a finite sequence of elementary deformations of Type* 4 *and Type* 6 *which deforms one to the other.*

Proof. The detail will be omitted (see Figure 2.2). ///

A *loop* means a simple closed curve. A loop is said to be *essential* in an annulus, if it does not bounds a disk in the annulus.

Let $f_0, f_3: F^2 \to M^3$ be nice maps, one of which is obtained from the other by a single h-move with a support pair (D^2, B^3). Suppose that each of the maps

f_0 and f_3 embeds a regular neighborhood N^2 of D^2 into the ball B^3 and that the union $f_0(D^2)\cup f_3(D^2)$ bounds a ball B^3_* in B^3. Then we say that f_0 deforms to f_3 by a *pushing-disk through the ball* B^3_*. If the ball B^3_* is a relative regular neighborhood of a simple arc L relative to ∂L (see [*Hd*] for the defihition), then we say that f_0 deforms to f_3 by a *pushing-disk along the curve* L. If the ball B^3_* is a relative regular neighborhood of a disk D^2_* relative to ∂D^2_* then we say that f_0 deforms to f_3 by a *pushing-disk-along the disk* D^2_* (see Figure 2.3).



Dimension is reduced to two.

Fig. 2.3

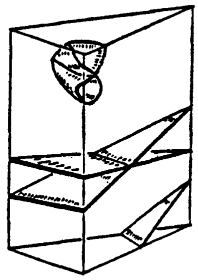

$\tau^2\times I$

Fig. 2.4

Lemma 2.3. (Pushing-disk lemma) *Let* $f_0, f_3: F^2\to M^3$ *be nice maps. If one deforms to the other by a pushing-disk, then there exists a finite sequence a finite sequence of elementary deformations which deforms* f_0 *to* f_3.

Proof. We use the same notations in the definition of pushing-disk. There are two cases.

Case 1. $B^3_*\cap f_0(N^2-D^2=\emptyset$: There exists a homeomorphism $\phi: B^3_*\to I\times I\times I$ such that $\phi^{-1}(I\times I\times 1\cup\partial(I\times I)\times I)=f_0(D^2)$ and $\phi^{-1}(I\times I\times 0)=f_3(D^2)$. If we choose ϕ suitably, by general position argument there exists a triangulation T of $I\times I$ such that

(C0) for each vertex v of T, $\phi^{-1}(v\times I)$ does not meet the singularities of $f_0|(F^2-\text{Int}(N^2))$,

(C1) for each 1-simplex ζ of T, $\phi^{-1}(\zeta\times I)$ does not meet the triple points nor branch points of $f_0|(F^2-\text{Int}(N^2))$, and each connected component of $\phi^{-1}(\zeta\times I)\cap f_0(F^2-\text{Int}(N^2))$ is an arc or homeomorphic to the set $J\times 0\cup 0\times J$,

(C2) for each 2-simplex τ^2 of T, each connected component of $\phi^{-1}(\tau^2\times I)\cap f_0(F^2-\text{Int}(N^2))$ is homeomorphic to a disk, a cone over figure 8, $(J\times 0\cup 0\times J)\times I$, or the set $J\times J\times 0\cup J\times 0\times J\cup 0\times J\times J$ (see Fugure 2.4),

(C3) if $\phi_i=\phi\circ(f_i|D^2)(i=0, 3)$, then $\phi_0\circ\phi_3^{-1}(v\times 0)=v\times 1$ for each vertex v of T in $\text{Int}(I\times I)$, and

(C4) if $X=I\times I\times 1\cup\partial(I\times I)\times I$, then $\phi_0\circ\phi_3^{-1}(\sigma\times 0)=\sigma\times I\cap X$ for each simplex σ of T $(\dim\sigma>0)$.

Let T_0 be the set of vertices of T, which do not lie in $\partial(I\times I)$. Let V^2_0 be a small regular neighborhood of T_0 in $I\times I$, and V_1^2 a small regular neighborhood of V_0^2 in $I\times I$. Then $V_1^2-\mathrm{Int}(V_0^2)$ consists of mutually disjoint annuli. Thanks to Conditions (C0), (C3), and (C4), by a finite sequence of elementary deformations of Type 1, we can push the disks $\phi^{-1}(V_1^2\times 1)$ along the curve $\phi^{-1}(v\times I)$ for each vertex v in T_0 to deform the map f_0 to a map f_1 such that

(1) f_1 and f_0 coincide outside the disks $\phi_0^{-1}(V_1^2\times 1)$,

(2) f_1 and f_3 coincide on the disks $\phi_0^{-1}(V_0^2\times 1)$,

(3) $\phi\circ f_1(D^2)=\partial(I\times I)\times I\cup(I\times I-V_0^2)\times 1\cup\partial V_0^2\times I\cup V_0^2\times 0$, and

(4) if $P^2=\mathrm{Cl}(I\times I-V_0^2)$ and $\phi_1=\phi\circ(f_1|D^2)$, then $\phi_1\circ\phi_3^{-1}((\zeta\cap P^2)\times 0)=\partial(\zeta\cap P^2)\times I\cup(\zeta\cap P^2)\times 1$ for each 1-simplex ζ of T (we may need elementary deformations of Type 0 to get properties (3) and (4)).

Let T_1 be the set of 1-simplexes of T, which do not lie in $\partial(I\times I)$. Let W_0^2 be a small regular neighborhood of $T_1\cap P^2$ in P^2. Then W_0^2 consists of mutually disjoint disks. Let W_1^2 be a small regular neighborhood of W_0^2 in P^2. Thanks to Conditions (C1), (C4), and (4) above, by a finite sequence of elementary deformations of Type 2 and 6, we can push the disks $\phi^{-1}(W_1^2\times 1\cup(W_1^2\cap$

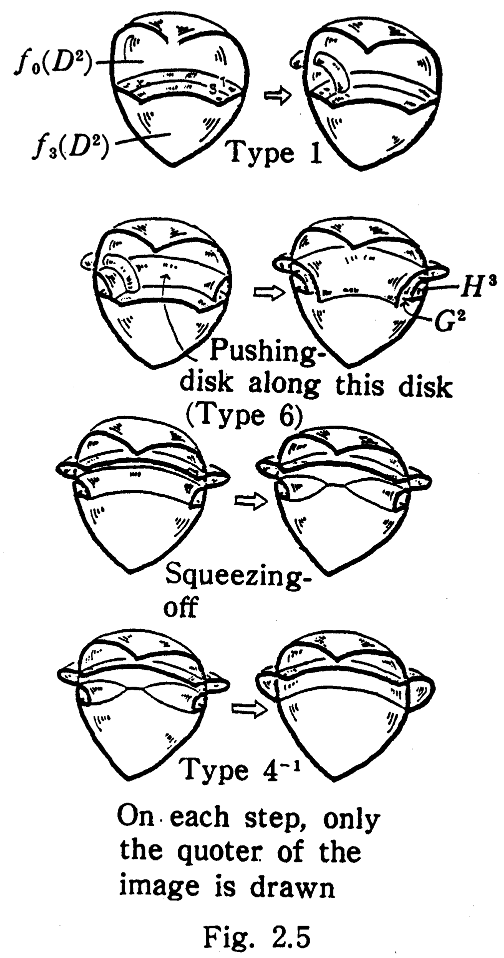


Fig. 2.5

$V_0^2)\times I)$ along the disks $\phi^{-1}(\zeta\cap P^2)\times I$ for each 1-simplex ζ of T to deform the map f_1 to a map f_2 such that

(1) f_2 and f_3 coincide on the set $\phi_3^{-1}(V_0^2\cup W_0^2)$, and

(2) if $Q^2=\mathrm{Cl}(I\times I-(V_0^2\cup W_0^2))$, then $\phi\circ f_2(D^2)=(V_0^2\cup W_0^2)\times 0\cup Q\times 1\cup \partial Q\times I$ (again we may need elementary deformation of Type 0 to get the property (2)).

Finally thanks to Conditions (C2) and (2) above, the map f_2 deforms to the map f_3 by a finite sequence of elementary deformations. Therefore the map f_0 deforms to the map f_2 by a finite sequence of elementary deformations.

Case 2: $f_0(N^2-D^2)\subset B_*^3$: Let S^1 be an essential loop in N^2-D^2 near ∂D^2, i.e., S^1 which does not bound a disk in N^2-D^2. Let A^2 be the annulus on N^2 bounded by the loops S^1 and ∂D^2. We push out the annulus $f_0(A^2)$ from the ball B_*^3 as follows (see Figure 2.5). Let x be a point in S^1. By an elementary deformation of Type 1, we can push out a regular neighborhood of x from the ball B_*^3 through the disk $f_0(D^2)$. By an elementary deformation f_0 of Type 6, we can push out the remaining of S^1. Now a regular neighborhood of S^1 is outside the ball B_*^3. Then on $f_0(D^2)$ two parallel double curves appear near $f_0(\partial D^2)$. The double curve nearer to $f_0(\partial D^2)$ splits off a torus G^2 from the deformed singular disk such that the solid torus H^3, bounded by the torus G^2, intersects with $f_0(F^2-N^2)$ only by mutually disjoint disks. Hence by squeezing-off, the inverse process of shifting-branch-point, and an elementary deformation of Type 4^{-1}, we can eliminate the torus, so that $f_0(A^2)$ is completely outside the ball B_*^3. By this modification on the annulus $\mathrm{Cl}(N^2-D^2)$, we can deform the map f_0 to a map f_0 and also the map f_3 to a map f_3' simultaneously. Set $N_1^2=D^2\cap A^2$. Then we have $B_*^3\cap f_0'(N_1^2-D^2)=\emptyset$. Hence by Case 1, the map f_0' can be deformed to the map f_3' by a finite sequence of elementary deformations. Therefore the result follows from the deformations $f_0\to f_0'\to f_3'\to f_3$. ///

An annulus is said to be *unknotted* in a 3-ball, if it lies on a disk in the 3-ball.

Lemma 2.4. (Rewinding lemma) *Let $f_1: F^2\to M^3$ be a nice map into the interior of M^3. Let S^1 be a loop in F^2 and A^2 a regular neighborhood of S^1 in F^2. Suppose that $f_1|A^2: A^2\to M^3$ is an embedding, and that $f_1(A^2)$ lies in a ball B^3 in M^3. If $f_1(S^1)$ is unknotted in B^3, then for any $\varepsilon>0$ there exists a nice map $f_2: F^2\to M^3$ such that* (1) *$d(f_1, f_2)<\varepsilon$,* (2) *the maps f_1 and f_2 coincids outside N^2,* (3) *there exists a finits sequence of elementary deformations of Type* 4 *which deforms the map f_1 to the map f_2, and* (4) *the map f_2 embeds a regular neighborhood of S^1 in A^2 onto an unknotted annulus in B^3, i.e., the annulus which lies on a disk in B^3.*

Proof. The map f_1 may deform to a map f_* by an elementary deformation of Type 4 such that (1) $f_*(F^2=f_1(F^2)\cup S_*^2$, where S_*^2 is a 2-sphere with

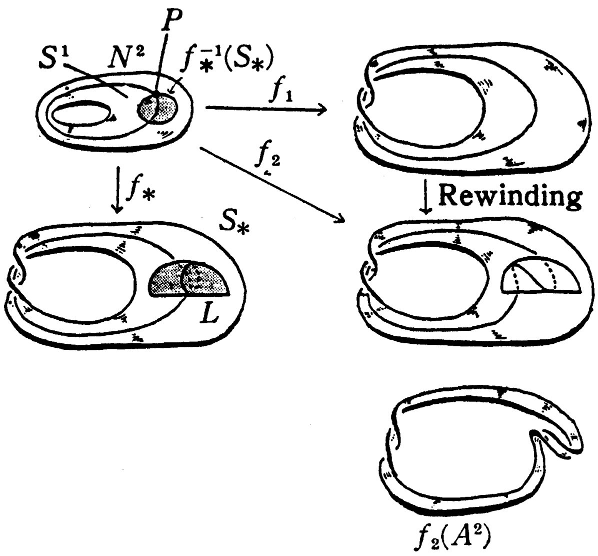


Fig. 2.6

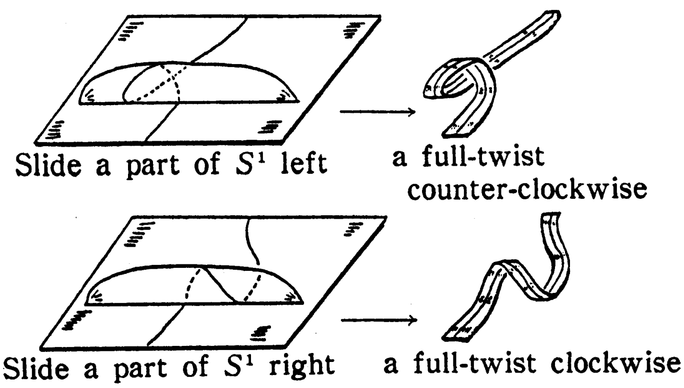


Fig. 2.7

$f_1(F^2)\cap S^2_*$ being the induced double curve of f_*, say L, (2) $f_*^{-1}(S^2_*)$ is a disk in A^2 and is penetrated by the loop S^1, and (3) $f_*(S^1)\supset f_1(S^1)$ (see Figure 2.6).

Let P be an intersection point of the two circles S^1 and $f_*^{-1}(L)$. If we slide S^1 left or right near the point P, then the image $f_*(S^1)$ may be a loop. Moreover for a small regular neighborhood A^2_1 of S^1 in F^2, this sliding of S^1 left or right makes $f_*(A^2_1)$ a full-twist clockwise or counter-clockwise with respect to $f_1(A^2_1)$ (see Figure 2.7). The decreases the twisting number of the annulus $f_1(A^2_1)$ by one. If we repeat this process, we get a desired map f_2.///

Lemma 2.5. *Let $f_1, f_4: F^2\to M^3$ be nice maps, one of which is obtained from the other by an h-move with a support pair (D^2, B^3). Suppose that $f_1|D^2: D^2\to M^3$ deforms to $f_4|D^2$ by an elementary deformation of Type 4. Then f_1 deforms to f_4 by a finite sequence of the elementary deformations (see Figure 2.8).*

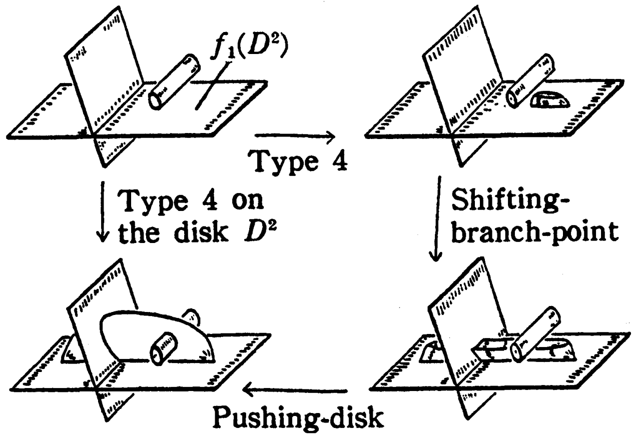


Fig. 2.8

Proof. By the assumption, the restriction $f_1|D^2$ is an embedding, and the restriction $f_4|D^2$ possesses a double curve L and two branch points. Now $(f_4|D^2)^{-1}(L)$ is a circle which bounds a disk D_1^2 in D^2. Then $S^2=f_4(D_1^2)$ is a sphere which bounds a ball B_*^3 in the ball B^3. The ball B_*^3 may intersects $f_4(F^2-D^2)$. First on the disk D^2 we deform the map f_1 to a map f_2 by an elementary deformation of Type 4 such that

(1) the double curve L' of f_2 is contained in the curve L,

(2) the maps $f_2|D^2$ and $f_4|D^2$ possess a branch point in common,

(3) $(f_2|D^2)^{-1}(L')$ bounds the disk D_1^2, and

(4) $f_2(D^2)$ is contained in the ball B_*^3.

Next we several times deform the map f_2 to a map f_3 by shifting-branch-point along the curve $\mathrm{Cl}(L-L')$ such that

(1) $f_3(D_1^2)$ is contained in the ball B_*^3,

(2) f_3 and f_4 differ only on a disk D_2^2 in Int D_1^2, and

(3) $f_3(D_2^2)\cup f_4(D_2^2)$ bounds a ball.

Then we can deform f_3 to f_4 by pushing-disk through the ball bounded by $f_3(D_2^2)\cup f_4(D_2^2)$.///

Lemma 2.6. *Let $f_1, f_2: F^2\to M^3$ be nice maps. Suppose that the map f_1 deforms to the map f_2 by a single h-move with a support pair (D^2, B^3). Suppose that $f_1|D^2: D^2\to B^3$ is an embedding, and that, for a regular neighborhood N^2 of D^2 in F^2, the restriction $f_2|N^2: N^2\to B^3$ is an embedding. Then the map f_1 deforms to the map f_2 by a finite sequence of the elementary deformations.*

Proof. There are two cases.

Case 1: The disk $f_2(N^2)$ is proper in B^3: Let $A^2=N^2-\mathrm{Int}(D^2)$. Let S_* be the set of singularities of $f_1|N^2$. Then S_* consists of loops and simple arcs connecting branch points on $f_1(\partial D^2)$. We may assume that each of the arcs are proper in both the disk $f_1(D^2)$ and the annulus $f_1(A^2)$. We use induction on the number of components of S_*.

If $S_*=\emptyset$, then it is clear that the map f_1 deforms to the map f_2 by pushing-disks.

Suppose that there exists a loop in S_* which is a non-essential loop in the annulus $f_1(A^2)$, i.e., the loop which bounds a disk in $f_1(A^2)$. Let L be a non-essential loop which is inner-most with respect to the annulus $f_1(A^2)$. Then L bounds two disks: a disk D_1^2 in $f_1(D^2)$ and a disk D_2^2 in $f_1(A^2)$. Let D_3^2 be a disk such that

(1) the disk D_3^2 is parallel to the disk D_2^2,

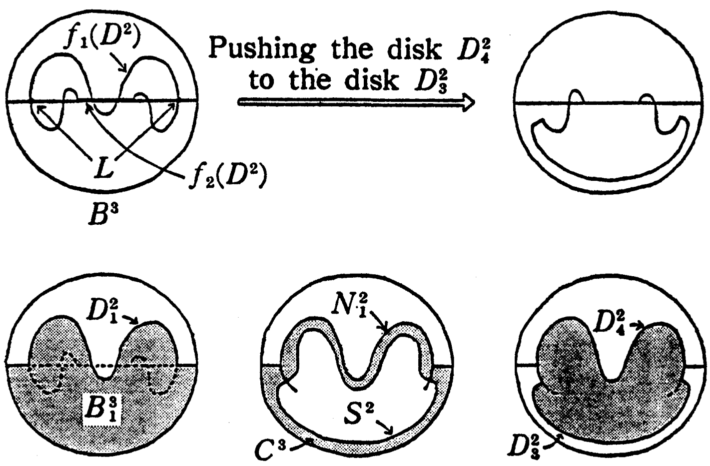
(2) $D_3^2 \cap f_1(D^2)=\partial D_3^2$,

(3) $D_3^2 \cap f_1(A^2)=\emptyset$, and

(4) ∂D_3^2 bounds the disk D_4^2 in $f_1(D^2)$ which contains the disk D_1^2.

Since $D_3^2 \cup D_4^2$ bounds the ball B_1^3 in B^3, by Pushing-disk lemma we can push the disk D_4^2 to the disk D_3^2 through the ball B_1^3 to eliminate the loop L from S_*. This decreases the number of the components of S_*. Thus we can assume that there are no non-essential loops.

Suppose that there exists a loop in S_* which is essential in the annulus $f_1(A^2)$, i.e., the loop which does not bound a disk in $f_1(A^2)$. Let L be the essential loop in S_* which is outer-most in $f_1(A^2)$. Then L bounds two disks: a disk D_1^2 in $f_1(D^2)$ and a disk D_2^2 in $f_2(N^2)$. Let B_1^3 be the complimentary ball of the proper disk $(f_2(N^2)-\mathring{D}_2^2)\cup D_1^2$ in B^3 which contains a boundary collar of D_2^2. The ball B_1^3 allows us to eliminate the loop L from S_* by Pushing-disk lemma as follows (see Figure 2.9): Let N_1^2 be a regular neighborhood of D_1^2



Dimension is reduced to two.

Fig. 2.9

in $f_1(D^2)$. Take a very thin boundary collar C^3 of B_1^3 which intersects N_1^2 by an annulus. Let S^2 be the connected component of ∂C^3 different from ∂B_1^3. Then $S^2 \cap N_1^2$ is a loop. Since S^2 is a 2-sphere, the loop $S^2 \cap N_1^2$ bounds the disk D_3^2 in S^2 which does not meet $f_1(A^2)$. Now D_3^2 misses $f_2(N^2)$ and intersects $f_1(D^2)$ by the loop $S^2 \cap N_1^2$ which is ∂D_3^2. The loop ∂D_3^2 bounds the disk D_4^2 in the disk $f_1(D^2)$. Since $D_3^2 \cup D_4^2$ bounds a ball in B^3, by Pushing-disk lemma we can push the disk D_4^2 to the disk D_3^2 through the ball to eliminate the loop L from S_*. This decreases the number of the components of S_*. Hence we can assume that there is no loop in S_*.

Suppose that there exists a simple arc in S_* which is an inner-most proper arc in $f_1(A^2)$. Then we can eliminate the arc from S_* by an elementary deformation of Type 4^{-1} or an inverse process of Lemma 2.5. Hence we have $S_* = \emptyset$. This case has been shown.

Case 2. The disk $f_2(N^2)$ is not proper in B^3: Let $A^2 = N^2 - \mathrm{Int}(D^2)$. Since $f_2(N^2)$ is a disk in B^3, there exists an annulus A_1^2 in B^3 such that

(1) $A_1^2 \cap f_2(N^2)$ is a boundary loop L_1 of A_1^2 and an essential loop on the open annulus $f_2(\mathrm{Int}(A^2))$, and

(2) $A_1^2 \cap \partial B^3$ is the boundary loop L_2 of A_1^2 different from L_1 (see Figure 2.10).

We will push a regular neighborhood of L_1 in $f_2(N^2)$ out of B^3 along the an-

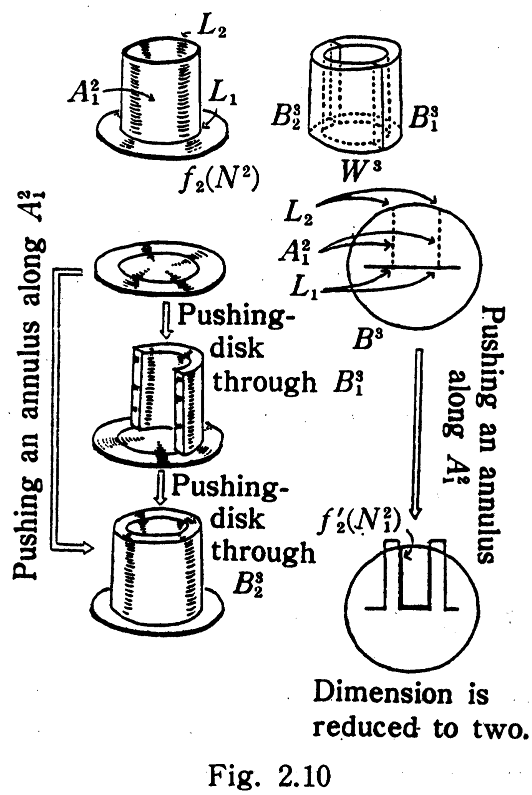


Fig. 2.10

nulus A_1^2 as follows (see Figure 2.10): Let W^3 be a very thin relative regular neighborhood of A_1^2 in M^3 relative to the boundary loop L_1 such that

(1) $W^3 \cap f_2(N^2) \subset \partial W^3$,

(2) $W^3 \cap f_2(N^2)$ is a regular neighborhood of L_1 in $f_2(A^2)$, and

(3) $\mathrm{Cl}(\partial W^3 - f_2(N^2)) \cap B^3$ consists of two annuli each of which is parallel to the annulus A_1^2 (see Figure 2.10).

Now W^3 is a union of two balls B_1^3 and B_2^3 such that $B_1^3 \cap B_2^3$ is a union of two disks. Then the map f_2 deforms to a map f_2' by two consecutive pushing-disks; a pushing-disk through B_1^3 and a pushing-disk through B_2^3. At the same time these two consecutive Pushing-disks deform the map f_1 to a map f_1'. The maps f_1' and f_2' send a small regular neighborhood of $(f_2|N^2)^{-1}(L_1)$ in A^2 outside the ball B^3. Let N_1^2 be the connected component of $(f_2'|N^2)^{-1}(B^3)$ which contains the disk D^2. Then N_1^2 is a disk such that $f_2'(N_1^2)$ is a proper disk in B^3 and $f_2'|N_1^2$ is an embedding. Furthermore f_1' and f_2' differ only on the disk D^2 and $f_1'|D^2$ is an embedding. Hence f_1' deforms to f_2' by a finite sequence of elementary deformations by Case 1. Therefore by the deformations $f_1 \to f_1' \to f_2' \to f_2$ the map f_1 deforms to the map f_2 by a finite sequence of elementary deformations.///

Corollary 2.7. (Disk-trade lemma) *Let f_1, $f_2\, F^2 \to M^3$ be nice maps, one of which deforms to the other by a single h-move with a support pair (D^2, B^3). Suppose that restrictions $f_1|D^2$, $f_2|D^2 : D^2 \to B^3$ are embeddings, and that for a regular neighborhood N^2 of D^2 in F^2, the image $f_1(N^2 - \mathrm{Int}(D^2))$ is an unknotted annulus in B^3. Then the map f_1 deforms to the map f_2 by a finite sequence of elementary deformations.*

Proof. Let $f_3 : F^2 \to M^3$ be a nice map such that f_3 and f_1 coincide outside the disk D^2 and such that $f_3|N^2$ is an embedding of N^2 into the ball B^3. Then each of the maps f_1 and f_2 deforms to the map f_3 by a finite sequence of elementary deformations by the above lemma. Therefore the result follows.///

In Case 2 Lemma 2.6 we have proved the following:

Lemma 2.8. *Let $f : F^2 \to M^3$ be a nice map which sends a disk D^2 in F^2 into the interior of a ball B^3 in M^3. Suppose that there exists a regular neighborhood N^2 of D^2 in F^2 such that*

(1) *$f(N^2) \subset B^3$, and*

(2) *if $A^2 = \mathrm{Cl}(N^2 - C^2)$, then $f(A^2)$ is an unknotted annulus in B^3.*

Let L be an essential loop in $\mathrm{Int}\, A^2$. Then there exists a nice map $f' : F^2 \to M^3$ such that

(1) *f and f' differ only on a small regular neighborhood of L in $\mathrm{Int}(A^2)$,*

(2) *f deforms to f' by a finite sequence of elementary deformations,*

(3) *for a regular neighborhood N_1^2 of D^2 in N^2, we have $f'(N_1^2) \cap \partial B^3 = f'(\partial N_1^2)$, and*

(4) *$f'(\mathrm{Cl}(N_1^2-D^2))$ is an unknotted annulus in B^3.*

Lemma 2.9. *Let $f; F^2\to M^2$ be a nice map. Suppose that the map f sends a disk N^2 in F^2 into the interior of a ball B^3 in M^3 and that for a boundary collar A^2 of N^2, $f(A^2)$ is an unknotted annulus in B^3. Then there exists a nice map $f': F^2\to M^2$ such that*
(1) *$f'(N^2)\subset B^3$,*
(2) *$f'|N^2$ is an immersion,*
(3) *maps f' and f coincide on a neighborhood of $\mathrm{Cl}(F^2-N^2)$, and*
(4) *f deforms to f' by a finite sequence of elementary deformations.*

Proof. Since $f(A^2)$ is unknotted, the number of branch points of $f|N^2$ is even. Choose mutually disjoint arcs on $f(N^3)$, each of which connects a pair of branch points. Apply shifting-branch-points along the arcs and inverse process of squeezing-off to eliminate the branch points. Since any nice map without branch points is an immersion, the result follows.///

§ 3. Decomposing a homotopy into h-moves.

Let d be a metric of M^3. Let T_1 be a triangulation of M^3. Let δ_1 be a Lebesgue number of the covering which consists of all open star neighborhoods of vertices of T_1. Let T_2 be a subdivision of T_1 with $\mathrm{mesh}(T_2)<\delta_1/10$. Let $\delta_2<\delta_1/10$ be a Lebesgue number of the covering which consists of all open star neighborhoods of vertices of T_2. Let T_3 be a subdivision of T_2 with $\mathrm{mesh}(T_3)<\delta_2/10$. Let $\delta<\delta_2/10$ be a Lebesgue number of the covering which consists of all open star neighborhoods of vertices of T_3.

Lemma 3.1. *Let $f_1, f_4: F^2\to M^3$ be nice immersions. If $d(f_1, f_4)<\delta$, then there exists a finite sequence of h-moves which deforms the map f_1 to the map f_4.*

Proof. Let K be a triangulation of F^2 such that for a subdivision T_4 of T_3 the maps $f_1, f_4: K\to T_4$ are simplicial. Note that for each simplex ζ of K the diameter of $f_4(\zeta)$ is less than $\delta_2/10$. Since $d(f_1(v), f_4(v))<\delta$ for each vertex v of K, the two points $f_1(v)$ and $f_4(v)$ lie in the open star neighborhood of a vertex of T_3 whose daimeter is less than $\delta_2/5$. Hence there exists a simple curve from $f_1(v)$ to $f_4(v)$ whose diameter is less than $\delta_2/5$. Applying pushing-disk's along curves, we have a finite sequence of h-moves which deforms the map f_1 to a nice map f_2 such that the maps f_2 and f_4 coincide on a regualr neighborhood of the 0-skeleton of K, and such that $d(f_1, f_2)<\delta_2/5$. Since $d(f_2, f_4)<3\delta_2/10$, we have the following: $\mathrm{diam}(f_2(\sigma^1)\cup f_4(\sigma^1))<\delta_2$ for each 1-simplex σ^1 of K. Hence $f_2(\sigma^1)\cup f_4(\sigma^1)$ lies in the open star neighborhood of a vertex of T_2. Thus there exists a finite sequence of h-moves which deforms the map

f_2 to a map f_3 such that the maps f_3 and f_4 coincide on a regular neighborhood of the 1-skeleton of K, and such that $d(f_2, f_3)<\delta_1/5$. Since $d(f_2, f_4)<3\delta_2/10<\delta_1/10$, we have the following: $d(f_3, f_4)<3\delta_1/10$. Hence $\mathrm{diam}(f_3(\tau^2)\cup f_4(\tau^2))<\delta_1$ for each 2-simplex τ of K. Thus $f_3(\tau^2)\cup f_4(\tau^2)$ lies in the open star neighborhood of a vertex of T_1. Therefore there exists a finite sequence of h-moves which deforms the map f_3 to the map f_4. This completes the proof of Lemma 3.1.///

Lemma 3.2. *Let $f_1: F^2\to M^3$ be a continuous map with $f_1(F^2)\subset \mathrm{Int}(M^3)$. Then for any $\varepsilon>0$ there exists a nice immersion f_3 with $d(f_1, f_3)<\varepsilon$.*

Proof. By the general position argument there exists a nice map f_2 with $d(f_2, f_1)<\varepsilon/2$. Suppose that P_1 and P_2 are branch points of f_2. Then on $f_1(F^2)$ there is a simple curve L from P_1 to P_2 such that the curve L does not contain any triple points nor any branch points except the two end points P_1 and P_2, and such that the curve L contains finitely many double points. Applying shifting-branch-point along the curve L, we may assume that the curve L does not contain any double points. Thus we can apply the inverse process of squeezing-off to diminish two branch points, if we choose the curve L suitabley (see Figure 3.1).///

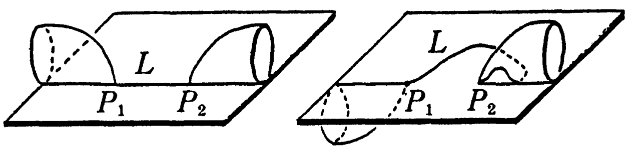


Fig. 3.1

Theorem. *Let $f, g: F^2\to M^3$ be nice maps. Suppose that the maps f and g are homotopic. Then thers exists a finite sequence of h-moves which deforms the map f to the map g.*

Proof. Let $H: F^2\times I\to M^3$ be a homotopy with $H_0=f$ and $H_1=g$, where for each $t\in J$, $H_t: F^2\to M^3$ is the map defined by $H_t(x)=H(x, t)$ for all $x\in F^2$. Let δ be a positive number in Lemma 3.1 for a triangulation of M^3. Since the surface F^2 is compact, there exists a partition of $I; 0=t_0<t_1\cdots<t_n$ with $d(H_{i-1}, H_i)<\delta/3$ for each $i=1, \cdots, n$. For each $i=1, \cdots, n-1$ let H^* be a nice immersion with $d(H_i, H_i^*)<\delta/3$ assured by Lemma 3.2. Since for $i=0$ or n there exists a finite sequence of elementary deformations which deforms H_i to a nice immersion by the same argument with the one in Lemma 3.2, the result follows from Lemma 3.1.///

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