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ON THE TOTAL CURVATURE OF NON-COMPACT RIEMANNIAN MANIFOLDS II

By

MASAO MAEDA

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1. Let M be a 2-dimensional connected complete Riemannian manifold with non-negative Guassian curvature K. Then by Cohn-Vossen, it was proved that M is isometric to a flat open Möbius band or a flat cylinder or a manifold which is diffeomorphic to an Euclidean plane and its total curvature $\iint_M K\,dv$ satisfies an inequality

$$0 \leqq \iint_M K\,dv \leqq 2\pi \tag{1}$$

where dv is the volume element of M induced from the Riemannian metric of M. And in [5], we gave a simple proof of the inequality (1) by showing that there exists a family of geodesic quadraterals $\{Q_i\}_{i=1,2,\dots}$ each of which has interior angles whose sum does not exeed 4π and satisfies (using the same notation Q_i to denote the domain bounded by Q_i)

$$Q_i \subset Q_{i+1,\, i=1,2,\dots} \quad \text{and} \quad \bigcup_{i=1}^{\infty} Q_i = M.$$

Then by applying the Gauss-Bonnet's Theorem to Q_i, we have

$$\iint_M K\,dv = \lim_{i\to\infty} \iint_{Q_i} K\,dv \leqq 2\pi\,.$$

And successively in [6], we gave another proof of the inequality (1) by giving a geometrical significance of the total curvature. It is stated as follows.

For a point $p \in M$, let $T_p^1(M)$ be the subset of the tangent space $T_p(M)$ at p of M consisting of all unit tangent vectors in $T_p(M)$. Then with the Riemannian metric induced from the inner product of $T_p(M)$, $T_p^1(M)$ is a 1-dimensional Riemannian manifold isometric to a unit circle in a 2-dimensional Euclidean plane R^2 ($\cong T_p(M)$). Thus we can consider the Riemannian measure on $T_p^1(M)$ and measure $T_p^1(M) = 2\pi$. Let $A(p)$ be the subset of $T_p^1(M)$ given by

$$A(p) := \{v \in T_p^1(M) : \text{geodesic } \gamma : [0, \infty) \to M,\ \gamma(t) = \exp_p tv \text{ is a ray}\}$$

where $\exp_p : T_p(M) \to M$ is the exponential mapping of M and geodesic $\gamma : [0, \infty)$

$\to M$ is called a ray when any subarc of γ is a shortest connection between its ends points. As is easily seen, $A(p)$ is a closed subset of $T_p^1(M)$ and hence we can consider the measure of $A(p)$. In this situation, we have proved in [4], [6] that for any point $p\in M$,

(2) $$\iint_M K\,dv \geqq 2\pi - \text{measure } A(p)$$

and for any $\varepsilon>0$, there exists a compact domain D such that for all point $q\in M-D$,

(3) $$\iint_M K\,dv \leqq 2\pi - \text{measure } A(q)+\varepsilon\,.$$

Thus combining (2) and (3), we have

(4) $$\iint_M K\,dv = 2\pi - \inf_{p\in M} \text{measure } A(p)\,.$$

From (2), (3) and (4), we can get a geometrical significance of the total curvature. And in particular, we have the inequality

(1) $$0 \leqq \iint_M K\,dv \leqq 2\pi\,,$$

because measure $A(p)\geqq 0$.

Now the purpose of this note is to give another simple proof of the inequality (1). And from its proof, we will give a slight generalization for the result mentioned above.

2. Let M be a 2-dimensional complete Riemannian manifold diffeomorphic to an Euclidean plane. In the following, we use the same notations mentioned in the above introduction. For a point $p\in M$, we assume that there exists a point $v_0\in T_p^1(M)$ which is an interior point of $A(p)$. Let $U(v_0)$ be an open connected neighbourhood of v_0 with boundary $\partial U(v_0)=\{v_1, v_2\}$ satisfying $U(v_0)\subset A(p)$ and $D\subset M$ where D is the domain defined by

$$D=\{q\in M : q=\exp_p tv \text{ for } v\in U(v_0),\ t>0\}.$$

Let $\tilde{U}(v_0)$ be the domain in $T_p(M)$ defined by

$$\tilde{U}(v_0)=\{v\in T_p(M) : v=tw \text{ for } w\in U(v_0) \text{ and } t>0\}.$$

Then by definition of $U(v_0)\subset A(p)$,

$$\exp_p|\tilde{U}(v_0) : \tilde{U}(v_0)\to D$$

is a diffeomorphism. So we can consider a polar coordinate system (r, θ) on D around p such that $0<r<\infty$ and $\theta_1<\theta<\theta_2$ where $\theta_1=\lim_{\substack{v\to v_1\\ v\in U(v_0)}}\theta(v)$ and $\theta_2=\lim_{\substack{v\to v_2\\ v\in U(v_0)}}\theta(v)$.

Using this polar coordinates, the volume element of M is expressed as

$$dv=\varphi(r, \theta)drd\theta .$$

Since $g(\partial/\partial r, \partial/\partial\theta)=0$ because of Gauss-Lemma and $g(\partial/\partial r, \partial/\partial r)=1$,

$$\varphi(r, \theta)=\sqrt{g(\partial/\partial\theta, \partial/\partial\theta)}$$

where g is the Riemannian metric of M. By definition, $\partial/\partial\theta$ is a Jacobi field Y along a geodesic $c:[0, \infty)\to M$ given by $\theta(c(t))=$constant θ_0 for all $t>0$ which satisfy $Y(0)=0$, $Y'(0)\perp c(0)$ and $\|Y'(0)\|=1$ where "$'$" denotes the covariant derivative along a curve and $\|\cdot\|$ the norm of the vectors. Thus $\varphi(r, \theta_0)$ is a solution of the Jacobi equation

$$f''+K(r, \theta_0)f=0$$

with the initial values $f(0)=0$ and $f'(0)=1$ where $K(r, \theta_0)$ is the Gaussian curvature of M at point (r, θ_0). If we put $\varphi(0, \theta)=0$ for all $\theta\in[\theta_1, \theta_2]$, then φ is extended continuously for all point (r, θ), $0\leqq r$, $\theta_1\leqq\theta\leqq\theta_2$. For each $s>0$, let D_s be the domain defined by

$$D_s=\{q\in\bar{D}: 0\leqq r(q)\leqq s \text{ and } \theta_1\leqq\theta(q)\leqq\theta_2\}.$$

By definition, $D_s\subset D_{s'}$ if $s\leqq s'$ and $\bigcup_{s\geqq 0} D_s=\bar{D}$.

Here we put further assumption that the Gaussian curvature of M is non-negative. Then

$$\iint_{D_s} K\, dv=\int_0^s\int_{\theta_1}^{\theta_2} K(r, \theta)\varphi(r, \theta)drd\theta$$

is a monotone non-decreasing function of s and hence

$$\int_{\bar{D}} K\, dv=\lim_{s\to\infty}\iint_{D_s} K\, dv .$$

Now for each $s\geqq 0$, let $\Phi_s:[\theta_1, \theta_2]\to R$ be the function defined by

$$\Phi_s(\theta)=\int_0^s K(r, \theta)\varphi(r, \theta)dr , \quad \theta\in[\theta_1, \theta_2] .$$

Clearly each Φ_s is a continuous function. Let $\Phi:[\theta_1, \theta_2]\to R$ be the function defined by

$$\Phi(\theta)=\lim_{s\to\infty}\Phi_s(\theta) \quad \left(=\int_0^\infty K(r, \theta)\varphi(r, \theta)dr\right)$$

for $\theta\in[\theta_1, \theta_2]$.

Lemma 1. *Φ is well defined for all $\theta\in[\theta_1, \theta_2]$ and $0\leqq\Phi(\theta)\leqq 1$ for all $\theta\in[\theta_1, \theta_2]$.*

Proof. Since $\varphi''(r, \theta)+K(r, \theta)\varphi(r, \theta)=0$ and $\varphi'(0, \theta)=1$ for all $\theta\in[\theta_1, \theta_2]$, it holds

$$\int_0^s K(r, \theta)\varphi(r, \theta)dr = -\int_0^s \varphi''(r, \theta)dr$$
$$= \varphi'(0, \theta) - \varphi'(s, \theta)$$
$$= 1 - \varphi'(s, \theta).$$

And also $\varphi''(r, \theta) = -K(r, \theta)\varphi(r, \theta) \leqq 0$ for all $r \geqq 0$. Because $\varphi(r, \theta) > 0$ for all $r > 0$. For, if $\varphi(r_0, \theta) = 0$ for some $r_0 > 0$, then the point (r_0, θ) is a conjugate point of p along the geodesic $c: \theta = \text{constant}$. This is a contradiction. Thus $\varphi'(r, \theta)$ is monotone non-decreasing with respect to r. Hence if there exists $r' > 0$ such that $\varphi'(r', \theta) < 0$, then $\varphi'(r, \theta) \leqq \varphi'(r', \theta) < 0$ for all $r \geqq r'$. So we can find $r > 0$ such that $\varphi(r, \theta) = 0$. This is a contradiction. Thus $0 \leqq \varphi'(r, \theta) \leqq 1$ for all $r \geqq 0$ and hence $\lim_{r\to\infty}(1 - \varphi'(r, \theta))$ exists for each $\theta \in [\theta_1, \theta_2]$. q. e. d.

From above lemma, for any $s > 0$, it holds

$$\iint_{D_s} K\, dv = \int_{\theta_1}^{\theta_2}\int_0^s K(r, \theta)\varphi(r, \theta)dr d\theta$$
$$= \int_{\theta_1}^{\theta_2} \Phi_s(\theta)d\theta$$
$$\leqq \int_{\theta_1}^{\theta_2} \Phi(\theta)d\theta \leqq \theta_2 - \theta_1.$$

So, since $\int_{D_s} K\, dv$ is monotone non-decreasing function of s, there exists the limit and

$$0 \leqq \lim_{s\to\infty}\iint_{D_s} K\, dv = \iint_{\bar{D}} K\, dv \leqq \theta_2 - \theta_1.$$

Summarizing the above, we have

Proposition 1. *Let M be a 2-dimensional complete connected Riemannian manifold with non-negative Gaussian curvature K. For a point $p \in M$, let U be a connected domain in $A(p)$ and D the domain in M defined by*

$$D = \{q \in M : q = \exp_p tv \ \textit{for} \ v \in U \ \textit{and} \ t > 0\}.$$

Then it holds

$$0 \leqq \iint_{\bar{D}} K\, dv \leqq \text{measure } U = \measuredangle_D(v, w),$$

where $\{v, w\} = \partial U$ and $\measuredangle_D(v, w)$ denotes the interior angle between the vectors v and w measured on D.

Now let $W \subset M_p - A(p)$ be a connected component and $v, w \in T_p^1(M)$ the boundary of W. Since $A(p)$ is closed, W is open and hence $v, w \in A(p)$. Let γ_1 and γ_2 are two rays defined by $\gamma_1(t) = \exp_p tv$ and $\gamma_2(t) = \exp_p tw$. Then, by geodesic $\gamma_1 \circ \gamma_2 : (-\infty, \infty) \to M$ defined by

$$\gamma_1\circ\gamma_2(t)=\begin{cases}\gamma_1(-t), & t\leqq 0\\ \gamma_2(t), & t>0,\end{cases}$$

M is devided into two mutually disjoint domaines D_1 and D_2 i. e. $\bar{D}_1\cup\bar{D}_2=M$, $D_1\cap D_2=\emptyset$ and $\partial D_1=\partial D_2=\gamma_1\circ\gamma_2((-\infty, \infty))$. Without loss of generality, we can assume that D_1 is the domain satisfying

$$\{q\in M: q=\exp_p tv',\ v'\in W,\ 0<t\leqq r(p)\}\subset D_1$$

where $r(p)$ is the convexity radius of p. Then in [6], we proved the following

Proposition 2. *In the above situation, it holds*

$$\iint_{\bar{D}_1} K\,dv=\text{measure } w=\sphericalangle_{D_1}(v, w).$$

Let $D\subset M$ be a domain whose boundary ∂D is an union of the images of two rays σ_1 and σ_2 starting from p i. e. $\partial D=\sigma_1([0, \infty))\cup\sigma_2([0, \infty))$ and $D_p\subset T_p^1(M)$ the set defined by

$$D_p=\{v'\in T_p^1(M): \exp_p tv'\in\bar{D} \text{ for } 0\leqq t\leqq r(p)\}.$$

Then measure $D_p=\sphericalangle_D(\dot\sigma_1(0), \dot\sigma_2(0))$ and D_p is the union of closed connected domaines $\{\bar{U}_\alpha\}_{\alpha\in A}$ and $\{\bar{W}_\beta\}_{\beta\in B}$ which satisfy the following conditions;

(i) $U_\alpha\subset A(p)\cap D_p$ is open and $\bar{U}_\alpha$ is a connected component of $A(p)\cap D_p$ for each $\alpha\in A$

(ii) $W_\beta\subset(M-A(p))\cap D_p$ is open and W_β is a connected component of $(M-A(p))\cap D_p$ for each $\beta\in B$

(iii) $\{U_\alpha\}_{\alpha\in A}$ and $\{W_\beta\}_{\beta\in B}$ are mutually disjoint, because $A(p)\subset T_p^1(M)$ is closed.

For each U_α, $\alpha\in A$, let D_α be the domain in M defined by

$$D_\alpha=\{q\in M: q=\exp_p tv \text{ for } v\in U_\alpha \text{ and } t>0\}.$$

Then from Proposition 1, it holds

$$0\leqq\iint_{\bar{D}_\alpha} K\,dv\leqq\text{measure } U_\alpha=\sphericalangle_{D_\alpha}(v_\alpha, w_\alpha). \tag{5}$$

Here $\{v_\alpha, w_\alpha\}=\partial U_\alpha$. And for each W_β, $\beta\in B$, let D'_β be the domain bounded by two rays γ_1 and γ_2 defined by $\gamma_1(t)=\exp_p tv'$ and $\gamma_2(t)=\exp_p tw'$ where $\partial W_\beta=\{v'_\beta, w'_\beta\}$ and satisfy

$$\{q\in M: q=\exp_p tv \text{ for } v\in W_\beta \text{ and } 0<t<r(p)\}\subset D'_\beta.$$

Then from Proposition 2, it holds

$$0\leqq\iint_{\bar{D}'_\beta} K\,dv=\text{measure } \bar{D}'_\beta=\sphericalangle_{D'_\beta}(v'_\beta, w'_\beta). \tag{6}$$

By definition, $\{D_\alpha\}_{\alpha\in A}$ and $\{D'_\beta\}_{\beta\in B}$ are mutually disjoint and $(\bigcup_{\alpha\in A}\bar{D}_\alpha)\cup(\bigcup_{\beta\in B}\bar{D}'_\beta)=\bar{D}$. So if we put $D_1:=\bigcup_{\alpha\in A}D_\alpha$ and $D_2:=\bigcup_{\beta\in B}D'_\beta$, then we have

$$\iint_{\bar{D}_1} K\,dv=\sum_{\alpha\in A}\text{measure }(U_\alpha)\,, \tag{7}$$

$$\iint_{\bar{D}_2} K\,dv\leqq\sum_{\beta\in B}\text{measure }(W_\beta) \tag{8}$$

and $\sum_{\alpha\in A}$ measure $(U_\alpha)+\sum_{\beta\in B}$ measure $(W_\beta)=$ measure $(D_p)=\measuredangle_D(\dot\sigma_1(0), \dot\sigma_2(0))$. Thus summarizing the above, we have

Theorem A. *Let M be a 2-dimensional complete Riemannian manifold with non-negative Gaussian curvature K diffeomorphic to an Euclidean plane. Let D be a domain in M bounded by two rays σ_1 and σ_2. Then it holds*

$$0\leqq\iint_{\bar{D}} K\,dv\leqq\measuredangle_D(\dot\sigma_1(0), \dot\sigma_2(0))\,.$$

This theorem is a generalization of the result obtained by Cohn-Vossen. Indeed from above theorem, we have

Corollary. *Let M be a 2-dimensional complete Riemannian manifold with non-negative Gaussian curvature K diffeomorphic to an Euclidean plane. Then it holds*

$$0\leqq\iint_M K\,dv\leqq 2\pi\,.$$

Proof. From Lemma 1 in [5; p. 96], there exists a point $p\in M$ such that there exists at least two rays σ_1 and σ_2 starting from p. Then by the geodesic $\sigma_1\circ\sigma_2:(-\infty, \infty)\to M$, M is devided into two mutually disjoint domains D_1 and D_2 whose boundary $\partial D_1=\partial D_2$ equals $\sigma_1\circ\sigma_2((-\infty, \infty))$. For these $D_{i,\,i=1,2}$, from above theorem we have

$$\iint_{\bar{D}_i} K\,dv\leqq\measuredangle_{D_i}(\dot\sigma_1(0), \dot\sigma_2(0))\,,\quad i=1, 2\,.$$

Thus

$$\begin{aligned}\iint_M K\,dv&=\iint_{\bar{D}_1} K\,dv+\iint_{\bar{D}_2} K\,dv\\ &\leqq\measuredangle_{D_1}(\dot\sigma_1(0), \dot\sigma_2(0))+\measuredangle_{D_2}(\dot\sigma_1(0), \dot\sigma_2(0))\\ &=2\pi\,.\end{aligned}$$

q. e. d.

3. In this section, we will give another generalization of the result by Cohn-Vossen (Corollary of Theorem A in §2).

Proposition 3. *Let M be a 2-dimensional complete Riemannian manifold.*

Then the Gaussian curvature K of M is non-negative if and only if the following condition (*) *holds*

(*) *for any point $p \in M$ and any geodesic $\gamma:[0, \infty) \to M$ starting from p, any Jacobi field Y along γ with initial values $Y(0)=0$, $Y'(0) \perp \dot{\gamma}(0)$ satisfies*

$$\|Y'(s)\| \leqq \|Y'(0)\|$$

as long as γ has no conjugate point of p in $(0, s)$.

Proof. If $K \geqq 0$, then by a proof of Rauch's comparison theorem, we easily see that the condition (*) holds, see [3; pp. 178~]. Conversely under the condition (*), we assume that there exist a point $p \in M$ such that $K(p)<0$. Let $\sigma:[0, \infty) \to M$ be a geodesic starting from p and Y a Jacobi field along σ satisfying $Y(0)=0$, $Y'(0) \perp \dot{\sigma}(0)$. Then Y is expressed as $Y=\varphi P$ where P is a parallel vector field along σ satisfying $P(0)=Y'(0)/\|Y'(0)\|$ and $\varphi:[0, \infty) \to R$ is a C^∞ solution of the Jacobi equation $\varphi''(s)+K(\sigma(s)\varphi(s)=0$ with initial values $\varphi(0)=0$, $\varphi'(0)=\|Y'(0)\|$. By continuity, we can find a constant s_0 such that $K \circ \sigma | [0, s_0]<0$ and $\varphi | [0, s_0) \geqq 0$. Thus $\int_0^{s_0} K(\sigma(s))\varphi(s)ds$ is negative. On the other hand, by definition of φ, we have

$$\begin{aligned}\int_0^{s_0} K(\sigma(s))\varphi(s)ds &= -\int_0^{s_0} \varphi''(s)ds \\ &= \varphi'(0)-\varphi'(s_0) \\ &= \|Y'(0)\|-\|Y'(s_0)\| \geqq 0 .\end{aligned}$$

And this is a contradiction. q. e. d.

Noticing this proposition, we will extend the result by Cohn-Vossen slightly in a following manner.

Theorem B. *Let M be a complete Riemannian manifold diffeomorphic to an Euclidean plane and satisfies the following two conditions;*

(i) *Gaussian curvature K of M is non-negative outside some compact subset C of M*

(ii) *there exists a point $p \in M$ such that for any geodesic $\gamma:[0, \infty) \to M$ starting from p, any Jacobi field Y along γ with initial values $Y(0)=0$, $Y'(0) \perp \dot{\gamma}(0)$ satisfies*

$$\|Y'(s)\| \leqq \|Y'(0)\|$$

as long as γ has no conjugate points of p in $(0, s)$.

Then it holds

$$0 \leqq \iint_M K\, dv \leqq 2\pi$$

For a while, we assume that M satisfies the condition (i) only. For a ray $c:[0, \infty)\rightarrow M$ starting from p, put

$$B_c:=\bigcup_{t\geqq 0} B_t(c(t))$$

where $B_r(q)$ denotes the open geodesic ball with radius $r\geqq 0$ centerd at $q\in M$. For a $t\geqq 0$, let $c_t:[0, \infty)\rightarrow M$ be a ray defined by $c_t(s)=c(t+s)$. Then we can easily check that the family of sets $\{B^c_{c_t}\}_{t\geqq 0}$, $B^c_{c_t}:=M-B_{c_t}$ satisfy the following;

(1) $B^c_{c_t}\subset B^c_{c_{t'}}$ if $t\leqq t'$ and

(2) $\bigcup_{t\geqq 0} B^c_{c_t}=M$.

From property (2), we can find a constant $t_0>0$ such that $B^c_{c_{t_0}}\supset C$. Then just as the proof of Theorem 1.2 in [1; p. 415], we can prove that B_{c_t} is totally convex for all $t>t_0$. Here a subset $A\subset M$ is called totally convex if for any two points p, $q\in A$ and for any geodesic $\beta:[0, d(p, q)]\rightarrow M$, connecting between p and q, $\beta([0, d(p, q)])\subset A$.

Now, let D be a domain in M bounded by two rays σ and τ starting from p satisfying the following condition;

any $v\in T^1_p(M)$, $v\neq\dot{\sigma}(0)$, $v\neq\dot{\tau}(0)$ such that $\exp_p tv\in D$ for $0<t<r(p)$, $v\notin A(p)$.

We do not exclude the case $\sigma=\tau$ (in this case $D=M-\sigma([0, \infty))$). For these σ and τ, we apply the above argument. Let t_0 and t_1 be two numbers such that $B^c_{\sigma_{t_0}}\supset C$ and $B^c_{\tau_{t_1}}\supset C$. Then for any $t>t':=\max(t_0, t_1)$, $B^c_{\sigma_t}$ and $B^c_{\tau_t}$ are totally convex. Put $D_t:=\bar{D}\cap B^c_{\sigma_t}\cap B^c_{\tau_t}$ for each $t>t'$. Then D_t is compact for each $t>t'$. For, if D_t is not compact, then there exists a divergent sequence $\{q_i\}_{i=1,2,\cdots}$ contained in D_t. Let $c_i:[0, d(p, q_i)]\rightarrow M$ be a shortest geodesic from p to q_i, $i=1, 2, \cdots$. Then $c_i([0, d(p, q_i)])\subset B^c_{\sigma_t}\cap B^c_{\tau_t}$ for all $i=1, 2, \cdots$, because $B^c_{\sigma_t}$ and $B^c_{\tau_t}$ are totally convex. Also for $i=1, 2, \cdots$, $c_i([0, d(p, q_i)])\subset\bar{D}$ because its boundary σ and τ are rays. Thus $c_i([0, d(p, q_i)])\subset D_t$ for all $i=1, 2, \cdots$. Choose a convergent subsequence $\{\dot{c}_{i_j}(0)\}_{j=1,2,\cdots}\subset\{\dot{c}_i(0)\}_{i=1,2,\cdots}$ and let $v\in T^1_p(M)$ be its limit vector. Then the geodesic $c:[0, \infty)\rightarrow M$ defined by $c(t)=\exp_p tv$ is a ray which is different from σ and τ. This contradicts the definition of D. So $\{D_t\}_{t>t'}$ is a family of compact connected convex subsets of M and satisfies the following properties;

(1) $D_{t_1}\subset D_{t_2}$ if $t'<t_1<t_2$

(2) $\bigcup_{t>t'} D_t=D$.

For the definition of convex set and its following properties, see [1; pp. 417-420]. From Theorem 1.6 [1; p. 418], $\partial D_t:=D_t-\mathrm{Int}.\, D_t$ is a connected 1-dimensional manifold (possibly non-smooth) for $t>t'$. We can easily see that $\sigma([0, t])\subset B^c_{\sigma_t}\cap B^c_{\tau_t}$ and $\tau([0, t])\subset B^c_{\sigma_t}\cap B^c_{\tau_t}$ for $t>t'$. So $\partial D_t-(\sigma([0, t])\cup\tau([0, t]))=$

$D\cap\partial(B^c_{\sigma_t}\cap B^c_{\tau_t})$ and from above construction, $D\cap\partial(B^c_{\sigma_t}\cap B^c_{\tau_t})$ is a 1-dimensional connected manifold with boundary $\{\sigma(t), \tau(t)\}$ for $t>t'$. Then putting $E_t:= D\cap\partial(B^c_{\sigma_t}\cap B^c_{\tau_t})$, just as the proof of the Lemma in [4; p. 2], we have

Lemma 2. $\iint_D K\,dv \geqq \sphericalangle_D(\dot\sigma(0), \dot\tau(0)) \geqq 0$.

In the proof of the Lemma in [4], assumption that M is non-negatively curved is only used to show the existance of $\{E_t\}_{t>t_0}$ having certain properties which is satisfied for the familly of $\{E_t\}_{t>t'}$ obtained as above.

Proof of Theorem B. As is easily checked, Lemma 1 in §2 remains valid under the assumption of Theorem B. Thus Proposition 1 in §2 holds good under the assumption of Theorem B. Then combining this with Lemma 2, just as the proof of Theorem A, we can prove that

$$0\leqq\iint_M K\,dv\,.$$

And in any cases, it holds

$$\iint_M K\,dv\leqq 2\pi\chi(M)=2\pi$$

by Cohn-Vossen. q. e. d.

References

[1] J Cheeger and D. Gromoll: *On the Structure of Complete Manifolds of Nonnegative Curvature,* Ann. of Math. **96** (1972), 413-443.

[2] S. Cohn-Vossen: *Kürzeste Wege und Total krümmung auf Flächen,* Compositio Math. **2** (1935), 63-133.

[3] D. Gromoll, W. Klingenberg and W. Meyer: *Riemannsche Geometrie im Grossen,* Springer-Verlag, 1968.

[4] M. Maeda: *On the Existance of Rays,* Sci. Rep. Yokohama National University, Sec. I, **26** (1979), 1-4.

[5] M. Maeda: *On the Total Curvature of Non-compact Riemannian Manifolds,* Kodai Math. Sem. Rep. **29** (1974), 95-99.

[6] M. Maeda: *A Geometric Significance of Total Curvature on Complete Open Surfaces,* Geometry of Geodesics and Related Topics, Advanced Studies in Pure Mathematics 3, Kinokuniya, Tokyo and North-Holland Publ.

[7] K. Shiga: *On a Relation Between the Total Curvature and the Measure of Rays,* Tsukuba J. Math. **6** (1982), 41-50.

Department of Mathematics
Faculty of Education
Yokohama National University