CONSTRUCTION OF GRAPHS WHICH ARE NOT UNIQUELY AND NOT FAITHFULLY EMBEDDABLE IN SURFACES

By

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ABSTRACT. A graph G is said to be *uniquely embeddable* in a surface F^2 if for any two embeddings $f_1, f_2: G \to F^2$, there is an automorphism $\sigma: G \to G$ and a homeomorphism $h: F^2 \to F^2$ such that $h \circ f_1 = f_2 \circ \sigma$. A graph G is said to be *faithfully embeddable* in a surface F^2 if G admits an embedding $f: G \to F^2$ such that for any automorphism $\sigma: G \to G$, there is a homeomorphism $h: F^2 \to F^2$ with $h \circ f = f \circ \sigma$. Given a hyperbolic closed surface F^2, an infinite number of 6-connected graphs which are not uniquely or not faithfully embeddable in F^2 will be constructed systematically.

1. Introduction

A graph G is said to be *uniquely embeddable* in a surface F^2 if there is only one way to embed it into F^2 up to equivalence and to be *faithfully embeddable* in a surface F^2 if it can be embedded so that all of its automorphisms extend to self-homeomorphisms of F^2. For example, Fig. 1(a) and (b) show two inequivalent embeddings of a graph in the plane, where the reversion of their right diamonds causes their difference, so this graph is not uniquely embeddable. On the other hand, Fig. 2 shows a graph which is not faithfully embedded in the plane. Also the turning of its right diamond cannot extend to a self-homeomorphism of the plane with the left diamond fixed. The detailed definitions of these concepts will be given in the next section.

The author has already discussed when a graph is uniquely or faithfully embeddable in a surface and found many classes of such graphs in a torus [2], a Klein bottle [5] and a projective plane [3], [4], [6]. Conversely, we shall develop a method to construct systematically those graphs whose embeddings do not possess uniqueness or faithfulness and show the complete answer for the problem which is set up as follows.

Given a closed surface F^2, consider the following two statements with parameters n and m:

$U(F^2; n)$: *Every n-connected graph embeddable in a closed surface F^2 is uniquely*

embeddable in F^2, with finitely many exceptions.

$F(F^2; m)$:   *Every m-connected graph embeddable in a closed surface F^2 is faith-fully embeddable in F^2, with finitely many exceptions.*

Then our problem is to determine the minimum values n and m, for each closed surface F^2, which make $U(F^2; n)$ and $F(F^2; m)$ true, respectively.
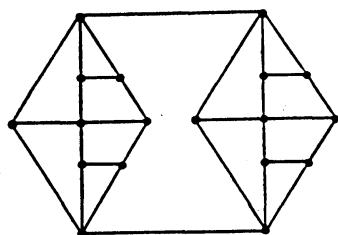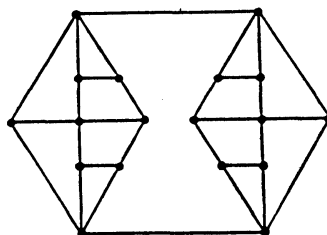

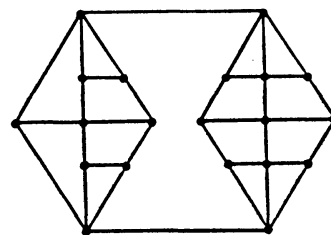Fig. 1(a).   Fig. 1(b).   Fig. 2.

As the previous examples in Fig. 1 and 2 suggest, if a graph G had a vertex-cut U with few vertices, then G might have either two inequivalent embeddings or a non-faithful embedding in F^2 because of the turning over of one of the parts into which U splits G. To exclude such a phenomenon, we assume that the connectivity of G is sufficiently large. Are the uniqueness and faithfulness of embedding of G guaranteed in this case? This question is our motivation.

It is however very doubtful that there would be sufficiently large numbers n and m such that every n- or m-connected graph is uniquely or faithfully embeddable in F^2, respectively. In fact, the complete graph K_p with $p \geqq 5$ can be embedded faithfully in no surface. Thus, if we did not allow a finite number of exceptions for $U(F^2; n)$ and $F(F^2; m)$, then our problem would be nonsense. So it may be said that our problem simply asks a rough correlation between the uniqueness and faithfulness and the connectivity of graphs.

Our goal in this paper is to complete the following table which presents the answers for n and m. For example, the first line reads that every 3-connected

Answer for General Graphs		
Surface	Uniqueness (n)	Faithfulness (m)
Sphere	3	3
Projective plane	6	6
Torus	6	6
Klein bottle	6	6
The others	7	7

Table 1.

planar graph is uniquely and faithfully embeddable in a sphere, with at most a finite number of exceptions. In fact, there is no such exception by the uniqueness of duals of 3-connected planar graphs [7].

The results in [3], [4] and [6] complete the second line for a projective plane. It has been already shown in [2] that every 6-connected toroidal graph is uniquely embeddable in a torus and that it is also faithfully embeddable unless it is isomorphic to K_7, $\overline{4K_2}$ or $\overline{3K_3}$. Hence, we have to construct infinitely many 5-connected toroidal graphs whose uniqueness and faithfulness break down in order to show that $n=m=6$ for a torus. It is the same in case of a Klein bottle; The classification of 6-regular Klein-bottlal graphs in [5] implies that every 6-connected Klein-bottlal graph is uniquely embeddable in a Klein bottle and that $\overline{K_3 \cup C_6}$ is the unique exception for the faithfulness. A infinite number of 5-connected examples for the toroidal and Klein-bottlal case will be obtained in Section 3.

Therefore, the remaining case is when a given closed surface has negative Euler characteristic, so in other words when a surface is *hyperbolic*. In this case, we need not prove theorems which give sufficient conditions for uniqueness and faithfulness in terms of connectivity, because of the scarcity of 7-connected graphs. Just by the calculation of the Euler characteristic $\chi(F^2)$ alone, it can be shown that if a graph G has minimum degree at least 7 and is embeddable in a closed surface F^2, then the number of vertices of G does not exceed $6|\chi(F^2)|$ and hence there are only finitely many 7-connected graphs which are embeddable in F^2. Even if there are exceptions for $U(F^2; n)$ or $F(F^2; m)$ with $n, m \geqq 7$, the number of them is finite. So it suffices to construct infinitely many 6-connected examples for non-uniqueness and non-faithfulness in order to complete the fifth line of Table 1; " 7 " and " 7 ".

After arguments on *splitting* and *sewing* of embeddings of graphs in Section 4, we shall propose two transformations of graphs, called *insertion of a skew, handle of K_6* and *of a skew cross-cap of K_5* in Section 5. The former plays a role in destroying uniqueness and faithfulness and in decreasing $\chi(F^2)$ by two, while the latter preserves uniqueness and faithfulness and decrease $\chi(F^2)$ by one. Using them, we shall derive many non-uniquely and non-faithfully embeddable graphs from 6-connected toroidal graphs and Klein-bottlal graphs in Section 6. To estimate the connectivity of those graphs, we shall discuss in Section 5 an operation, called *n-path-splitting*, which transforms a graph without changing its connectivity. Unfortunately, our method does not work for one case, namely when F^2 is a non-orientable closed surface with $\chi(F^2)=-1$, so we shall deal with this exceptional case individually.

In all hyperbolic cases but this, the graphs constructed in Section 6 triangulate F^2, so if we restrict our objects to only triangulations of F^2, then the answer to our problem will be obtained as Table 2: By the result on projective-

planar triangulations in [4], the second line changes to "5" and "5". The answer for uniqueness in the exceptional case is still unknown.

Answer for Triangulations		
Surface	Uniqueness (n)	Faithfulness (m)
Sphere	3	3
Projective plane	5	5
Torus	6	6
Klein bottle	6	6
Non-orientable genus 3	?	7
The others	7	7

Table 2.

Our graph is finite, undirected and simple and has the canonical topology as a 1-complex. Our terminology for graph theory can be found in [1] and that for topology is quite standard.

2. Uniqueness and faithfulness of embedding

In this section, we shall formulate our subjects, the uniqueness and faithfulness of embedding of graphs into surfaces. We shall consider graphs with additional structure, called *peripheral cycles*, to make "cut and paste" on embeddings of graphs easily done. If we neglect each occurrence of ∂G, then the definitions below can be read as for the ordinary case.

Let G be a graph and ∂G a union of pairwise disjoint cycles in G. We call the pair $(G, \partial G)$ a *graph with peripheral cycles* and call each component of ∂G a *peripheral cycle* of G (or $(G, \partial G)$). A vertex or an edge is said to be *peripheral* if it lies on ∂G. An isomorphism between two graphs $(G_i, \partial G_i)$ $(i=1, 2)$ is a homeomorphism $\sigma : G_1 \rightarrow G_2$ which sends each vertex of G_1 to a vertex of G_2 and ∂G_1 onto ∂G_2, and is denoted by $\sigma : (G_1, \partial G_1) \rightarrow (G_2, \partial G_2)$. When two graphs are identical, an isomorphism is called an *automorphism* of $(G_1, \partial G_1)$ $(=(G_2, \partial G_2))$.

Clearly, the collection of all automorphisms of $(G, \partial G)$ is a group and consists of uncountably many elements. Then we classify them up to isotopy relative to the vertex set $V(G)$ and denote the set of such isotopy classes of automorphisms of $(G, \partial G)$ by $\mathrm{Aut}(G, \partial G)$. Also $\mathrm{Aut}(G, \partial G)$ is a group and is finite in turn. Precisely speaking, an automorphism of $(G, \partial G)$ does not belong to $\mathrm{Aut}(G, \partial G)$ but its isotopy class does. We shall however deal with an automorphism as if it would be a member of $\mathrm{Aut}(G, \partial G)$.

An *embedding* of a graph $(G, \partial G)$ with peripheral cycles in or into a surface

F^2 with boundary ∂F^2 is a continuous map $f : G \to F^2$ such that G and its image $f(G)$ are homeomorphic via f and $f(\partial G) = \partial F^2$. We denote such an embedding by $f : (G, \partial G) \to (F^2, \partial F^2)$. An embedding $f : (G, \partial G) \to (F^2, \partial F^2)$ is called a 2-*cell embedding* if each component of $F^2 - f(G)$, called a *face*, is homeomorphic to an open 2-cell $\{x \in \mathbf{R}^2 : |x| < 1\}$.

Let $(G_1, \partial G_1)$, $(G_2, \partial G_2)$ be two graphs with peripheral cycles and let F_1^2, F_2^2 be two surfaces. Two embeddings $f_i : (G_i, \partial G_i) \to (F_i^2, \partial F_i^2)$ are *equivalent* if there exist a homeomorphism $h : F_1^2 \to F_2^2$ and an isomorphism $\sigma : (G_1, \partial G_1) \to (G_2, \partial G_2)$ such that $h \circ f_1 = f_2 \circ \sigma$.

When we work in the category of labeled graphs, this isomorphism $\sigma : (G_1, \partial G_1) \to (G_2, \partial G_2)$ should be taken so as to preserve the labels. If both $(G_1, \partial G_1)$ and $(G_2, \partial G_2)$ are the same graph $(G, \partial G)$, then such a label-preserving isomorphism is nothing but the identity map of $(G, \partial G)$.

In the labeled sense, the two embeddings shown in Fig. 3(a) and (b) are not equivalent even if they have the same appearance, which is contrary to our expectation. If we modify the labeling of vertices in the graph itself, then the image of one embedding will be carried onto the other by a homeomorphism between the surfaces so that their labelings coincide. Clearly, neither Fig. 3(a) nor (b) is equivalent to (c), which meets our expectation. That is why we do not define the equivalence of embeddings by the formula $h \circ f_1 = f_2$ and why we prepare such an isomorphism between graphs which plays the role in permuting the labels of vertices.
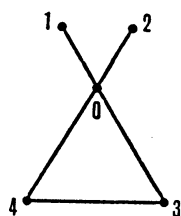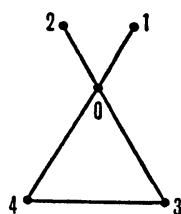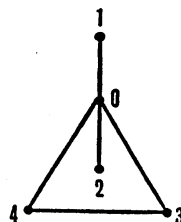

Fig. 3(a).   Fig. 3(b).   Fig. 3(c).

Proposition 2.1. *Let $(G_1, \partial G_1)$ and $(G_2, \partial G_2)$ be two graphs with peripheral cycles which contain no vertex of degree 2. Then two embeddings $f_i : (G_i, \partial G_i) \to (F_i^2, \partial F_i^2)$ in surfaces F_i^2 $(i = 1, 2)$ are equivalent if and only if there is a homeomorphism $h : F_1^2 \to F_2^2$ such that $h(f_1(G_1)) = f_2(G_2)$.* $\square$

This proposition is clearly true, but is false in general if there are some vertices of degree 2, as Fig. 4 suggests.

By the compactness of a surface, a graph admits at most a finite number of embeddings into the surface, up to equivalence. If there are n equivalence classes of embeddings of a graph $(G, \partial G)$ with peripheral cycles into a surface F^2, then $(G, \partial G)$ is said to be n-*way embeddable* in F^2. When $n > 0$, $(G, \partial G)$ is
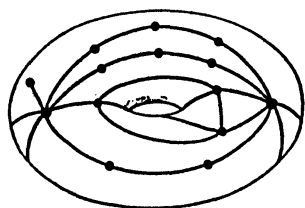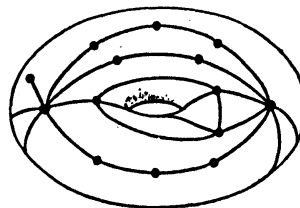
Fig. 4(a).   Fig. 4(b).

said to be *embeddable* in F^2. In particular, if $(G, \partial G)$ is only one-way embeddable in F^2, then $(G, \partial G)$ is said to be *uniquely embeddable* in F^2.

Now we shall define the faithfulness of embedding. Let $(G, \partial G)$ be a graph with peripheral cycles and F^2 a surface. An embedding $f:(G, \partial G) \to (F^2, \partial F^2)$ is *faithful* if for each automorphism $\sigma:(G, \partial G) \to (G, \partial G)$, there is a homeomorphism $h: F^2 \to F^2$ such that $h \circ f = f \circ \sigma$. For such a homeomorphism $h: F^2 \to F^2$, we say that h *realizes* σ under f or that σ *extends* to h via f. A graph $(G, \partial G)$ is said to be *faithfully embeddable* in a surface F^2 if $(G, \partial G)$ has a faithful embedding into F^2.

Consider the following embedding of C_n, a cycle of length n, in a disk. Let D^2 be the unit disk in the complex plane C and embed C_n in D^2 as the unit circle with vertices placed at nth roots of 1. Let $h: D^2 \to D^2$ be the self-homeomorphism of D^2 defined by

$$h(re^{i\theta}) = r^2 e^{i(\theta + (2\pi/n))}.$$

Then $h|_{C_n}$ is an automorphism of C_n of period n, but h itself has infinite period. Although each element of a cyclic subgroup of $\text{Aut}(C_n)$ extends to a power of h, this extension does not preserve the group structure of $\text{Aut}(C_n)$. Hewever, we can extend each automorphism of C_n cone-like so that $\text{Aut}(C_n)$ is realized by a group of self-homeomorphisms of D^2.

In general, after modifying extension, any faithful embedding $f:(G, \partial G) \to (F^2, \partial F^2)$ induces an injective homomorphism from $\text{Aut}(G, \partial G)$. It may be said that a faithful embedding embeds a graph $(G, \partial G)$ into a surface F^2 so that the symmetry of $(G, \partial G)$, with respect to its automorphisms, can be presented by the symmetry on F^2. In other words, the automorphism group of $(G, \partial G)$ has a faithful representation in the group of self-homeomorphisms of F^2 when $(G, \partial G)$ is faithful embeddable in F^2.

An automorphism $\sigma:(G, \partial G) \to (G, \partial G)$ of G is called a *symmetry* of f if it can be realized by a homeomorphism $h: F^2 \to F^2$ under an embedding $f:(G, \partial G) \to (F^2, \partial F^2)$, that is, if $h \circ f = f \circ \sigma$. Let $\text{Sym}(f)$ denote the set of all symmetries of f. (Strictly, $\text{Sym}(f)$ should be defined as the collection of all isotopy classes of symmetries of f.) Clearly, $\text{Sym}(f)$ is a subgroup of the automorphism group $\text{Aut}(G, \partial G)$. We call $\text{Sym}(f)$ the *symmetry group* of f. When $(G, \partial G)$ is already

embedded in F^2, the symmetry group of its inclusion map may be denoted by Sym$(G, \partial G)$ and is called the *symmetry group* of $(G, \partial G)$. An embedding $f:(G, \partial G) \to (F^2, \partial F^2)$ is faithful if and only if Sym$(f)=$Aut$(G, \partial G)$.

Notice that a faithful embedding is not a symmetrical embedding. If a graph has a higher symmetry then its image under a faithful embedding also has a higher symmetry as well as the graph itself, while if the automorphism group of $(G, \partial G)$ is trivial then every embedding of $(G, \partial G)$ into F^2 is faithful necessarily, with no symmetry, since Sym$(f)=$Aut$(G, \partial G)=\{$identity map$\}$.

Proposition 2.2. *The following three statements are equivalent to one another:*

(i) *A graph $(G, \partial G)$ is uniquely and faithfully embeddable in a surface F^2.*

(ii) *For any two embeddings $f_1, f_2:(G, \partial G) \to (F^2, \partial F^2)$ and any automorphism $\sigma:(G, \partial G) \to (G, \partial G)$, there is a homeomorphism $h:F^2 \to F^2$ such that $h \circ f_1 = f_2 \circ \sigma$.*

(iii) *For any two embeddings $f_1, f_2:(G, \partial G) \to (F^2, \partial F^2)$, there is a homeomorphism $h:F^2 \to F^2$ such that $h \circ f_1 = f_2$.*

When a graph $(G, \partial G)$ is already embedded in a surface F^2, the next is also equivalent to the above:

(iv) *For any embedding $f:(G, \partial G) \to (F^2, \partial F^2)$, there is a homeomorphism $h:F^2 \to F^2$ such that $h|_G = f$.*

Proof. (i) implies (ii): Since $(G, \partial G)$ is uniquely embeddable in F^2, there is an automorphism $\sigma_0:(G, \partial G) \to (G, \partial G)$ and a homeomorphism $h_0:F^2 \to F^2$ such that $h_0 \circ f_1 = f_2 \circ \sigma_0$. Let $\sigma:(G, \partial G) \to (G, \partial G)$ be any automorphism of $(G, \partial G)$. Since the unique embedding $f_1 = h_0^{-1} \circ f_2 \circ \sigma_0$ is faithful, there is a homeomorphism $h':F^2 \to F^2$ such that $h' \circ f_1 = f_1 \circ (\sigma_0^{-1} \circ \sigma) = h_0^{-1} \circ f_2 \circ \sigma$. Take $h_0 \circ h'$ as h, then $h \circ f_1 = f_2 \circ \sigma$.

(ii) implies (iii): Assign the identity map of $(G, \partial G)$ to σ.

(iii) implies (i): The condition of (iii) assures that $(G, \partial G)$ is uniquely embeddable in F^2 in the labeled version, and hence also in the unlabeled version. Let $f_1:(G, \partial G) \to (F^2, \partial F^2)$ be a unique embedding of $(G, \partial G)$. For any automorphism $\sigma:(G, \partial G) \to (G, \partial G)$, set $f_2 = f_1 \circ \sigma$, then there is a homeomorphism $h:F^2 \to F^2$ such that $h \circ f_1 = f_1 \circ \sigma$. Thus, f_1 is faithful and $(G, \partial G)$ is faithfully embeddable in F^2.

Assume that $(G, \partial G)$ is already embedded in F^2. Define two embeddings $f_1, f_2:(G, \partial G) \to (F^2, \partial F^2)$ to be *related* if there is a homeomorphism $h:F^2 \to F^2$ with $h \circ f_1 = f_2$, then this is an equivalence relation over all embeddings of $(G, \partial G)$ in F^2. In this term, (iii) states that any two embeddings are related to each other and hence equivalently that any embedding is related to the inclusion map $(G, \partial G) \subset (F^2, \partial F^2)$. This is nothing but the statement of (iv), so (iv) is equivalent to (iii) and also to (i) and (ii). $\square$

3. Skew vertices in triangulations

An embedding $f:(G, \partial G) \to (F^2, \partial F^2)$ is said to be *triangular* if each face of $f(G)$ is bounded by precisely three edges, that is, if $f(G)$ yields a triangulation of F^2. By only calculation of the Euler characteristic, it is easy to show that if a graph $(G, \partial G)$ has a triangular embedding in F^2, then any other embedding of $(G, \partial G)$ in F^2 is triangular. For this reason, we call such a graph $(G, \partial G)$ itself a *triangulation* of a surface F^2, not referring to its embedding.

In [2] and [4]. The author has proposed a concept, called *skew vertices*, to discuss the uniqueness and faithfulness of triangular embeddings. Here we shall generalize and purify it, and construct an infinite number of 5-connected toroidal and Klein-bottlal triangulations which are not uniquely or not faithfully embeddable in a torus or a Klein bottle.

Let $(G, \partial G)$ be a graph with peripheral cycles embedded in a surface F^2, and $f:(G, \partial G) \to (F^2, \partial F^2)$ another embedding of $(G, \partial G)$ into F^2. A face A of $(G, \partial G)$ is said to be *extendable* for f if there is an embedding $h: G \cup A \to F^2$ such that $h|_G = f$.

Proposition 3.1. *If all faces of G are extendable for f, then there is a homeomorphism $h: F^2 \to F^2$ such that $h|_G = f$.* □

Let $(G, \partial G)$ be a triangulation embedded in a surface F^2. Define the *star neighborhood* $st(v)$ of a vertex v as the closure of the union of triangular faces meeting at v, which is homeomorphic to a 2-cell, and the *link* $lk(v)$ of v as the induced subgraph in $(G, \partial G)$ by all edges in $st(v)$ but not incident to v. Then $lk(v)$ is a hamiltonian cycle in the subgraph $\langle N(v) \rangle$ induced by the neighbors of v in G unless v lies on ∂G. If v belongs to ∂G, then $lk(v)$ is a hamiltonian path of $\langle N(v) \rangle$.

A vertex v of $(G, \partial G)$ not lying on ∂G is *skew* if there is another hamiltonian cycle in $\langle N(v) \rangle$, that is, if $\langle N(v) \rangle$ contains at least two hamiltonian cycles. A peripheral vertex v is *skew* if there is a hamiltonian path of $\langle N(v) \rangle$, different from $lk(v)$, whose ends lie on the peripheral cycle containing v. A triangle uvw, a cycle of length 3, in G is *skew* if all three vertices u, v, w are skew. A triangular face A of $(G, \partial G)$ is *skew* if all three corners are skew vertices. Notice that the skewness of a vertex can be recognized only from the structure of a graph without its embedding.

Let G be a triangulation embedded in a closed surface F^2 and let v be a skew vertex of G. Then there exist two triangles Δ_1, Δ_2 in G which cross each other at v, and $lk(v) \cup \Delta_1 \cup \Delta_2$ is a subdivision of the complete graph K_5. Since such a subgraph is not planar, one of Δ_1 and Δ_2 cuts a handle or a cross-cap of F^2. This implies the following:

Proposition 3.2. *Let G be a triangulation of a closed surface F^2. The maximum number of skew vertices of G any pair of which has distance at least 3 in G does not exceed the genus of F^2.* □

It has been already proved in [4] that if a face A of G in F^2 is not skew, then A is extendable for any embedding of G in F^2 and that if G has at most four skew vertices, then G is uniquely and faithfully embeddable in F^2. As all skew vertices do not cause the uniqueness and faithfulness to break, we would like to pick up only essential ones among them.

Let S be the set of skew vertices of a triangulation G. If for a skew vertex v in S, the induced subgraph $\langle S \rangle$ contains no two skew triangles which intersect in only v, then remove v out of S and rest S to be $S-\{v\}$. Repeat this reduction of S as long as possible. Then we call a skew vertex remaining in the final S a *core skew vertex* of G. The set of core skew vertices is determined uniquely, not depending on the choice of a skew vertex v in each stage; for if there are no two skew triangles which contact at v, then there are no such triangles in later stages.

Theorem 3.3. *If a triangulation of a closed surface contains no core skew vertex, then it is uniquely and faithfully embeddable in the surface.*

Proof. Let G be a triangulation embedded in a closed surface F^2 and let S be a set of skew vertices obtained in a stage of the above process. Assume that if one of three vertices on the boundary triangle of a face A does not belong to S, then A is extendable for any embedding $f : G \to F^2$. Let v be a skew vertex to be removed from S in the next stage. Consider the case that v is adjacent to three skew vertices u, w, x which belong to S and two skew triangles uvw and vwx bound two faces A_1 and A_2, respectively, which meet in the edge vw. Let $st(v)$ consist of faces $A_3, \cdots, A_n$ together with A_1 and A_2, where $n=\deg(v)$. (See Fig. 5.) By our assumption, $A_3, \cdots, A_n$ are extendable for f, so the path between u and x in $lk(v)$ missing w, that is, $lk(v)-\{w\}$ is sent to $lk(f(v))-\{f(w)\}$ by f. Since $lk(f(v))$ is a unique hamiltonian cycle in $\langle N(f(v)) \rangle$ which contains $lk(f(v))-\{f(w)\}$, it follows that $f(lk(v))$ $=lk(f(v))$, and hence that A_1 and A_2 are extendable for f. In the other case, we can get easily the same conclusion.
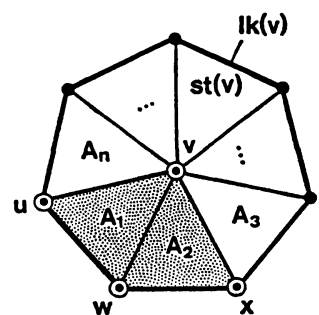

Fig. 5

Therefore, the number of faces whose extendability have been known increases stage by stage, and if S becomes empty finally then all faces of G will be extendable for f. By Proposition 2.2 (iv) and 3.1, G is uniquely and faithfully embeddable in F^2. □

Note that the converse of this theorem does not hold. In fact, there is a triangulation all of whose vertices are core skew but which is uniquely and faithfully embeddable. One of such examples can be found in [2] as the 6-regular toroidal graph denoted by $T(9, 2, 1)$.

Since two triangles which contact at only one vertex consist of five vertices, if there are at most four skew vertices then there is no core skew vertex. Thus we have:

Corollary 3.4. (S. Negami [4]) *If a triangulation of a closed surface contains at most four skew vertex, then it is uniquely and faithfully embeddable in the surface.* $\square$

If a triangulation G embedded in F^2 contains a skew vertex, then there is a triangle which bounds no 2-cell in F^2. Subdivide the triangulation as a 2-complex to eliminate such a triangle, then the resulting refinement contains no skew vertex, so it is uniquely and faithfully embeddable in F^2. In this way, we can .find a lot of triangulations which are uniquely and faithfully embeddable in a given surface.

Here we shall construct infinitely many 5-connected triangulations which are not uniquely or not faithfully embeddable in a torus or a Klein bottle, in order to complete the third and fourth lines in Table 1 and 2.

Fig. 6(a) and (b) illustrate 5-connected isomorphic graphs G_1 and G_2, respectively, triangularly embedded in a torus. Turning over the hexagon 254163 around two vertices 1 and 2 causes the difference between their appearances. Let H_i be the hexagonal cycle 254163 in G_i ($i=1, 2$) and let $(D_i, \partial D_i)$ and $(E_i, \partial E_i)$ be the graphs, with peripheral cycles $\partial D=\partial E=H_i$, outside and inside H_i, respectively.
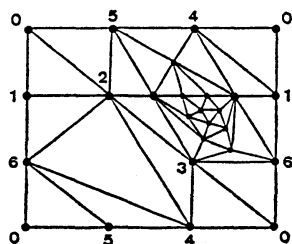

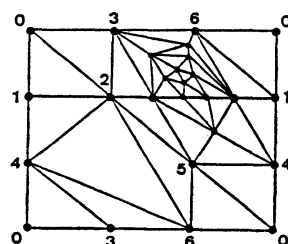Fig. 6(a).   Fig. 6(b).

In each graph, there is no skew vertex inside H_i and all vertices 0, 1, 2, 3, 4, 5, 6 are skew. Thus, any isomorphism $f: G_1 \to G_2$ sends the unique vertex 0 to 0 which is adjacent to no skew vertex, H_1 onto H_2, and D_1 onto D_2. By the arguments of skew vertices, each face of $(D_1, \partial D_1)$ is extendable for $f|_{D_1}$, and hence f preserves the cyclic order of vertices on H_1. This implies that f sends the seven vertices 0, 1, 2, 3, 4, 5, 6 to the vertices with the same labels, respec-

tively. However, the triangle 025 bounds a face in G_1 but does not in G_2, and so f cannot extend to a homeomorphism of the torus. Therefore, the two embeddings in Fig. 6(a) and (b) are not equivalent and G_1 or G_2 is not uniquely embeddable in a torus.

Now, consider the graph G_3 triangularly embedded in a torus, shown in Fig. 7(a) and (b) with only difference of labeling. The inside $(D, \partial D)$ of the hexagon 254163 of G_3 contains no skew vertex, and the outside $(E, \partial E)$ is the same as that of G_1. The two labelings of vertices given in Fig. 7(a) and (b) induce a unique non-trivial automorphism σ of G_3, but σ cannot extend to a self-homeomorphism of the torus, for the same reason on the triangle 025. Thus, this embedding of G_3 is not faithful. To see that G_3 is not faithfully embeddable in a torus, it suffices to observe that there is no other embedding of G_3 in a torus.
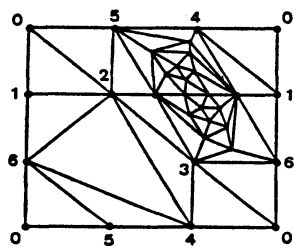

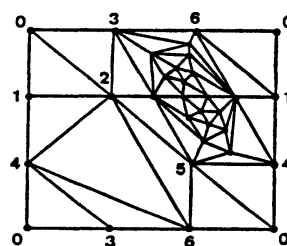Fig. 7(a)   Fig. 7(b).

Any embedding of G_3 in a torus always embeds $(D, \partial D)$ uniquely within a hexagonal 2-cell, for the lack of skew vertices. So if $(E, \partial E)$ is uniquely embeddable in a punctured torus, then G_3 itself is uniquely embeddable in a torus.

The uniqueness for $(E, \partial E)$ can be checked as follows. The graph E is isomorphic to the complete graph with seven vertices 0, 1, 2, 3, 4, 5, 6 minus three edges 13, 35 and 51. So an embedding of the complete graph K_7 can be obtained from the embedding of G_3 by replacing the part of D with three edges 13, 35 and 51. Embed $(E, \partial E)$ into a punctured torus in another way, and attach to ∂E a hexagonal 2-cell with three edges 13, 35 and 51 included, then another embedding of K_7 in a torus can be constructed. By Uniqueness Theorem for toroidal graphs [2], these two embeddings of K_7 are equivalent to the unique triangular embedding $T(7, 2, 1)$.

Since any two faces of $T(7, 2, 1)$ are transferable by a symmetry on the torus, there is an isomorphism between the two embeddings of K_7 which sends the face 135 onto 135, and hence which sends the hexagon 254163 onto 254163. Since the permutation (246) (135) induces a symmetry of the original embedding of $(E, \partial E)$, that is, of the embedding of G_3 with the inside of the hexagon ∂E deleted, any two embedding of $(E, \partial E)$ in a punctured torus are equivalent in

the labeled sense.

Therefore, G_3 is uniquely embeddable in a torus and has only the embedding, shown in Fig. 7, which is not faithful in a torus.

Starting from the above graphs, we shall construct infinitely many examples:

Theorem 3.5. *There exist an infinite number of 5-connected toroidal triangulations which are not uniquely embeddable in a torus, and ones which are not faithfully embeddable in a torus.*

Proof. In Fig. 6 and 7, fill each face inside the hexagon 254163 with the configuration of Fig. 8(a) or (b), according to whether or not the face meets the hexagon in an edge. (The bottom bold edge in Fig. 8(a) has to coincide with an edge on the hexagon.) Repeating this process, we shall create two infinite series
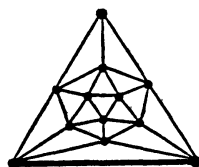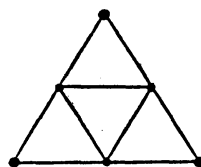

Fig. 8(a).   Fig. 8(b).

beginning from G_1 and from G_3. By the same logic as above, we can conclude that each graph belonging to the former and latter series is not uniquely and is not faithfully embeddable in a torus, respectively. All of those graphs have no cycle of length 4 which bounds a 2-cell containing at least one vertex in the torus. This implies that they are 5-connected. □

The two embeddings in Klein bottles as given in Fig. 9(a) and (b) are ones of two isomorphic 5-connected graphs G_4 and G_5, respectively. In either graph, the labeled vertices 0, 1, 2, 3, 4, 5, but 6, are skew and all vertices inside the hexagon 125634 are not skew. Thus, there is a unique isomorphism between G_4 and G_5 which sends necessarily 0, 1, 2, 3, 4, 5, 6 to the vertices with the same labels. Since 023 bounds a face in G_4 but does not in G_5, the isomorphism cannot extend to a homeomorphism between Klein bottles. Therefore, these embeddings are not equivalent and G_4 or G_5 is not uniquely embeddable in a Klein bottle.
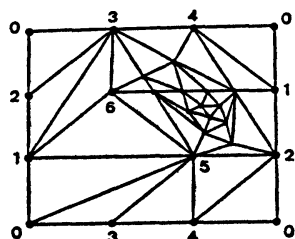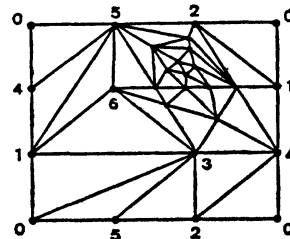

Fig. 9(a).   Fig. 9(b).

The graph G_6 in Fig. 10 is 5-connected and has an automorphism which interchanges only two vertices 0 and 5 fixing the other. Since the vertex 2 has degree 7, this interchanging cannot be realized by a dihedral action of order 14 on seven neighbors of 2. Thus, such an automorphism is not a symmetry of any embedding of G_6 in a Klein bottle and hence G is not faithfully embeddable in a Klein bottle.
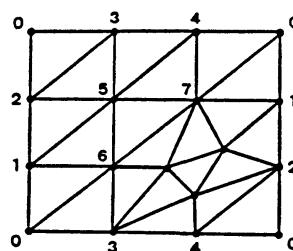

Fig. 10

To create an infinite number of examples, fill each face of G_4 inside the hexagon 125634 and of G_6 inside 124367 with the configulations in Fig. 8(a) and (6) under the same rule as in the proof of Theorem 3.5.

Theorem 3.6. *There exist an infinite number of 5-connected Klein-bottlal triangulations which are not uniquely embeddable in a Klein bottle and ones which are not faithfully embeddable in a Klein bottle.* □

4. Splitting and sewing of embeddings

Here we shall develop a method to produce those graphs which are are not uniquely or not faithfully embeddable in a closed surface. Roughly speaking, we shall construct such graphs from graphs with peripheral cycles whose uniqueness and faithfulness are recognized, sewing them together along their peripheral cycles.

Let $(C_1, \partial G_1)$ and $(G_2, \partial G_2)$ be two graphs with isomorphic peripheral cycles, and let $\phi: \partial G_2 \to \partial G_1$ be an isomorphism. Then we denote by $G_1 \cup_\phi G_2$ the graph obtained from $(G_1, \partial G_1)$ and $(G_2, \partial G_2)$ by sewing them together along their peripheral cycles via ϕ. That is, each point x on ∂G_2 is identified with the point $\phi(x)$ on ∂G_1 in $G_1 \cup_\phi G_2$. We call such ϕ a *sewing map*.

If we choose various sewing maps, then a lot of graphs can be constructed from two graphs with peripheral cycles. Of course, isomorphic graphs will be often produced:

Proposition 4.1. *Let $(G_1, \partial G_1)$ and $(G_2, \partial G_2)$ be two graphs with peripheral cycles, and let $\phi, \psi: \partial G_2 \to \partial G_1$ be two sewing maps. If there exist automorphisms $\sigma \in \mathrm{Aut}(G_1, \partial G_1)$ and $\tau \in \mathrm{Aut}(G_2, \partial G_2)$ such that $\sigma \circ \phi = \psi \circ \tau |_{\partial G_2}$, then $G_1 \cup_\phi G_2$ and $G_1 \cup_\psi G_2$ are isomorphic. If there is an isomorphism between $G_1 \cup_\phi G_2$ and $G_1 \cup_\psi G_2$ which sends ∂G_2 onto ∂G_2, then the converse is also true.* □

Now we shall sew up two embeddings of graphs with peripheral cycles. Let $f_1: (G_1, \partial G_1) \to (F_1^2, \partial F_1^2)$ and $f_2: (G_2, \partial G_2) \to (F_2^2, \partial F_2^2)$ be two embeddings, and let $\phi: \partial G_2 \to \partial G_1$ be a sewing map. Identify each point x on ∂F_2^2 with $f_1 \cdot \phi \cdot f_2^{-1}(x)$ and denote the resulting closed surface by F^2. Then we obtain a well-defined embedding $f_1 \cup_\phi f_2: G_1 \cup_\phi G_2 \to F^2$ so that $f_1 \cup_\phi f_2(x) = f_i(x)$ if $x \in F_i^2$. Note that

the homeomorphism type of F^2 does not depend on the choice of sewing maps.

Proposition 4.2. *Let $f_1: (G_1, \partial G_1) \to (F_1^2, \partial F_1^2)$ and $f_2: (G_2, \partial G_2) \to (F_2^2, \partial F_2^2)$ be two embeddings from graphs with peripheral cycles into surfaces, and let ϕ, $\psi: \partial G_2 \to \partial G_1$ be two sewing maps. If there exist automorphisms $\sigma \in \mathrm{Sym}(f_1)$ and $\tau \in \mathrm{Sym}(f_2)$ such that $\sigma \circ \phi = \psi \circ \tau|_{\partial G_2}$, then $f_1 \cup_\phi f_2$ and $f_1 \cup_\psi f_2$ are equivalent. If there is a homeomorphism $h: F^2 \to F^2$ which maps $f_1 \cup_\phi f_2(\partial G_2)$ onto $f_1 \cup_\psi f_2(\partial G_2)$ and induces an isomorphism between the image of $f_1 \cup_\phi f_2$ and $f_1 \cup_\psi f_2$, then the converse is also true.* $\square$

Let G be a graph and let G_1, G_2 be two subgraphs of G such that $G_1 \cap G_2$ consists of a disjoont union of cycles, denoted by ∂G_1 and also by ∂G_2. An automorphism $\sigma: G \to G$ is said to *split* (with respect to $\{G_1, G_2\}$) if $\sigma(G_1) = G_1$ and $\sigma(G_2) = G_2$. An embedding $f: G \to F^2$ of G into a closed surface F^2 is said to *split* (with respect to $\{G_1, G_2\}$) if F^2 decomposes into two subsurfaces F_1^2 and F_2^2 so that $F_1^2 \cap F_2^2 = \partial F_i^2$ and $f|_{G_i}$ induces an embedding $f_i: (G_i, \partial G_i) \to (F_i^2, \partial F_i^2)$ $(i=1, 2)$. Then we say that f split into f_1 and f_2 and write it by $f = f_1 \cup f_2$.

Lemma 4.3. *Let $G = G_1 \cup G_2$ be a connected graph with $G_1 \cap G_2 = \partial G_i$ $(i=1, 2)$ disjoint cycles. Suppose that all automorphisms of G and all embeddings of G into a closed surface $F^2 = F_1^2 \cup F_2^2$ split and suppose that $(G_1, \partial G_1)$ is uniquely and faithfully embeddable in F_1^2 and $(G_2, \partial G_2)$ is uniquely embeddable in F_2^2. Suppose that an embedding $f: G \to F^2$ splits into $f_1: (G_1, \partial G_1) \to (F_1^2, \partial F_1^2)$ and $f_2: (G_2, \partial G_2) \to (F_2^2, \partial F_2^2)$. Then:*

(i) *G is uniquely embeddable in F^2 if and only if for any automorphism $\psi \in \mathrm{Aut}(G_2, \partial G_2) - \mathrm{Sym}(f_2)$, there exist automorphisms $\sigma \in \mathrm{Aut}(G_1, \partial G)$ and $\tau \in \mathrm{Sym}(f_2)$ such that $\sigma|_{\partial G_1} = \psi \circ \tau|_{\partial G_2}$.*

(ii) *f is faithful if and only if for any automorphism $\tau \in \mathrm{Aut}(G_2, \partial G_2) - \mathrm{Sym}(f_2)$, there is no automorphism $\sigma \in \mathrm{Aut}(G_1, \partial G_1)$ such that $\sigma|_{\partial G_1} = \tau|_{\partial G_2}$.*

Proof. (i) From all of hypotheses, we conclude that any embedding of G in F^2 can be obtained as $f_1 \cup_\psi f_2$ with sewing map $\psi: \partial G_2 \to \partial G_1$. We have to choose a sewing map ψ so that $G_1 \cup_\psi G_2$ is isomorphic to $G = G_1 \cup_{id} G_2$ with sewing map the identity map. Thus, it follows from Proposition 4.1 that for some $\sigma \in \mathrm{Aut}(G_1, \partial G_1)$ and $\tau \in \mathrm{Aut}(G_2, \partial G_2)$,

$$\sigma|_{\partial G_1} = \psi \circ \tau|_{\partial G_1},$$

and hence

$$\sigma \circ \tau^{-1}|_{\partial G_1} = \psi \circ id_{G_2}|_{\partial G_2}.$$

This implies that $G_1 \cup_\psi G_2$ and $G_1 \cup_{\tau^{-1}} G_2$ are isomorphic via an isomorphism which splits. Thus, a sewing map ψ may be assumed to extend to an automorphism of $(G_2, \partial G_2)$. We use the same symbol ψ for the extension of ψ

$(\phi \in \mathrm{Aut}(G_2, \partial G_2))$.

By Proposition 4.2, all embeddings of G in F^2 are equivalent to $f_1 \cup f_2$ with sewing map the identity map, that is, G is uniquely embeddable in F^2 if and only if for any automorphism $\phi \in \mathrm{Aut}(G_2, \partial G_2)$, there exist automorphisms $\sigma \in \mathrm{Sym}(f_1) = \mathrm{Aut}(G_1, \partial G_1)$ and $\tau \in \mathrm{Sym}(f_2)$ such that $\sigma|_{\partial G_1} = \phi \circ \tau|_{\partial G_2}$. When ϕ belongs to $\mathrm{Sym}(f_2)$, then we can take the identity map of ∂G_1 as σ and ϕ^{-1} as τ, and so the above criterion is automatically true for ϕ. Thus, the range of ϕ may be restricted to $\mathrm{Aut}(G_2, \partial G_2) - \mathrm{Sym}(f_2)$.

(ii) Since all automorphisms of G split, each $\rho \in \mathrm{Aut}(G)$ can be regarded as a pair of $\sigma \in \mathrm{Aut}(G_1, \partial G_1)$ and $\tau \in \mathrm{Aut}(G_2, \partial G_2)$ for which $\sigma|_{\partial G_1} = \tau|_{\partial G_2}$;

$$\mathrm{Aut}(G) = \{(\sigma, \tau) : \sigma \in \mathrm{Aut}(f_1), \ \tau \in \mathrm{Aut}(f_2), \ \sigma|_{\partial G_1} = \tau|_{\partial G_2}\}.$$

If ρ belongs to $\mathrm{Sym}(f)$, then σ and τ belong to $\mathrm{Sym}(f_1)$ and $\mathrm{Sym}(f_2)$, respectively. Thus,

$$\mathrm{Sym}(f) = \{(\sigma, \tau) : \sigma \in \mathrm{Sym}(f_1), \ \tau \in \mathrm{Sym}(f_2), \ \sigma|_{\partial G_1} = \tau|_{\partial G_2}\}.$$

Since f_1 is a faithful embedding by the hypothesis for $(G_1, \partial G_1)$, $\mathrm{Sym}(f_1) = \mathrm{Aut}(G_1, \partial G_1)$ and hence $\mathrm{Aut}(G)$ coincides with $\mathrm{Sym}(f)$ if and only if no member of $\mathrm{Aut}(G_2, \partial G_2) - \mathrm{Sym}(f_2)$ is compatible with an automorphism of $(G_1, \partial G_1)$. That is the conclusion of (ii). □

Here we should reduce the above abstract statement into an applicable type of a theorem, adding a condition which ensures the splittability of automorphisms and embeddings of a graph:

Theorem 4.4. *Let G be a triangulation embedded in a closed surface F^2 and let G_2 be the subgraph induced by all skew vertices in G. Suppose that:*

(i) *G_2 is isomorphic to the complete graph K_6 with six vertices.*
(ii) *A hamiltonian cycle ∂G_2 of G_2 separates F^2 into F_1^2 and F_2^2.*
(iii) *F_2^2 is a punctured torus and $F_2^2 \cap G = G_2$.*

Let G_1 denote $F_1^2 \cap G$. Then if G_1 has an automorphism of period 2 which reflects ∂G_1, then G is not faithfully embeddable in F^2. Otherwise, G is not uniquely embeddable in F^2.

Proof. Check the conditions of Lemma 4.3 for this G. By the skewness of vertices, each automorphism of G splits into an automorphism of $(G_1, \partial G_1)$ and of $(G_2, \partial G_2)$, and the restriction $f|_{G_1}$ of each embedding $f : G \to F^2$ extends to an embedding $h : F_1^2 \to F^2$. When F_1^2 is non-orientable, then there might be the possibility for $F^2 - F_1^2$ and $F^2 - h(F_1^2)$ not to be homeomorphic.

If they were not, then the closure of $F^2 - h(F_1^2)$ would be a punctured Klein bottle which admits an embedding of $(K_6, \partial K_6)$ with peripheral hamiltonian cycle ∂K_6. Cap off its boundary with a 2-cell including the seventh vertex, and join

the vertex to all six vertices on ∂K_6. Then an embedding of the complete graph K_7 in a Klein bottle could be obtained. It is however contrary to the fact that K_7 is not embeddable in a Klein bottle. (By the results in [5], every 6-regular Klein-bottlal graph has at least 9 vertices.) Therefore, both $F^2-F_1^2$ and $F^2-h(F_1^2)$ are homeomorphic to a punctured torus, and f splits.

Since $(G_1, \partial G_1)$ has no skew face, it is uniquely and faithfully embeddable in F_1^2. The uniqueness of embedding of $(G_2, \partial G_2)$ can be concluded from that of K_7. First construct embeddings of K_7 in tori from two embeddings of $(G_2, \partial G_2)$ in punctured tori by capping off the boundaries with 2-cells and adding new vertices adjacent to all vertices on each ∂G_2. Since K_7 is uniquely embeddable in a torus, there is a homeomorphism h between the two tori which carries one embedding of $(G_2, \partial G_2)$ onto the other. Moreover, h can be assumed to send one of the additional vertices to the other since K_7 is symmetrically embeddable in a torus. Then the restriction of h to the punctured tori makes the given two embeddings of $(G_2, \partial G_2)$ equivalent.

Now we have found all of the hypotheses of Lemma 4.3. What to do next is to determine $\mathrm{Aut}(G_2, \partial G_2)-\mathrm{Sym}(f_2)$. Label the six vertices of $(G_2, \partial G_2)$ with 1, 2, 3, 4, 5, 6 according to their cyclic order on ∂G_2. Then $\mathrm{Aut}(G_2, \partial G_2)$ is the dihedral group of order 12 generated by (26)(35) and (12)(36)(45), and $\mathrm{Sym}(f_2)$ is the cyclic subgroup of index 2 generated by (123456). (See Fig. 11, where ∂G_2 is indicated by bold edges.) Thus all member of $\mathrm{Aut}(G_2, \partial G_2)-\mathrm{Sym}(f_2)$ are listed up as the six reflections of ∂G_2; namely (26)(35), (12)(36)(45), (13)(46), (23)(14)(56), (24)(15) and (34)(25)(16).
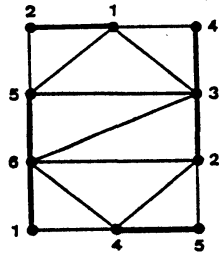

Fig. 11

Suppose that there is an automorphism σ of $(G_1, \partial G_1)$ which reflects ∂G_1. Then $\sigma|_{\partial G_1}$ coincides with one of the above six reflections none of which extends to a symmetry of f_2. By (ii) in Lemma 4.3, G is not faithfully embedded in F^2. Since our argument has proceeded for an arbitrary embedding of G, G is not faithfully embeddable in F^2.

Now suppose that there is no automorphism of $(G_1, \partial G_1)$ which reflects ∂G_1. Then none of the six reflections listed above extends to an automorphism of $(G_1, \partial G_1)$ and hence none of them satisfies the criterion of (i) in Lemma 4.3. Thus, G is not uniquely embeddable in F^2. $\square$

We shall call the unique embedding of $(K_6, \partial K_6)$ in a punctured torus or the graph itself the *skew handle* of K_6. As all orientable closed surfaces are constructed from a sphere by attaching handles, the skew handle of K_6 will play a role in creating a lot of graphs which are not uniquely embeddable and ones which are not faithfully embeddable in a given orientable surface.

On the other hand, all non-orientable closed surfaces are obtained as a

sphere with several cross-caps added. Then consider the unique embedding of $(K_5, \partial K_5)$ with hamitonian peripheral cycle ∂K_5 in a Möbius band or a cross-cap and call it the *skew cross-cap* of K_5. Notice that the skew cross-cap of K_5 is faithfully embedded in a Möbius band, illustrated in Fig. 12, in contrast with the skew handle of K_6 not being faithfully embedded.
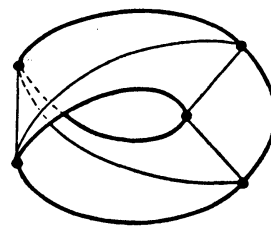

Fig. 12.

Proposition 4.5. *Let G be a triangulation embedded in a non-orientable closed surface F^2 with splitting $G = G_1 \cup G_2$ such that $G_1 \cap G_2$ is a cycle, denoted by ∂G_1 and also by ∂G_2, and that $(G_2, \partial G_2)$ is isomorphic to $(K_5, \partial K_5)$ and is induced by all skew vertices of G. Then G is uniquely and faithfully embedded in F^2.*

Proof. Let F_1^2 and F_2^2 be the subsurface of F^2 where $(G_1, \partial G)$ and $(G_2, \partial G_2)$ are embedded respectively and let $f : G \to F^2$ be another embedding of G into F^2. Then f is also triangular. Since $(G_1, \partial G)$ has no skew vertex, $f|_{G_1}$ extends to an embedding $h : F_1^2 \to F^2$. (Each vertex on ∂G_1 is skew in G but is not in $(G_1, \partial G_1)$.) In this case, the Möbius band F_2^2 is homeomorphic to $h(F_2^2)$ and h extends to a homeomorphism of the whole of F^2 which realizes f since the skew cross-cap of K_5 is uniquely and faithfully embeddable in a Möbius band. Thus, G is uniquely and faithfully embeddable in F^2. $\square$

It is easy to generalize our arguments above for triangulations which include more than one skew handles and skew cross-caps.

Corollary 4.6. *Let G be a triangulation of a closed surface F^2. Suppose that the subgraph of G induced by all skew vertices consists of a disjoin union of several skew handles of K_6 and skew cross-caps of K_5. If there is an automorphism of G which reflects some peripheral cycles of the skew handles, then G is not faithfully embeddable in F^2. Otherwise, G is not uniquely embeddable in F^2.* $\square$

The skew cross-cap cannot be used alone to produce graphs not uniquely or not faithfully embeddable in a surface, but it is available to increase the non-orientable genus of graphs by one. What we should do next is to discuss the connectivity of graphs including several skew handles of K_6 or skew cross-caps of K_5 and to find out infinitely many candidates for G_1 in Theorem 4.4.

5. n-Path-splitting of n-connected graphs

Here we shall define a new operation to transform a graph without changing its connectivity, in order to create a lot of n-connected graphs from one. The

n-vertex-splitting is known as one of such operations and is useful to make many graphs embedded in a common surface. However, we want now one which increases the genus of a graph aggressively.

Let G be a graph and Q a path, given as a sequence $\{a, x_1, \cdots, x_k, b\}$ of vertices in G, which joins two vertices a and b at distance $k+1$. First, remove all of inner vertices $x_1, \cdots, x_k$ of Q from G and join a to b by two new paths Q_1 and Q_2, respectively given as $\{a, x_{11}, \cdots, x_{1k}, b\}$ and $\{a, x_{21}, \cdots, x_{2k}, b\}$. Next, add new edges $x_{1i}x_{2j}$ ($|i-j|\leq 1$) and join each neighbor $w \in N(x_i)$ of x_i to precisely one of x_{1i} and x_{2i} ($i=1, \cdots, k$) so that both x_{1i} and x_{2i} have degree at least n in the resulting graph G'. This procedure to transform G into G' is called n-*path-splitting* along Q. (See Fig. 13.) The choice of x_{1i} or x_{2i} to be jointed to w gives rise to the ambiguity of n-path-splitting along Q. Note that we require Q to be one of the shortest paths between a and b.
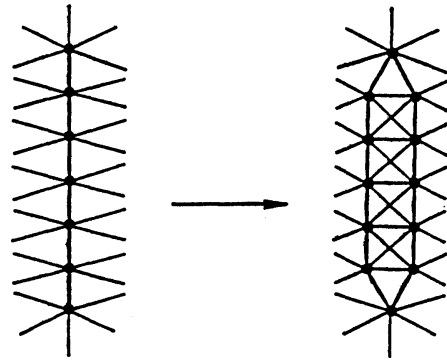
n-path-splitting

Fig. 13.

Proposition 5.1. *Every graph obtained from an n-connected graph by n-path-splitting along a path is also n-connected.*

Proof. Let G be an n-connected graph and G' a graph obtained from G by n-path-splitting along a path Q in G. (We use the same notation as above.) Let $p: G' \to G$ be the canonical projection for which $p(x_{1i})=p(x_{2i})=x_i$ ($i=1, \cdots, k$). Suppose that the removal of less than n vertices, say $U=\{u_1, \cdots, u_k\}$ ($k<n$), separates G' into two disjoint non-empty subgraphs H_1 and H_2, that is, G' is not n-connected and $G'-U=H_1\cup H_2$. Set respectively;

$$X=\{x_1, \cdots, x_k\},$$
$$X_1=\{x_{11}, \cdots, x_{1k}\},$$
$$X_2=\{x_{21}, \cdots, x_{2k}\}.$$

Then we have

$$G-p(U)=(p(H_1)-p(U)\cap X)\cup(p(H_2)-p(U)\cap X).$$

Since x_{1i} and x_{2i} are adjacent, they do not belong to H_1 and H_2 separately if both of them remain in $G'-U$. This implies that $(p(H_1)-p(U)\cap X)$ and $(p(H_2)-p(U)\cap X)$ are disjoint from each other. Since G is n-connected, $G-p(U)$ is connected and hence one of $p(H_1)-p(U)\cap X$ and $p(H_2)-p(U)\cap X$ must be empty, say $p(H_1)-p(U)\cap X$. In other words, H_1 is contained in the subgraph $\langle X_1\cup X_2\rangle$ induced by X_1 and X_2 in G'.

If both x_{1i} and x_{2i} belonged to H_1, then $p(H_1)-p(U)\cap X$ would contain x_i and would be non-empty. Thus, at most one of x_{1i} and x_{2i} belongs to H_1 and so we may assume that $H_1 \subset \langle X_1 \rangle$ after renumbering and renaming. Notice that $\langle X_1 \rangle$ coincides with $Q_1 - \{a, b\}$ and does not contain an edge of the form $x_{1i}x_{1j}$ ($|i-j| \geq 2$) since Q is chosen to be shortest.

To cut off a path $x_{1i} \cdots x_{1j}$ ($i \leq j$) from G', one has to remove at least $n-1$ vertices in $N(x_{1i}) - \{x_{1i+1}\}$ and one more x_{1j} or b. Thus, U would have to contains at least n vertices but it is contrary to the hypothesis of U. Therefore, G' is n-connected. □

A triangulation including a skew handle of K_6 or a skew cross-cap of K_5 can be obtained from a triangulation by adding edges after n-path-splitting along a path of length 3. However, an arbitrary way of n-path-splitting does not yield a triangulation in general. We should apply n-path-splitting to a triangulation compatibly with its embedding.

Let G be a triangulation embedded in a closed surface F^2 and Q a path of length 3, given as $\{a, x, y, b\}$ which joins two vertices a and b at distance 3. Assume that the rotations around x and y are presented as

$$x. \quad au \cdots u_k y v_s \cdots v_1$$

$$y. \quad xu \cdots u_h b v_t \cdots v_s \quad (1<k<h, \ 1<s<t).$$

First apply 6-path-splitting along Q to G so that $u_1, \cdots, u_h$ are joined to Q_1 and $v_1, \cdots, v_t$ to Q_2 after Q splits into two paths Q_1 and Q_2 with common ends and add two new edges ay_1 and ay_2. Then the resulting graph has a skew cross-cap of K_5, induced by $\{a, x_1, x_2, y_1, y_2\}$, and can be triangularly embedded in a non-orientable closed surface with Euler characteristic $\chi(F^2)-1$. This process is called *insertion of a skew cross-cap of K_5 along a path Q*.
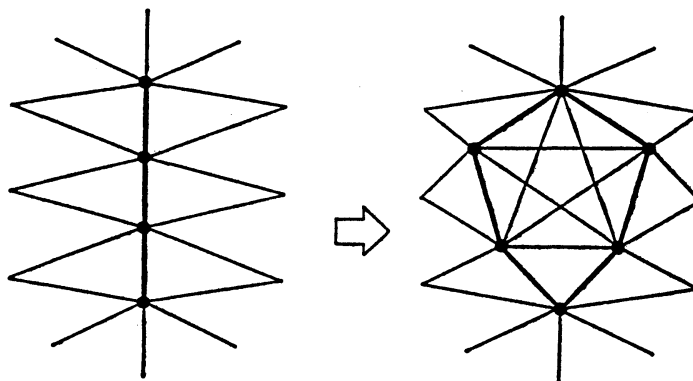
Insertion of skew cross-cap of K_5

Fig. 14.

Furthermore, add three edges ab, bx_1 and bx_2 so that $\{a, x_1, x_2, y_1, y_2, b\}$ induces a complete subgraph. Then the resulting graph has a skew handle of K_6 in turn and can be triangularly embedded in a closed surface with Euler characteristic $\chi(F^2)-2$ and with the same orientability as F^2. This process is called *insertion of a skew handle of K_6 along a path Q*.
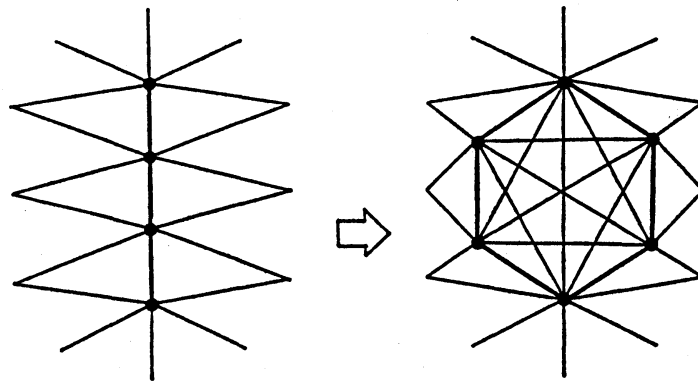
Insertion of a skew handle of K_6

Fig. 15.

Corollary 5.2. *A graph obtained from a 6-connected triangulation of a closed surface by insertion of a skew handle of K_6 or of a skew cross-cap of K_5 is also a 6-connected triangulation of a closed surface.* □

6. Examples.

Combining our previous arguments with the results in [2] and [5], we shall construct an infinite number of examples to complete the answer to our problem in the hyperbolic case. Section 4 gives us a logic to deny the uniqueness or faithfulness of embedding and Section 5 assures the connectivity of graphs constructed below.

First, we shall use the 6-regular toroidal graphs $T(p, q, r)$, as material for examples, constructed as follows. First prepare $r+1$ cycles $C_0, \cdots, C_r$ of the same length p and let u_{ij} $(j \equiv 0, \cdots, p-1 \pmod{p})$ be p vertices lying on C_i in order. Join each vertex u_{ij} $(1 \leq i \leq r-1)$ to u_{i-1j}, u_{i-1j-1}, u_{i+1j} and u_{i+1j+1}, then the vertices on C_0 and C_r have degree 4 and the other 6 in the resulting graph H_r^p. We shall call H_r^p the (p, r)-*drum*. Next paste the two end cycles C_0 and C_r so that u_{0q} is identified with u_{r0}, then a 6-regular graph will be obtained. This graph is $T(p, q, r)$ and can be triangularly embedded in a torus so that $C_0, \cdots, C_r$ are placed on the torus in parallel. These parallel cycles $C_0, \cdots, C_r$ are called *geodesic cycles* of $T(p, q, r)$.

In [2], the author classified $T(p, q, r)$'s, up to isomorphism, with the translation rule of parameters and showed that if $pr \geq 10$, then $T(p, q, r)$ contains no

skew vertex and hence that it is uniquely and faithfully embeddable in a torus. Notice that a toroidal graph is 6-connected if and only if it is 6-regular.

Theorem 6.1. *Let G be the graph obtaind from the 6-regular toroidal graph $T(p, q, r)$ $(pr \geqq 10)$ by insertion of h skew handles of K_6 $(h > 0)$ along h paths on a geodesic cycle of length p in $T(p, q, r)$ so that any two of the skew handles have distance at least 2. Then G is 6-connected and the genus of G is equal to $h+1$. If $2q \equiv -r$ (mod p) then G is not faithfully embeddable in an orientable closed surface of genus $h+1$. Otherwise, G is not uniquely embeddable in an orientable closed surface of genus $h+1$.*

Theorem 6.2. *Let G be the graph obtained from the 6-regular toroidal graph $T(p, q, r)$ $(pr \geqq 10)$ by insertion of h skew handles of K_6 and q skew cross-caps of K_5 $(h, q > 0)$ along $h+q$ paths on a geodesic cycle of length p in $T(p, q, r)$ so that any two of the skew handles or skew cross-caps have distance at least 2. Then G is 6-connected and the non-orientable genus of G is equal to $2h+q+2$. If $2q \equiv -r$ (mod p) then G is not faithfully embeddable in a non-orientable closed surface of genus $2h+q+2$. Otherwise, G is not uniquely embeddable in a non-orientable closed surface of genus $2h+q+2$.*

Proof of Theorem 6.1 and 6.2. Since $T(p, q, r)$ with at least 10 vertices has no skew vertex, the subgraph of G induced by all skew vertices, in either theorem, consists of the disjoint union of skew handles and skew cross-caps inserted. By Corollary 5.2, G is 6-connected. It is easy to evaluate the genus or the non-orientable genus of G. If we check conditions for the existence of an automorphism which reflects some peripheral cycles of skew handles, then we can conclude the theorems from Corollary 4.6.

Such an automorphism of G induces an automorphism of $T(p, q, r)$ leaving the geodesic cycle along which skew handles and skew cross-caps are inserted fixed and the automorphism extends to an orientation-reversing self-homeomorphism of the torus. This implies that the 6-regular torus graph obtained from $T(p, q, r)$ by the translation (III) of Theorem 3.6* in [2] has the same parameters as $T(p, q, r)$; namely

$$p = p$$

$$q \equiv -(q+r) \ (\text{mod } p)$$

$$r = r.$$

Thus, there is an automorphism of G in question if and only if $2q \equiv -r$ (mod p). $\square$

Now take the 6-regular Klein-bottlal graphs $Kc(p, k)$ and $Kh(p, k)$. When p is even, $Kc(p, k)$ can be constructed from H_k^p by identify u_{0j} with u_{0-j} and

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* Replace the second formula in (V) of Theorem 3.6 [2, p. 172] with $q' \equiv \beta r$ (mod p').

u_{kj} with u_{k-j} ($j \equiv 0, \cdots, p-1$ (mod p)), respectively, on C_0 and C_k. On the other hand, if p is an odd number $2m+1$, use the (p, $k-1$)-drum H^2_{k-1} and join u_{0j} to u_{0j+m} and u_{k-1j} to u_{k-1j+m} ($j \equiv 0, \cdots, p-1$ (mod p)), respectively. The resulting graph is $Kc(p, k)$. In either case, $Kc(p, k)$ can be triangularly embedded in a Klein bottle so that the part of its drum lies in an annulus and the two ends are placed on two cross-caps separately. The latter $Kh(p, k)$ is the 6-regular graph obtained from H^2_k by identifying each u_{0j} with u_{k-j} and also can be embedded in a Klein bottle naturally.

In [5], it has been shown that if $p \geq 6$, then both $Kc(p, k)$ and $Kh(p, k)$ are 6-connected and have no skew vertex. The important fact is that the system of specified cycles, derived from $C_0, \cdots, C_k$ (or C_{k-1}), in each of $Kc(p, k)$ and $Kh(p, k)$ is invariant under any automorphism of it. Such a unique system is called the *geodesic 2-factor*.

Theorem 6.3. *Let G be the graph obtained from the 6-regular Klein-bottlal graph $Kc(p, k)$ ($p \geq 6$) by insertion of h skew handles of K_6 and q skew cross-cap of K_5 ($h > 0$, $q \geq 0$) along $h+q$ paths on a component of the unique geodesic 2-factor in $Kc(p, k)$ so that any two of the skew handles or skew cross-caps have distance at least 2. Then G is 6-connected and the non-orientable genus of G is equal to $2h+q+2$. Furthermore, G is not uniquely or not faithfully embeddable in a non-orientable closed surface of genus $2h+q+2$, depending on the choice of $h+q$ paths.*

Proof. There are automorphisms of $Kc(p, k)$ which reflect each cycle of the geodesic 2-factor. If one choose $h+q$ paths so as to be invariant under such an automorphism σ of $Kc(p, k)$, then σ induces an automorphism of G which reflects the peripheral cycles of the skew handles and the faithfulness for G is denied by Corollary 4.6. Otherwise, the uniqueness for G breaks. The restriction of $p \geq 6$ is one for $Kc(p, k)$ to be 6-connected. $\square$

Theorm 6.4. *Let G be the graph obtained from the 6-regular Klein-bottlal graph $Kh(p, k)$ or $Kc(p, k)$ ($p \geq 6$) by insertion of h skew handles of K_6 and q skew cross-caps of K_5 ($h > 0$, $q \geq 0$) along $h+q$ geodesic paths none of which lies on the unique geodesic 2-factor so that any two of the skew handles or skew cross-caps have distance at least 2. Then G is 6-connected and the non-orientable genus of G is equal to $2h+q+2$. Furthermore, G is not uniquely embeddable in a non-orientable closed surface of genus $2h+q+2$.*

Proof. If G had an automorphism which reflects some peripheral cycles of the skew handles, then it would induce an automorphism of $Kh(p, k)$ or $Kc(p, k)$ which interchanges the geodesic 2-factor and other geodesic walks. This is contrary to the uniqueness of the geodesic 2-factor of $Kh(p, k)$ or $Kc(p, k)$. $\square$

The four theorems above have presented infinitely many graphs which are not uniquely embeddable and ones which are not faithfully embeddable in each hyperbolic closed surface F^2, orientable or non-orientable, with $\chi(F^2) \leq -2$. The only remaining case is when the surface is non-orientable and has genus 3. Since the insertion of the skew handle of K_6 increases the non-orientable genus of a graph by two, if we used the same logic as above for this case, we would have to insert one skew handle of K_6 to a 6-connected projective-planar triangulation. There is however no 6-connected projective-planar graph since it has a vertex of degree at most 5. Thus, we have to construct examples for a non-orientable closed surface of genus 3 in a way different from the other cases.

Let H_k^6 be the $(6, k)$-drum $(k \geq 1)$ with $k+1$ cycles $C_0, \cdots, C_k$ of length 6 and identify each u_{0j} with u_{0-j} on C_0. Let $(R_k, \partial R_k)$ denote the resulting graph with peripheral cycle $\partial R_k = C_k$. Then $(R_k, \partial R_k)$ can be triangularly embedded in a Möbius band and is uniquely and faithfully embeddable there since $(R_k, \partial R_k)$ has no skew vertex.

Sew $(R_k, \partial R_k)$ and $(K_6, \partial K_6)$ together along their peripheral cycles. Not depending on the choice of a sewing map, a unique graph G_k will be obtained up to isomorphism.

Theorem 6.5. *The graph G_k $(k \geq 1)$ is a 6-connected triangulation of a non-orientable closed surface of genus 3 and is not faithfully embeddable there.*

Proof. Since any automorphism of H_k^6 which reflects each C_i induces an automorphism of $(R_k, \partial R_k)$ which reflects ∂R_k. Thus, G_k satisfies the conditions of Corollary 4.6 and hence G_k is not faithfully embeddable. To see that G_k is 6-connected, check that there is no cycle of length 5 which bounds a 2-cell including at least one vertex. $\square$

Let x_i $(i=0, 1, 2, 3, 4, 5)$ be the six vertices lying on ∂R_k in this order and let N_k denote the graph obtained from R_k by identify x_0 with x_3 and adding four edges x_1x_4, x_1x_5, x_2x_4, x_2x_5. Then:

Theorrem 6.6. *The graph N_k $(k \geq 2)$ is 6-connedted with non-orientable genus 3, and is not uniquely embeddable in a non-orientable closed surface of genus 3.*

Proof. To check the connectivity of N_k is a routine work. Since N_k has $6k+2$ vertices and $18k+7$ edges, if it is embedded in a non-orientable closed surface of genus q with F faces, then

$$6k+2-(18k+7)+F=2-q$$

$$2(18k+7) \geq 3F.$$

From these, it follows that $q \geq \dfrac{7}{3}$ and hence the non-orientable genus of N_k is

equal to or more than 3. In fact, N_k can be embedded into a non-orientable closed surface of genus 3 in two ways below.

In the obvious way, R_k with x_0 and x_3 identified can be embedded in a projective plane triangularly so that two triangles $x_0 x_1 x_2$ and $x_0 x_4 x_5$ bound faces. To obtain a non-orientable closed surface of genus 3, join those faces by a tube. Fig. 16(a) and (b) show instruction for such tubing in two different ways; attach a tube to two circles in triangles so that the numbering along the circles is coherent.
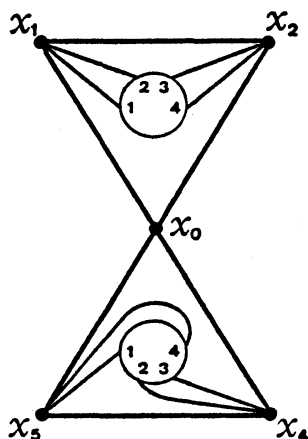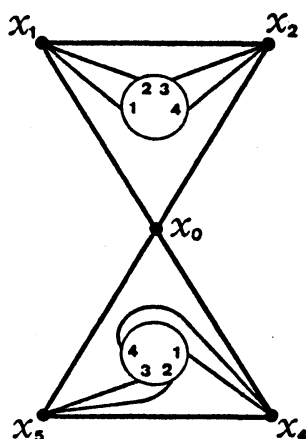

Fig. 16(a).   Fig. 16(b).

Suppose that the two embeddings (a) and (b) by tubing above were equivalent. Then there would be a homeomorphism between two non-orientable closed surfaces of genus 3 which carries N_k onto N_k. Necessarily, this homeomorphism would send the triangular part of one to that of the other and the cycle C_i onto C_i ($i=0, \cdots, k$). However, C_{k-1} separates the surface into a Möbius band and a punctured torus in (a) while into a Möbius band and a punctured Klein bottle in (b). Thus, it is impossible for such a homeomorphism to exist and N_k has at least two inequivalent embeddings in a non-orientable closed surface of genus 3. □

Let G be a 6-connected graph embedded in a non-orientable closed surface F^2 of genus 3, and suppose that G splits into H_1 and H_2 along a cycles C so that each vertex v on C has degree at least 4 in both H_1 and H_2. One of H_1 and H_2, say H_1, is embedded in a subsurface of F^2 which is homeomorphic to a Möbius band. Sew up two copies of the embedding of H_1 in the Möbius band by the identity map of C, then there will be obtained an embedding of $H_1 \cup H_1$ in a Klein bottle. By the choice of C, the minimum degree of $H_1 \cup H_1$ is equal to or more than 6. Since the mean value of degrees of vertices in a Klein-bottlal graph does not exceed 6, $H_1 \cup H_1$ is 6-regular Klein-bottle graph and C lies there as a geodesic cycle. We conclude, from the classification of 6-regular Klein-bottle

graphs, that $H_1 \cup H_1$ is equivalent to $Kc(p, k)$ for some p and k and hence that H_1 is isomorphic to R_k, defined above.

This phenomena causes the difficulty in construction of non-uniquely embeddable 6-connected triangulations of a non-orientable closed surface of genus 3. Since $(R_k, \partial R_k)$ has automorphisms which reflect ∂R_k, our criterions for the non-uniqueness of embedding in Section 4 cannot be used.

To construct 5-connected triangulations of non-orientable genus 3 whose embeddings are not unique or not faithful, insert one skew cross-cap of K_6 inside the hexagons of the toroidal or Klein-bottlal graphs, G_1 to G_6, presented in Section 3.

Proposition 6.7. *There are an infinite number of 5-connected triangulations which are not uniquely embeddable and ones which are not faithfully embeddable in a non-orientable closed surface of genus* 3. $\square$

By this proposition, the unknown answer " ? " in the fifth line of Table 2 is " 7 " or " 6 ". Are there infinitely many 6-connected triangulations not uniquely embeddable in a non-orientable closed surface of genus 3 ?

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