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THE ARENS PRODUCT AND QUASI-MULTIPLIERS

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ABSTRACT. Let A be a Banach algebra satisfying conditions weaker than having a two-sided bounded approximate identity. Let $QM(A)$ denote the Banach space of all quasi-multipliers on A. We construct a Banach space B_0 whose conjugate space B_0^* is a left B_1^*-module and right B_2^*-module under modified Arens product. It is shown that for Banach algebras satisfying this weaker condition, $QM(A)$ can be embedded isometrically isomorphically into B_0^*. We also show that this embedding of $QM(A)$ in B_0^* is an extension of the embedding of $M_r(A)$ ($M_l(A)$) the algebra of all right multipliers (algebra of left multipliers) on A in $B_1^*(B_2^*)$ [6]. It is also shown that if A is a dual A^*-algebra of the first kind, then $QM(A)$ is isometrically isomorphic to B_0^*.

1. Introduction. Let A be a Banach algebra. McKennon [7] studied quasi-multipliers of Banach algebras with two-sided bounded approximate identity and suggested the question of embedding $QM(A)$ in the second conjugate space A^{**} of A. In [14] for Banach algebras with two-sided bounded approximate identity an embedding of $QM(A)$ in A^{**} is obtained. A similar problem for multipliers was considered by Máté [6] for Banach algebras satisfying conditions weaker than a weak right identity. In Máté's paper [6] a Banach space B_1 is constructed whose conjugate space B_1^* is a Banach algebra under modified Arens product and it is shown that $M_r(A)$, the algebra of all right multipliers, is embedded isometrically isomorphically into B_1^*. In this paper a Banach space B_0 is constructed whose conjugate space B_0^* is a left B_1^*-module and right B_2^*-module under modified Arens product. For Banach algebras satisfying conditions (α), (β) and (γ) which are stated in §2, it is shown that $QM(A)$ is embedded isometrically isomorphically into B_0^*. We also show that this embedding is an extension of the embedding of $M_r(A)$ ($M_l(A)$) in B_1^* (B_2^*) due to Máté [6]. We also show that if A is a dual A^*-algebra of the first kind then $QM(A)$ can be

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identified with B_0^*. § 5 deals with the Banach algebras with two-sided bounded approximate identity in which we prove that B_0^* is isometric and isomorphic with B^* and we give a sharper form of the main result of [14].

2. Definitions and Preliminaries. For any Banach algebra A McKennon [7] defined a quasi-multiplier as a bilinear continuous mapping $m: A\times A\to A$, satisfying

(1) $$m(ab, cd)=am(b, c)d, \quad \forall a, b, c, d\in A.$$

We define a bilinear continuous mapping $m: A\times A\to A$, as a quasi-multiplier if

(2) $$m(ab, c)=am(b, c) \quad \text{and} \quad m(a, bc)=m(a, b)c, \quad \forall a, b, c\in A.$$

If A is a Banach algebra with two-sided bounded approximate identity then obviously both definitions are equivalent [7]. It is shown in this paper that if A satisfies conditions weaker than two-sided bounded approximate identity then the two definitions coincide (See Remark 5.2). Throughout the paper we adopt the definition of quasi-multiplier as given by (2). Let $QM(A)$ denote the set of all quasi-multipliers on A. $QM(A)$ is a Banach space under usual operations for operators and operator bound norm. $QM(A)$ is also a two-sided Banach A-module under the bilinear operations, $\circ: A\times QM(A)\to QM(A)$ and $\circ': QM(A)\times A\to QM(A)$ as defined below:

For $m\in QM(A)$ and $x, y, a\in A$

$$(a\circ m)(x, y)=m(xa, y) \quad \text{and} \quad (m\circ' a)(x, y)=m(x, ay).$$

Following Máté [6] we call a bounded, linear operator T mapping A into itself, a right (left) multiplier if $T(xy)=xT(y)$ $((Tx)y)$ for all $x, y\in A$. The set $M_r(A)$ $(M_l(A))$ of all right (left) multipliers on A is a Banach algebra.

Arens [1] has defined two products on A^{**} which makes A^{**} into a Banach algebra. We denote them by $\circ_1$ and $\circ_2$ and for completeness we sketch them here. These are done in stages as follows: For $x, y\in A$, $f\in A^*$ and $F, G\in A^{**}$,

define $f\circ_1 x\in A^*$ by $(f\circ_1 x)(y)=f(xy)$,
define $F\circ_1 f\in A^*$ by $(F\circ_1 f)(x)=F(f\circ_1 x)$,
define $F\circ_1 G\in A^{**}$ by $(F\circ_1 G)(f)=F(G\circ_1 f)$,
define $x\circ_2 f\in A^*$ by $(x\circ_2 f)(y)=f(yx)$,
define $f\circ_2 F\in A^*$ by $(f\circ_2 F)(x)=F(x\circ_2 f)$,
define $F\circ_2 G\in A^{**}$ by $(F\circ_2 G)(f)=G(f\circ_2 F)$.

A^{**} is a Banach algebra under the product $\circ_1$ as well as under $\circ_2$ such that the canonical mapping π of A into A^{**} is an isomorphism of A into $(A^{**}, \circ_1)$ and $(A^{**}, \circ_2)$.

Let Y be the linear hull of the set $\{f\circ_1 x: f\in A^* \text{ and } x\in A\}$ and let B_1 be the set

$$\{h=\sum_{k=1}^{\infty} h_k \circ_1 a_k : \sum_{k=1}^{\infty} \|h_k\| \|a_k\| < \infty,\ h_k \in A^*,\ a_k \in A\}.$$

By [6, Theorem 1, p. 229], B_1 is a Banach space under the norm $\| \|_1$ defined by

$$\|h\|_1 = \inf\{\sum_{k=1}^{\infty} \|h_k\| \|a_k\| : h=\sum_{k=1}^{\infty} h_k \circ_1 a_k\}$$

and $\|h\| \leqq \|h\|_1$, $\forall h \in B_1$. If B_2 is the set

$$\{h=\sum_{k=1}^{\infty} a_k \circ_2 h_k : \sum_{k=1}^{\infty} \|a_k\| \|h_k\| < \infty,\ h_k \in A^*,\ a_k \in A\},$$

then B_2 is also a Banach space under the norm $\| \|_2$ defined by

$$\|h\|_2 = \inf\{\sum_{k=1}^{\infty} \|a_k\| \|h_k\| : h=\sum_{k=1}^{\infty} a_k \circ_2 h_k\}$$

and

$$\|h\| \leqq \|h\|_2, \quad \forall h \in B_2.$$

Máté [6] has introduced a product in B_1^* (called the modified Arens product) which makes B_1^* into a Banach algebra. This product is defined as follows: For x, $y \in A$, $h \in B_1$ and F, $G \in B_1^*$,

define $h \circ_1 x \in B_1$ by $(h \circ_1 x)(y) = h(xy)$,
define $F \circ_1 h \in B_1$ by $(F \circ_1 h)(x) = F(h \circ_1 x)$,
define $F \circ_1 G \in B_1^*$ by $(F \circ_1 G)(h) = F(G \circ_1 h)$.

Then $(B_1^*, \circ_1)$ is a Banach algebra. Let $\rho_1(a) = \pi(a)|_{B_1} : B_1 \to \boldsymbol{C}$, $\boldsymbol{C}$ being the field of complex numbers, such that $\rho_1(a)(h) = h(a)$, $\forall h \in B_1$, then the mapping $a \to \rho_1(a)$ is an algebraic homomorphism of A into $(B_1^*, \circ_1)$, the kernel of the homomorphism being equal to $\{a \in A : Aa = (0)\}$ [6].

Similarly B_2^* is a Banach algebra under modified Arens product $F \circ_2 G \in B_2^*$ such that $(F \circ_2 G)(f) = G(f \circ_2 F)$, $\forall F$, $G \in B_2^*$, $\forall f \in B_2$. Define $\rho_2(a) = \pi(a)|_{B_2} : B_2 \to \boldsymbol{C}$ such that $\rho_2(a)(h) = h(a)$, $\forall h \in B_2$. Then the mapping $a \to \rho_2(a)$ is also an algebraic homomorphism of A into $(B_2^*, \circ_2)$, the kernel of the homomorphism being equal to $\{a \in A : aA = (0)\}$.

In § 4 we will consider Banach algebras with the following conditions:

Condition (α): $h_k \in A^*$, $a_k \in A$, $\sum_{k=1}^{\infty} \|h_k\| \|a_k\| < \infty$ and $\sum_{k=1}^{\infty} (h_k \circ_1 a_k) = 0$ implies that $\sum_{k=1}^{\infty} h_k(a_k) = 0$,

Condition (β): $h_k \in A^*$, $a_k \in A$, $\sum_{k=1}^{\infty} \|h_k\| \|a_k\| < \infty$ and $\sum_{k=1}^{\infty} a_k \circ_2 h_k = 0$ implies that $\sum_{k=1}^{\infty} h_k(a_k) = 0$,

Condition (γ): A is without a right and a left annihilator $\Leftrightarrow AxA = (0) \Rightarrow x = 0$.

These conditions together are weaker than the condition of having a two-sided bounded approximate identity. In fact if there is a two-sided bounded approximate identity then conditions (α), (β), (γ) are satisfied [6]. Máté [6] has also proved that if A has the condition (α) then $(B_1^*, \circ_1)$ has a right identity I_1 such that $I_1\left(\sum_{k=1}^{\infty} h_k \circ_1 a_k\right) = \sum_{k=1}^{\infty} h_k(a_k)$.

Similarly it can be shown that if A has the condition (β) then $(B_2^*, \circ_2)$ has a left identity I_2 such that $I_2\left(\sum_{k=1}^{\infty} a_k \circ_2 h_k\right) = \sum_{k=1}^{\infty} h_k(a_k)$.

Let $(A, \|\ \|, |\ |)$ be an A^*-algebra where $\|\ \|$ is a Banach algebra norm, and $|\ |$ is the auxiliary norm. The Banach algebra $A=(A, \|\ \|)$ is of the first kind if A is a two-sided ideal of its C^*-algebra completion $\mathfrak{a}=(\mathfrak{a}, |\ |)$. For further details on A^*-algebras see [9].

Lemma. *Let A be a dual A^*-algebra of the first kind. Then A has the properties* (α), (β), (γ).

Proof. See [12, Lemma 3.1, p. 283].

3. If $\pi(A)$ is the canonical embedding of A into A^{**} and $F \in A^{**}$ is such that $\pi(a) \circ_1 F \circ_2 \pi(b) \in \pi(A)$ for every $a, b \in A$, then the mapping $m: A \times A \to A$ defined by

$$\pi(m(a, b)) = \pi(a) \circ_1 F \circ_2 \pi(b), \quad a, b \in A \tag{3}$$

is a quasi-multiplier.

Deffinition 3.1. B is the linear hull of the set $\{a \circ_2 h \circ_1 b : h \in A^*,\ a, b \in A\}$ and $B^{\perp} = \{F \in A^{**} : F(a \circ_2 h \circ_1 b) = 0,\ \forall h \in A^*,\ a, b \in A\}$ is the orthogonal complement of B in A^{**}.

Lemma 3.1. *Let $F \in A^{**}$ be such that $\pi(a) \circ_1 F \circ_2 \pi(b) \in \pi(A)$ for every $a, b \in A$ and let m be the quasi-multiplier determined by F. Then $m=0 \Leftrightarrow F \in B^{\perp}$.*

Proof.
$$\begin{aligned} F \in B^{\perp} &\Leftrightarrow F(b \circ_2 h \circ_1 a) = 0\ \forall a, b \in A,\ \forall h \in A^* \\ &\Leftrightarrow [\pi(a) \circ_1 F \circ_2 \pi(b)](h) = 0\ \forall a, b \in A,\ \forall h \in A^* \\ &\Leftrightarrow \pi(m(a, b))(h) = 0\ \forall a,\ b \in A,\ \forall h \in A^* \\ &\Leftrightarrow \pi(m(a, b)) = 0\ \forall a, b \in A \\ &\Leftrightarrow m = 0. \end{aligned}$$

Lemma 3.2. *In order for A to satisfy the condition (γ), it is necessary and sufficient that $\pi(A) \cap B^{\perp} = \{0\}$.*

Proof. Proof is straightforward.

Definition 3.2. Let B be the set

$$\left\{h = \sum_{k=1}^{\infty} a_k \circ_2 h_k \circ_1 b_k : \sum_{k=1}^{\infty} \|a_k\| \|h_k\| \|b_k\| < \infty,\ h_k \in A^*,\ a_k, b_k \in A\right\}. \tag{$*$}$$

Then B_0 becomes a linear subset of A^*.

Theorem 3.1. *B_0 is a Banach space with the norm $\| \|'$ given by*

$$\|h\|'=\inf\{\sum_1^{\infty}\|a_k\|\|h_k\|\|b_k\| : h=\sum_{k=1}^{\infty}a_k\circ_2 h_k\circ_1 b_k\}.$$

Moreover, $\|h\|\leqq\|h\|'$ for each $h\in B_0$.

Proof. It is obvious that $\| \|'$ is a norm. If $\{h_n\}$ $(h_n\in B_0)$ is a Cauchy sequence, then there is a subsequence $\{h_k\}$ such that

$$\|h_k-h_{k-1}\|'<(1/2^k)\forall k.$$

Hence if $h=h_1+\sum_{k=2}^{\infty}(h_k-h_{k-1})$, then $\|h\|'<\infty$, h is in form $(*)$ and considering

$$h-h_{n+1}=h-h_1-(\sum_{i=1}^{n}h_{i+1}-h_i)=\sum_{i=n+1}^{\infty}(h_{i+1}-h_i),$$

we get, $\|h-h_{n+1}\|'\leqq\sum_{i=n+1}^{\infty}\|h_{i+1}-h_i\|\leqq\sum_{i=n+1}^{\infty}1/2^i$. Hence $\lim h_n=h$.

Now it is obvious that $h=\sum_{k=1}^{\infty}a_k\circ_2 h_k\circ_1 b_k\in A^*$ by definitions of o_1 and o_2. Further

$$\begin{aligned}\|h\|&=\|\sum_{k=1}^{\infty}a_k\circ_2 h_k\circ_1 b_k\|\\&=\|\lim_{n\to\infty}\sum_{k=1}^{n}a_k\circ_2 h_k\circ_1 b_k\|\\&=\lim_{n\to\infty}\|\sum_{k=1}^{n}a_k\circ_2 h_k\circ_1 b_k\|\\&\leqq\lim_{n\to\infty}\sum_1^{n}\|a_k\|\|h_k\|\|b_k\|\\&=\sum_1^{\infty}\|a_k\|\|h_k\|\|b_k\|.\end{aligned}$$

Then $\|h\|\leqq\sum_1^{\infty}\|a_k\|\|h_k\|\|b_k\|$ for every such representation of h. Therefore

$$\|h\|\leqq\inf\{\sum_1^{\infty}\|a_k\|\|h_k\|\|b_k\|=\|h\|' : h=\sum_1^{\infty}a_k\circ_2 h_k\circ_1 b_k\}.$$

Then $\|h\|\leqq\|h\|'$.

Remark. It is obvious that B supplied with the norm $\| \|'$ is a dense subset of B_0.

Let us consider the dual space B_0^* of B_0. Máté [6] has defined a modified Arens product o_1 with the help of which we are going to show that B_0^* is a left B_1^*-module and similarly it can be shown that B_0^* is a right B_2^*-module under modified Arens product $\circ_2$.

Theorem 3.2. *If A is a Banach algebra then B_0^* is a left B_1^*-module under modified Arens product $\circ_1$.*

Proof. The proof depends on the following which can easily be verified:

(i) $h\circ_1 a\in B_0$ and $\|h\circ_1 a\|'\leqq\|a\|\|h\|'$

(ii) $F\circ_1 h\in B_1$ and $\|F\circ_1 h\|_1\leqq\|F\|\|h\|'$ for all $a\in A$, $h\in B_0$ and $F\in B_0^*$.

From (i) and (ii) it follows that for $F\in B_1^*$ and $G\in B_0^*$, $F\circ_1 G\in B_0^*$ and $\|F\circ_1 G\|\leqq\|F\|\|G\|$. Hence it is shown that o_1 is a module multiplication and thus B_0^* is a left B_1^*-module.

Definition 3.3. Define a mapping $\rho: A\to B_0^*$ such that $\rho(a)=\pi(a)|_{B_0}: B_0\to \boldsymbol{C}$ and $\rho(a)(h)=h(a)$, $\forall a\in A$ and $\forall h\in B_0$.

Theorem 3.3. *Let A be a Banach algebra satisfying condition (γ) then ρ is a continuous, linear module-isomorphism of A into B_0^*.*

Proof. That ρ is continuous, linear and one-to-one is obvious. ρ is also a module-homomorphism for which we prove the following:

(i) $\rho(ab)=\rho_1(a)\circ_1\rho(b)$, for all $a, b\in A$.

(ii) $\rho(ab)=\rho(a)\circ_2\rho_2(b)$, for all $a, b\in A$.

For $h\in B_0$, we have that

$$\begin{aligned}[\rho_1(a)\circ_1\rho(b)](h)&=\rho_1(a)[\rho(b)\circ_1 h]\\ &=[\rho(b)\circ_1 h](a)\\ &=(h\circ_1 a)(b)\\ &=h(ab)\\ &=\rho(ab)(h),\end{aligned}$$

which proves (i). Similarly (ii) can be proved.

4. Embedding of $QM(A)$ in B_0^*.

In this section we prove the main result of this paper that is, if A is a Banach algebra satisfying conditions (α), (β) and (γ), then $QM(A)$ can be embedded isometrically isomorphically into B_0^*. Before coming to this result, first we establish some of the results needed in its proof.

Theorem 4.1. *If m is a quasi-multiplier on a Banach algebra A, then there exists a continuous bilinear mapping $S_1: B_1^*\times B_2^*\to B_0^*$ satisfying*

$$S_1(\rho_1(a)\circ_1 F,\ G\circ_2\rho_2(b))=\rho_1(a)\circ_1 S_1(F, G)\circ_2\rho_2(b)$$

for all $a, b\in A$, $F\in B_1^*$ and $G\in B_2^*$. Further S_1 is an extension of m in the following sense

$$S_1(\rho_1(a),\ \rho_2(b))=\rho(m(a, b)),\quad a, b\in A\,.$$

Proof. Let $m: A\times A\to A$ be a quasi-multiplier on A. Then $m^*: A^*\times A\to A^*$

is a continuous, bilinear mapping. Define $S=m^*|_{B_0\times A}:(B_0, \|\ \|')\times A\rightarrow(B_2, \|\ \|_2)$. Let us prove that S is a well defined continuous bilinear mapping.

Let $h\in B_0$, then $h=\sum_1^\infty a_k\circ_2 h_k\circ_1 b_k$, $\forall a_k, b_k\in A$ and $\forall h_k\in A^*$. For $a, x\in A$, consider

$$\begin{aligned}S(h, a)(x)&=S(\sum_1^\infty a_k\circ_2 h_k\circ_1 b_k, a)(x)\\&=\sum_1^\infty[(S(a_k\circ_2 h_k\circ_1 b_k), a)(x)]\\&=\sum_1^\infty[(a_k\circ_2 h_k\circ_1 b_k)m(a, x)]\\&=\sum_1^\infty[(h_k\circ_1 b_k)(m(a, x)a_k)]\\&=\sum_1^\infty[(h_k\circ_1 b_k)m(a, xa_k)]\\&=\sum_1^\infty[m^*(h_k\circ_1 b_k a)(xa_k)]\\&=\sum_1^\infty[a_k\circ_1 m^*(h_k\circ_1 b_k, a)](x)\\&=[\sum_1^\infty a_k\circ_2 m^*(h_k\circ_1 b_k, a)](x)\,.\end{aligned}$$

Then $S(h, a)=\sum_1^\infty a_k\circ_2 m^*(h_k\circ_1 b_k, a)$. And

$$\begin{aligned}\sum_1^\infty\|a_k\|\|m^*(h_k\circ_1 b_k, a)\|&\leqq\sum_1^\infty\|a_k\|\|m^*\|\|h_k\circ_1 b_k\|\|a\|\\&\leqq\|a\|\|m^*\|\sum_1^\infty\|a_k\|\|h_k\|\|b_k\|<\infty\,,\end{aligned}$$

then $S(h, a)\in B_2$. Therefore we have

$$\begin{aligned}\|S(h, a)\|_2&=\inf\{\sum_1^\infty\|c_k\|\|\phi_k\|:S(h, a)=\sum_1^\infty c_k\circ_2\phi_k,\ c_k\in A,\ \phi_k\in A^*\}\\&\leqq\inf\{\sum_1^\infty\|a_k\|\|m^*(h_k\circ_1 b_k, a)\|:h=\sum_1^\infty a_k\circ_2 h_k\circ_1 b_k\}\\&\leqq\inf\{\sum_1^\infty\|a_k\|\|m^*\|\|h_k\|\|b_k\|\|a\|:h=\sum_1^\infty a_k\circ_2 h_k\circ_1 b_k\}\\&=\|a\|\|m^*\|\inf\{\sum_1^\infty\|a_k\|\|h_k\|\|b_k\|:h=\sum_1^\infty a_k\circ_2 h_k\circ_1 b_k\}\\&=\|a\|\|m^*\|\|h\|'.\end{aligned}$$

Hence $\|S(h, a)\|_2\leqq\|a\|\|m^*\|\|h\|'$ implying that S is a continuous mapping.

Now $S^*:B_2^*\times(B_0, \|\ \|')\rightarrow A^*$ is a continuous bilinear mapping. Also, it can easily be shown that

$$S^*: B_2^* \times (B_0, \| \|') \to (B_1, \| \|_1)$$

is a well defined continuous mapping.

Therefore $S^{**}: B_1^* \times B_2^* \to B_0^*$ is obviously a continuous, bilinear mapping. Put $S_1 = S^{**}$, then we easily have

$$S_1(\rho_1(a) \circ_1 F,\ G \circ_2 \rho_2(b)) = \rho_1(a) \circ_1 S_1(F, G) \circ_2 \rho_2(b),$$

where $F \in B_1^*$, $G \in B_2^*$, in view of the following whose proofs are straightforward:

(i) $S(b \circ_2 h, a) = b \circ_2 S(h, a)$

(ii) $S^*(G \circ_2 \rho_2(b), h) = S^*(G, b \circ_2 h)$

(iii) $S^*(G, h) \circ_1 a = S^*(G, h \circ_1 a)$

(iv) $S^{**}(F, G \circ_2 \rho_2(b)) = S^{**}(F, G) \circ_2 \rho_2(b)$

(v) $S^{**}(\rho_1(a) \circ_1 F, G) = \rho_1(a) \circ_1 S^{**}(F, G)$

for all $a, b \in A$, $h \in B_0$, $F \in B_1^*$ and $G \in B_2^*$. Also for any $h \in B_0$,

$$\begin{aligned} S_1(\rho_1(a), \rho_2(b))(h) &= S^{**}(\rho_1(a), \rho_2(b))(h) \\ &= S(h, a)(b) = h(m(a, b)) \\ &= \rho(m(a, b))(h), \end{aligned}$$

so that $S_1(\rho_1(a), \rho_2(b)) = \rho(m(a, b))$. Hence S_1 is an extension of m.

Theorem 4.2. *Let A be a Banach algebra satisfying conditions (α) and (β). Then given $m \in QM(A)$, there exists F^m in B_0^* such that*

$$\rho(m(a, b)) = \rho_1(a) \circ_1 F^m \circ_2 \rho_2(b) \in \rho(A) \quad \text{for all} \quad a, b \in A.$$

Proof. Given an $m \in QM(A)$, there exists an extension S_1 of m such that

$$\rho(m(a, b)) = S_1(\rho_1(a), \rho_2(b)) \quad \text{(by Theorem 4.1).}$$

Let I_1 be the right identity for B_1^* and I_2 the left identity of B_2^*, then

$$\begin{aligned} S_1(\rho_1(a), \rho_2(b)) &= S_1(\rho_1(a) \circ_1 I_1, I_2 \circ_2 \rho_2(b)) \\ &= \rho_1(a) \circ_1 S_1(I_1, I_2) \circ_2 \rho_2(b) \quad \text{(by Theorem 4.1).} \end{aligned}$$

Hence $\rho(m(a, b)) = \rho_1(a) \circ_1 S_1(I_1, I_2) \circ_2 \rho_2(b)$. Let $F^m = S_1(I_1, I_2) \in B_0^*$. Therefore $\rho(m(a, b)) = \rho_1(a) \circ_1 F^m \circ_2 \rho_2(b)$. Also $\rho(m(a, b)) \in \rho(A)$. Finally we have that there exists $F^m \in B_0^*$ such that

$$\rho(m(a, b)) = \rho_1(a) \circ_1 F^m \circ_2 \rho_2(b) \in \rho(A).$$

Theorem 4.3. *Let A be a Banach algebra satisfying conditions (α), (β) and (γ), then there exists an isometric isomorphism of $QM(A)$ onto the sub-module*

$$E = \{F \in B_0^* : \rho_1(a) \circ_1 F \circ_2 \rho_2(b) \in \rho(A) \quad \text{for all} \ a, b \in A\}$$

of B_0^*.

Proof. E is a linear subspace of B_0^* follows from the fact that $\circ_1$ and $\circ_2$ are bilinear mappings and ρ is a linear mapping. We define $a\circ_1 F=\rho_1(a)\circ_1 F$ and $F\circ_2 a=F\circ_2\rho_2(a)$ for each $a\in A$ and $F\in B_0^*$. Then E is a two-sided A-module. In fact for $F\in E$, $a, b, a_1\in A$,

$$\begin{aligned}\rho_1(a)\circ_1(a_1\circ_1 F)\circ_2\rho_2(b)&=\rho_1(a)\circ_1(\rho_1(a_1)\circ_1 F)\circ_2\rho_2(b)\\&=\rho_1(aa_1)\circ_1 F\circ_2\rho_2(b)\in\rho(A),\end{aligned}$$

so that $a_1\circ_1 F\in E$ and also $F\circ_2 a_1\in E$ similarly.

Define $\sigma: QM(A)\to E$, by $\sigma(m)=S_1(I_1, I_2)$, for $m\in QM(A)$. Then σ is a one-to-one mapping because of Theorems 3.3 and 4.2. Also it is easy to prove that σ is linear. Let us prove that σ is an onto mapping.

σ is an onto mapping: Let $F\in E$. Then we have $\rho_1(a)\circ_1 F\circ_2\rho_2(b)\in\rho(A)$ for all $a, b\in A$. Define $m: A\times A\to A$, by $\rho(m(a, b))=\rho_1(a)\circ_1 F\circ_2\rho_2(b)$ for every $a, b\in A$. Then it can easily be checked that m is a quasi-multiplier. By Theorem 4.2, $\rho(m(a, b))=\rho_1(a)\circ_1\sigma(m)\circ_2\rho_2(b)$ for all $a, b\in A$. We now show $\sigma(m)=F$. But to do this, we have to prepare some lemmas.

In the following, we keep the notations: m, F and let S, S_1 be as defined in Theorem 4.1.

Lemma 4.1. *Let $a, b\in A$, $f\in A^*$ and $G\in B_2^*$. Then $S^*(G, a\circ_2 f\circ_1 b)=m^{**}(G\times a, f)\circ_1 b$, where $G\times a$ is an element of A^{**} such that $(G\times a)(g)=G(a\circ_2 g)$ for all $g\in A^*$.*

Proof. It is easily checked that $G\times a$ is well-defined and $G\times a\in A^{**}$. Note also that for each $x\in A$, $S(a\circ_2 f\circ_1 b, x)=a\circ_2 m^*(f, bx)$. Therefore we have

$$\begin{aligned}S^*(G, a\circ_2 f\circ_1 b)(x)&=G(S(a\circ_2 f\circ_1 b, x))\\&=G(a\circ_2 m^*(f, bx))\\&=(G\times a)(m^*(f, bx))\\&=m^{**}(G\times a, f)(bx)\\&=(m^{**}(G\times a, f)\circ_1 b)(x)\end{aligned}$$

for all $x\in A$ and the proof is complete.

Lemma 4.2. *Let $a, b\in A$ and $f\in A^*$. Then $I_1(S^*(I_2, a\circ_2 f\circ_1 b))=m^*(f, b)(a)$.*

Proof. By Lemma 4.1 and the definitions of $\circ_1$ and $\circ_2$, we have

$$\begin{aligned}I_1(S^*(I_2, a\circ_2 f\circ_1 b))&=I_1(m^{**}(I_2\times a, f)\circ_1 b)\\&=m^{**}(I_2\times a, f)(b)\\&=(I_2\times a)(m^*(f, b))\\&=I_2(a\circ_2 m^*(f, b))\\&=m^*(f, b)(a).\end{aligned}$$

Lemma 4.3. *Let $A^2=\{ab: a, b\in A\}$. If either A satisfies the condition (α) or (β), then the linear hull of A^2 is dense in A.*

Proof. Suppose that A satisfies (α). If $f\in A^*$ is such that $f(ab)=0$ for all $a, b\in A$, then $f\circ_1 a=0$ for all $a\in A$. So (α) implies that $f(a)=0$ for all $a\in A$, that is $f=0$. Then it follows from the Hahn-Banach separation theorem that the linear hull of A^2 is dense in A. Similarly for the case of (β).

Lemma 4.4. *For $a, b, x, y\in A$ and $f\in A^*$, $F((xy)\circ_2 f\circ_1(ab))=f(m(ab, xy))$.*

Proof. It can easily be checked that $(\rho_1(c)\circ_1 F\circ_2\rho_2(d))(g)=F(d\circ_2 g\circ_1 c)$ for each $c, d\in A$ and $g\in B_0$. Therefore

$$\begin{aligned} F((xy)\circ_2 f\circ_1(ab)) &= F(x\circ_2(y\circ_2 f\circ_1 a)\circ_1 b) \\ &= (\rho_1(b)\circ_1 F\circ_2\rho_2(x))(y\circ_2 f\circ_1 a) \\ &= \rho(m(b, x))(y\circ_2 f\circ_1 a) \\ &= (y\circ_2 f\circ_1 a)(m(b, x)) \\ &= f(m(ab, xy)). \end{aligned}$$

Lemma 4.5. *For $a, b\in A$ and $f\in A^*$, $f(m(a, b))=F(b\circ_2 f\circ_1 a)$.*

Proof. Define $\phi: A\times A\to \boldsymbol{C}$ by $\phi(a, b)=f(m(a, b))$ and define $\psi: A\times A\to \boldsymbol{C}$ by $\psi(a, b)=F(b\circ_2 f\circ_1 a)$. Then ϕ and ψ are bilinear continuous mappings. Moreover by Lemma 4.4, $\phi|_{A^2\times A^2}=\psi|_{A^2\times A^2}$ and hence $\phi=\psi$ because the linear hull of A^2 is dense in A from Lemma 4.3, and the proof is complete.

As some of the lemmas needed in the proof has been prepared, we shall continue the original proof.

Let $h\in B_0$, then $h=\sum_1^\infty a_k\circ_2 h_k\circ_1 b_k$ for $a_k, b_k\in A$ and $h_k\in A^*$. Of course $\lim_{n\to\infty}\|h-\sum_1^n a_k\circ_2 h_k\circ_1 b_k\|=0$. Therefore

$$\begin{aligned} \sigma(m)(h) &= \lim_{n\to\infty}(S_1(I_1, I_2))\Big(\sum_1^n a_k\circ_2 h_k\circ_1 b_k\Big) \\ &= \sum_1^\infty (S_1(I_1, I_2))(a_k\circ_2 h_k\circ_1 b_k) \\ &= \sum_1^\infty I_1(S^*(I_2, a_k\circ_2 h_k\circ_1 b_k)) \\ &= \sum_1^\infty m^*(h_k, b_k)(a_k) && \text{(by Lemma 4.2)} \\ &= \sum_1^\infty h_k(m(b_k, a_k)) \\ &= \sum_1^\infty F(a_k\circ_2 h_k\circ_1 b_k) && \text{(by Lemma 4.5)} \\ &= F(h), \end{aligned}$$

which shows that $F=\sigma(m)$. Thus it has been shown that σ is an onto mapping.

σ is a module-homomorphism: For $m\in QM(A)$ and $a\in A$, $a\circ m\in QM(A)$. Let $\sigma(a\circ m)=S^{**}(I_1, I_2)$ and $\sigma(m)=T^{**}(I_1, I_2)$. For $h\in B_0$,

$$\begin{aligned}\sigma(a\circ m)(h) &= S^{**}(I_1, I_2)[\sum_1^\infty a_k\circ_2 h_k\circ_1 b_k]\\ &= \sum_1^\infty h_k[(a\circ m)(b_k, a_k)]\\ &= \sum_1^\infty (h_k\circ_1 b_k)m(a, a_k)\\ &= \sum_1^\infty m^*(h_k\circ_1 b_k, a)(a_k)\\ &= I_2(\sum_1^\infty [a_k\circ_2 m^*(h_k\circ_1 b_k, a)])\\ &= \sum_1^\infty T^*(I_2, a_k\circ_2 h_k\circ_1 b_k)(a)\\ &= I_1(\sum_1^\infty [T^*(I_2, a_k\circ_2 h_k\circ_1 b_k)\circ_1 a])\\ &= T^{**}(I_1, I_2)(\sum_1^\infty a_k\circ_2 h_k\circ_1 b_k a)\\ &= [a\circ_1\sigma(m)](h)\,.\end{aligned}$$

Hence $\sigma(a\circ m)=a\circ_1\sigma(m)$ which proves that σ is a left A-module homomorphism. Similarly, we can show that σ is a right A-module homomorphism.

σ is an isometry: Let $m\in QM(A)$ and $\sigma(m)=S^{**}(I_1, I_2)$, then for $h\in B_0$ we have

$$\begin{aligned}|\sigma(m)(h)| &= |\sigma(m)(\sum_1^\infty a_k\circ_2 h_k\circ_1 b_k)|\\ &= |S^{**}(I_1, I_2)(\sum_1^\infty a_k\circ_2 h_k\circ_1 b_k)|\\ &= |\sum_1^\infty h_k[m(b_k, a_k)]|\\ &\leqq \sum_1^\infty |h_k[m(b_k, a_k)]|\\ &\leqq \|m\|(\sum_1^\infty \|a_k\|\|h_k\|\|b_k\|)\,.\end{aligned}$$

Since it is true for every such representation of h, therefore

$$|\sigma(m)(h)|\leqq \inf\|m\|(\sum_1^\infty \|a_k\|\|h_k\|\|b_k\|)=\|m\|\|h\|',$$

then $\|\sigma(m)\|\leqq\|m\|$.

Further since $S^{**}(I_1, I_2)(b\circ_2 h\circ_1 a)=h[m(a, b)]$ as can be observed in the proof of the surjectivity of σ, with the help of it one can show that

$$\|m\|\leqq\|\sigma(m)\| .$$

Thus σ is an isometric isomorphism of $QM(A)$ onto the submodule E of B_0^*.

We now prove that this embedding σ is an extension of the following embeddings due to Máté.

Máté [6] has proved that if A is a Banach algebra without a right annihilator and satisfying condition (α) then $M_r(A)$ can be embedded into B_1^* by the mapping $\sigma_1: M_r(A)\to B_1^*$ such that for each $T\in M_r(A)$, $\sigma_1(T)=(T^*|_{B_1})^*(I_1)$, where I_1 is the right identity for B_1^*.

Similarly it can be shown that when A is a Banach algebra without a left annihilator and satisfying condition (β) then $M_l(A)$ can be embedded into B_2^* by the mapping $\sigma_2: M_l(A)\to B_2^*$ such that for each $T\in M_l(A)$, $\sigma_2(T)=(T^*|_{B_2})^*(I_2)$, where I_2 is the left identity for B_2^*.

Let us now define the mappings $\lambda: M_l(A)\to QM(A)$, and $\mu: M_r(A)\to QM(A)$ by

$$\lambda(T)(x, y)=xTy \quad \text{and} \quad \mu(T)(x, y)=(Tx)y$$

for all $x, y\in A$.

Theorem 4.4. *If A is a Banach algebra without a right annihilator, then λ is a linear, one-to-one and continuous mapping.*

Proof. That λ is linear is evident. Let T be in $M_l(A)$. Then

$$\begin{aligned}\|\lambda(T)\|&=\sup\{\|xTy\|:\|x\|=\|y\|=1\}\\&=\sup\{\|Ty\|:\|y\|=1\}=\|T\|,\end{aligned}$$

which shows that λ is continuous. Next let $T_1, T_2\in M_l(A)$ such that $\lambda(T_1)=\lambda(T_2)$. Then $xT_1y=xT_2y$ for all $x, y\in A$, so that $T_1=T_2$ because T_1y-T_2y is a right annihilator of A for each $y\in A$.

Similarly it can be shown that when A is a Banach algebra without a left annihilator then μ is a linear one-to-one continuous mapping. Hence $M_l(A)$ and $M_r(A)$ can be identified algebraically with the subspace $\lambda(M_l(A))$ and $\mu(M_r(A))$ of $QM(A)$, respectively.

Theorem 4.5. *Let A be a Banach algebra satisfying conditions (α), (β) and (γ). Let σ, σ_1, σ_2 and λ, μ are as defined above, then $\sigma(\mu(T))=\sigma_1(T)|_{B_0}$ and $\sigma(\lambda(T))=\sigma_2(T)|_{B_0}$.*

Proof. Let $T\in M_r(A)$ and set $m=\mu(T)$. Construct m^*, S, S^* and S_1 as in Theorem 4.1. Further given $a\in A$, $h\in B_0$, $G\in B_2^*$ and $F\in B_1^*$, define $(G\circ_1 h)(a)=G(h\circ_1 a)$ and $(F\circ_1 G)(h)=F(G\circ_1 h)$. We can easily see that these are well-defined and so $B_2^* \circ_1 B_0 \subset B_1$ and $B_1^* \circ_1 B_2^* \subset B_0^*$. Note that $T^*(B_1) \subset B_1$, so one can establish that the followings hold:

(i) $S(h, a) = h \circ_1 Ta$,

(ii) $S^*(G, h) = T^*(G \circ_1 h)$,

(iii) $S_1(F, G) = (T^*|_{B_1})^*(F) \circ_1 G$

for all $a \in A$, $h \in B_0$, $F \in B_1^*$ and $G \in B_2^*$. Now if we put $F = I_1$ and $G = I_2$ in (iii), then $S_1(I_1, I_2) = (T^*|_{B_1})^*(I_1) \circ_1 I_2$. Furthermore noting that $I_2 \circ_1 h = h$ for all $h \in B_0$, we have

$$(T^*|_{B_1})^*(I_1) \circ_1 I_2 = (T^*|_{B_1})^*(I_1)|_{B_0}.$$

Therefore

$$\sigma(\mu(T)) = \sigma(m) = S_1(I_1, I_2) = (T^*|_{B_1})^*(I_1)|_{B_0} = \sigma_1(T)|_{B_0},$$

which implies that σ is an extension of σ_1 to $QM(A)$. Similarly, we can prove that $\sigma(\lambda(T)) = \sigma_2(T)|_{B_0}$ for all $T \in M_l(A)$, which in turn implies that σ is also an extension of σ_2.

Theorem 4.6. *Let A be a dual A^*-algebra of the first kind, then $\sigma: QM(A) \to B_0^*$ as defined in Theorem 4.3 is an onto mapping.*

Proof. Let $F \in B_0^*$. Define $T_F: A \times A \to A^{**}$ by $T_F(a, b)(f) = F(b \circ_2 f \circ_1 a)$ for all $a, b \in A$ and $f \in A^*$. The mapping T_F is obviously linear and continuous. Also it can be proved easily that

$$T_F(ab, c) = \pi(a) \circ_1 T_F(b, c), \quad \forall a, b, c \in A.$$

We claim that $T_F(a, b) \in \pi(A)$. Since every $a \in A$ can be expressed in the form $a = \sum_\alpha e_\alpha a$, where $\{e_\alpha\}$ is a maximal orthogonal family of self adjoint minimal idempotents in A [8, Theorem 16, p. 30] and $\pi(A)$ is a two-sided ideal of A^{**} [15], it follows that $T_F(a, b) \in \pi(A)$, $\forall a, b \in A$. Define $m: A \times A \to A$, by $\pi(m(a, b)) = T_F(a, b)$, $\forall a, b \in A$. It can now easily be shown that m is a quasi-multiplier. Let us now show that $\sigma(m) = F$. Let $\sigma(m) = S_1(I_1, I_2)$. Then for every $h = \sum_1^\infty a_k \circ_2 h_k \circ_1 b_k \in B_0$, we have

$$\begin{aligned}
\sigma(m)(h) &= S_1(I_1, I_2)\Big(\sum_1^\infty a_k \circ_2 h_k \circ_1 b_k\Big) \\
&= \sum_1^\infty h_k m(b_k, a_k) \\
&= \sum_1^\infty \pi(m(b_k, a_k))(h_k) \\
&= \sum_1^\infty T_F(b_k, a_k)(h_k)
\end{aligned}$$

$$=\sum_1^\infty F(a_k\circ_2 h_k\circ_1 b_k)$$

$$=F(\sum_1^\infty a_k\circ_2 h_k\circ_1 b_k)$$

$$=F(h),$$

and so $\sigma(m)=F$. Hence σ is an onto mapping.

5. Let us suppose that there is a two-sided bounded approximate identity $\{e_\alpha\}$ in A. It will be shown that the Banach space $QM(A)$ in this case is closely related to B^*.

Theorem 5.1. *Let A be a Banach algebra with two-sided bounded approximate identity $\{e_\alpha\}$ bounded by one, then B_0^* is isometrically isomorphic to B^*.*

Proof. Let $F\in B^*$. Since $(B, \|\ \|')$ is dense in $(B_0, \|\ \|')$, F can be extended to $\tilde{F}$ on $(B_0, \|\ \|')$ and it is easy to see that $\|\tilde{F}\|'\leqq\|F\|$.

Conversely if $F\in B_0^*$, then

$$\begin{aligned}|F(e_\alpha\circ_2 h\circ_1 e_\alpha)| &\leqq \|F\|'\|e_\alpha\circ_2 h\circ_1 e_\alpha\|'\\ &\leqq \|F\|'\|e_\alpha\|\|h\|\|e_\alpha\|\\ &=\|F\|'\|h\| \quad (h\in A^*). \qquad \text{........(I)}\end{aligned}$$

Define $\psi: A\times A\to \boldsymbol{C}$ by $\psi(a, b)=F(a\circ_2\phi\circ_1 b)$ for fixed $\phi\in A^*$. Since $|\psi(a, b)|\leqq\|F\|\|a\|\|b\|\|\phi\|$ for all $a, b\in A$, ψ is jointly continuous. Now

$$\begin{aligned}\lim_\alpha F(e_\alpha\circ_2[a\circ_2\phi\circ_1 b]\circ_1 e_\alpha) &=\lim_\alpha F(e_\alpha a\circ_2\phi\circ_1 b e_\alpha)\\ &=\lim_\alpha \psi(e_\alpha a, b e_\alpha)=\psi(a, b)\\ &=F(a\circ_2\phi\circ_1 b)\end{aligned}$$

for all $a, b\in A$ and $\phi\in A^*$. Thus $\lim_\alpha F(e_\alpha\circ_2 h\circ_1 e_\alpha)=F(h)$ for every $h\in B$. Moreover, taking into account (I), we have

$$|F(h)|\leqq\|F\|'\|h\|,\ h\in B, \text{ i.e. } \|F\|_B\leqq\|F\|'<\infty$$

implying that $F\in B^*$ and $\|F|_B\|\leqq\|F\|'$. Hence B_0^* is isometrically isomorphic to B^*.

Corollary 5.1. *Let A be a Banach algebra with two-sided bounded approximate identity bounded by one, then there exists an isometric isomorphism of $QM(A)$ onto the submodule*

$$E=\{F|_B\in B^*: F\in B_0^* \ \text{ and } \ \rho_1(a)\circ_1 F\circ_2\rho_2(b)\in\rho(A) \ \text{ for all } \ a, b\in A\}$$

of B^*.

Proof. Proof follows from Theorem 4.3 and Theorem 5.1.

Corollary 5.2. *Let* A *be a* B^**-algebra. Then there exists an isometric isomorphism of* $QM(A)$ *onto the submodule*

$$E=\{F\in A^{**}: \pi(a)\circ_1 F\circ_2\pi(b)\in\pi(A) \quad \text{for all } a, b\in A\}$$

of A^{**}.

Proof. Since A is a B^*-algebra, $A^*=B$ [13]. Now rest follows from Corollary 5.1.

Collorary 5.3. *Let* A *be a dual* B^**-algebra. Then* $QM(A)$ *is isometrically isomorphic to* A^{**}.

Proof. It follows from Theorem 4.6 and Corollary 5.2.

Example 1. Let $A=LC(H)$, the algebra of compact operators. Then $QM(A)\approx L(H)$, the algebra of all bounded operators, where $[LC(H)]^{**}=L(H)$ [11].

Let G be an arbitrary locally compact topological group, abelian or non-abelian, with left Haar measure m. For $p\in[1, \infty)$, we let, as is customary, $L_p(G)$ be the Banach space of m-measurable functions on G whose p^{th} powers are absolutely integrable. Now $[L_1(G)]^*=L_\infty(G)$, the space of measurable essentially bounded functions on G. Also for $p\in(1, \infty)$, $[L_p(G)]^*=L_q(G)$, where $1/p+1/q=1$.

Example 2. Let G be a compact topological group. Then $L_p(G)$ is a dual Banach algebra [9]. For $p>1$, $L_p(G)\subset L_1(G)$.

Consider the special Hilbert space $L_2(G)$. There exists a one-to-one left regular $*$-representation of $L_1(G)$ on $L_2(G)$ [9]. The restriction of the left regular $*$-representation to $L_p(G)$ is a one-to-one left regular $*$-representation and so $L_p(G)$ is an A^*-algebra. Thus $L_p(G)$ is a dual A^*-algebra. By Theorem 4.3 $QM(L_p(G))$ can be embedded isometrically isomorphically into B_0^*.

Let us now identify the space B_0. Let $f\in L_p(G)$ and $F\in[L_p(G)]^*$. Then for $h\in L_p(G)$, we have

$$\begin{aligned}[f\circ_2 F](h)&=F(h*f)\\ &=\int_G F(t)(h*f)(t)dt\\ &=\int_G F(t)\left[\int_G h(y)f(y^{-1}t)dy\right]dt\\ &=\int_G h(y)\left[\int_G F(t)f(y^{-1}t)dt\right]dy,\end{aligned}$$

so that for $y\in G$,

$$[f\circ_2 F](y)=\int_G F(t)f(y^{-1}t)dt$$
$$=\int_G F(t)\tilde{f}(t^{-1}y)dt$$
$$=(F*f)(y)\,,$$

where $\tilde{f}(t)=f(t^{-1})$. We then conclude that $f\circ_2 F=F*\tilde{f}$. Also

$$(F\circ_1 f)(h)=F(f*h)$$
$$=\int_G F(t)\left[\int_G f(s)h(s^{-1}t)ds\right]dt$$
$$=\int_G h(t)\left[\int_G f(s)F(st)ds\right]dt\,,$$

so that for $t\in G$,

$$(F\circ_1 f)(t)=\int_G f(s)F(st)ds$$
$$=\int_G f(s^{-1})F(s^{-1}t)ds$$
$$=(\tilde{f}*F)(t)\,.$$

We then conclude that $F\circ_1 f=\tilde{f}*F$. Therefore for a_k, $b_k\in L_p(G)$ and $h_k\in[L_p(G)]^*$ $=L_q(G)$,

$$a_k\circ_2 h_k\circ_1 b_k=a_k\circ_2(\tilde{b}_k*h_k)=(\tilde{b}_k*h_k)*\tilde{a}_k=\tilde{b}_k*h_k*\tilde{a}_k\,.$$

Also $a_k\in L_p(G)\Leftrightarrow\tilde{a}_k\in L_p(G)$. Let

$$C_p=\{h=\sum_1^\infty a_k*h_k*b_k : \sum_1^\infty \|a_k\|\|h_k\|\|b_k\|<\infty$$
$$\text{for } a_k,\ b_k\in L_p(G) \text{ and } h_k\in L_q(G)\}\,.$$

Thus $C_p=B_0$. Hence $QM(L_p(G))$ can be embedded isometrically isomorphically into $(C_p)^*$.

Remark. The above result is similar to the one proved by Figa-Talamanca for multipliers.

Example 3. Let G be a locally compact abelian topological group. Then $L_1(G)$ is a Banach algebra with two-sided bounded approximate identity. By Theorem 5.1, $QM(L_1(G))$ can be embedded isometrically isomorphically into B^*. Let us now identify B. For $f\in L_1(G)$ and $F\in[L_1(G)]^*$, as in Example 2, $f\circ_2 F$ $=F*\tilde{f}$, and $F\circ_1 f=\tilde{f}*F$. Then

$$L_\infty(G)\circ_1 L_1(G)=[L_1(G)]^{\sim}*L_\infty(G)$$
$$=L_1(G)*L_\infty(G)=C_u(G)\,,$$

where $C_u(G)$ is the space of uniformly continuous, bounded complex-valued functions on G. And

$$L_1(G)\circ_2 C_u(G)=C_u(G)*[L_1(G)]^{\sim}$$
$$=C_u(G)*L_1(G)=C_u(G)\,.$$

Hence $B=L_1(G)\circ_2 L_\infty(G)\circ_1 L_1(G)=C_u(G)$. Therefore if G is a compact abelian topological group, then $QM(L_1(G))$ can be embedded isometrically isomorphically onto $M(G)$, the measure algebra of G, with the help of Theorem 4.5 and the fact that $(C(G)^*\cong M(G)\cong M(L_1(G))$, the multiplier algebra of $L_1(G)$ (cf. [4, p. 9]).

Remark. 5.1. If a Banach algebra A satisfies the conditions (α) and (β), then A necessarily satisfies the condition (γ): $(\alpha)+(\beta)\Rightarrow(\gamma)$.

Proof. Suppose that A satisfies (α) and (β) and let $x\in A$ such that $AxA=(0)$. Then $f(axb)=0$ for all $a, b\in A$ and $f\in A^*$. Therefore $(xb)\circ_2 f=0$, so that by (β) $f(xb)=0$ for all $b\in A$ and $f\in A^*$. Hence $f\circ_1 x=0$, so that by (α) $f(x)=0$ for all $f\in A^*$, that is $x=0$. Then A satisfies (γ).

Remark 5.2. If a Banach algebra A satisfies the conditions (α) and (β), then both definitions of quasi-multipliers on A given by (1) and (2) are equivalent.

Pooof. In fact, let $m: A\times A\to A$ be a continuous bilinear mapping. Suppose that A satisfies (α) and (β) and m has the property (1). Given $a, b, c, d, x\in A$, we have

$$m(ab, cd)x=am(b, c)dx=m(ab, cdx)=am(b, cd)x\,.$$

Then $[m(ab, cd)-am(b, cd)]A=(0)$, so that $m(ab, cd)=am(b, cd)$ because A satisfies (γ) from Remark 5.1. This means that $m(ab, y)=am(b, y)$ for all $a, b\in A$ and $y\in A^2$. Note also that the linear hull of A^2 is dense in A from Lemma 4.3. Therefore $m(ab, c)=am(b, c)$ for all $a, b, c\in A$ from the continuity of m. Similarly we see that $m(a, bc)=m(a, b)c$ for all $a, b, c\in A$. Then m has the property (2). It is obvious that $(2)\Rightarrow(1)$.

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