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INFINITE DIVISIBILITY AND RANDOM SUMS OF RANDOM VECTORS

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ABSTRACT In the first part of this paper we prove a theorem relating the asymptotic behavior of a multivariate infinitely divisible d.f. to that of its Levy measure.

The second part of the paper is devoted to subordination and more generally to random sums of random vectors. Using a sequence of first passage times we then apply our results to obtain a multivariate limit theorem for the partial maxima of normed sums of random vectors.

1. Introduction.

Although there is an extensive literature on one-dimensional infinitely divisible (i.d.) probability distributions, there is much less available on the multivariate case. For an investigation of i.d. distributions in $\boldsymbol{R}_+^n$ we refer to [6].

Now suppose F is a d.f. in $\boldsymbol{R}_+^n$ such that $F(\vec{x})>0$ for all $\vec{x}$ in the interior of $\boldsymbol{R}_+^n$. Then [6, Th. 2.4] F is i.d. if and only if there is a nonnegative measure ν on $\boldsymbol{R}_+^n$ such that for $i=1, \cdots, n$ and all $\vec{x}\in\boldsymbol{R}_+^n$,

$$\int_{\vec{y},\, \vec{x}-\vec{y}\in\boldsymbol{R}_+^n} y_i F(dy)=\int_{\vec{y},\, \vec{x}-\vec{y}\in\boldsymbol{R}_+^n} F(\vec{x}-y) y_i \nu(dy)\,.$$

Moreover under these conditions we have $\nu\{y \mid y-\vec{x}\in\boldsymbol{R}_+^n\}<\infty$ for all $\vec{x}$ in the interior of $\boldsymbol{R}_+^n$.

We will call ν the Lévy measure of F. In the first part of this paper we want to compare the asymptotic behavior of F with that of ν. A typical example of an i.d. d.f. F is the compound Poisson distribution, i.e. for a d.f. G on $\boldsymbol{R}_+^n$ and $\lambda>0$ we have

$$F(\cdot)=e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k\,!}G^{*k}(\cdot) \tag{1.1}$$

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where G^{*k} denotes the k-th convolution of G. Then F is i. d. with Lévy measure $\nu(\cdot)=\lambda G(\cdot)$.

The second part of the paper will be devoted to generalizations of (1.1). Among others we will examine asymptotics between F and G where

$$F(\cdot)=\sum_{k=0}^{\infty} a_k G^{*k}(\cdot) \tag{1.2}$$

for some discrete probability distribution $\{a_k\}_N$.

Basic in our study are the papers of Stam [8] and Embrechts et al. [2]. In section 4 finally we apply our results to obtain a multidimensional limit theorem for maxima of normed sums.

2. Asymptotics for I. D. Distributions.

We start from the following one-dimensional result obtained in [2].

Lemma 2.1. *Suppose F is i.d. on $\mathbf{R}_+^1$ with Lévy measure ν, then for $\alpha>0$ the following statements are equivalent:*

(i) $1-F(x)\in RV_{-\alpha}$
(ii) $\nu([x, \infty))\in RV_{-\alpha}$

Both imply

(iii) $1-F(x)\sim\nu([x, \infty))$ $(x\to\infty)$. ■

Recall that a measurable and positive function $f:\mathbf{R}_+^1\to\mathbf{R}_+^1$ is regularly varying with index $\alpha\in\mathbf{R}$ $(f\in RV_\alpha)$ if for each $x>0$, $\lim_{t\to\infty} f(tx)/f(t)=x^\alpha$. A sequence of positive numbers $\{a_n\}_N$ belongs to RV_α if $f(x):=a_{[x]}\in RV_\alpha$. For i.d. distributions in $\mathbf{R}_+^n$, lemma 2.1 generalises as follows:

Thorem 2.2. *Suppose F is i.d. on $\mathbf{R}_+^n$ with Levy measure ν. For $\alpha>0$, let $h(x)\in RV_{-\alpha}$ and for $\vec{x}>0$, define $A(\vec{x})=\{\vec{u}\in\mathbf{R}_+^n \mid \vec{u}\leq\vec{x}\}$ and $A^c(\vec{x})=\mathbf{R}_+^n\setminus A(\vec{x})$. Then the following statements are eqnivalent: for a measure λ on $\mathbf{R}_+^n$,*

(i) *for all* $\vec{x}>0$, $\lim_{t\to\infty}\frac{F(tA^c(\vec{x}))}{h(t)}=\lambda(A^c(\vec{x}))$,

(ii) *for all* $\vec{x}>0$, $\lim_{t\to\infty}\frac{\nu(tA^c(\vec{x}))}{h(t)}=\lambda(A^c(\vec{x}))$.

Remark. The measure λ in the theorem then satisfies

$$t^{-\alpha}\lambda(A^c(\vec{x}))=\lambda(tA^c(\vec{x}))$$

for all $t>0$ and $\vec{x}>0$. This follows from the regular variation of h.

Before proving the theorem we first state

Corollary 2.3. *Suppose* (1.1) *holds. If* $h \in RV_{-\alpha}$, $\alpha > 0$, *the followings tatements are equivalent: for a measure* μ *on* $\boldsymbol{R}^n_+$

(i) *for all* $\vec{x} > 0$, $\lim_{t\to\infty} \frac{F(tA^c(\vec{x}))}{h(t)} = \mu(A^c(\vec{x}))$

(ii) *for all* $\vec{x} > 0$, $\lim_{t\to\infty} \frac{G(tA^c(\vec{x}))}{h(t)} = \frac{1}{\lambda}\mu(A^c(\vec{x}))$. ■

Proof of Theorem 2.2. For a measure M on $\boldsymbol{R}^n_+$ and for $a \in \boldsymbol{R}^n_+$, $x \in \boldsymbol{R}^1_+$ define

$$M_{\vec{a}}(x) = M(\{\vec{u} \in \boldsymbol{R}^n_+ \mid 0 \leqq a \cdot \vec{u} \leqq x\})$$

where $a \cdot \vec{u} = \sum_{i=1}^{n} a_i \cdot u_i$. Then it follows that if F is i.d. in $\boldsymbol{R}^n_+$ with Levy measure ν, then also $F_{\vec{a}}(x)$ is i.d. in $\boldsymbol{R}^1_+$ with Lévy measure $\nu_{\vec{a}}(\cdot)$. To prove the theorem it is sufficient to show that (i) (or (ii)) is equivalent to regular variation of $1 - F_{\vec{a}}(x)$ (or to regular variation of $\nu_{\vec{a}}([x, \infty))$). An application of Lemma 2.1 then yields the desired result. W.l.o.g. we may and do assume $h(x)$ decreases with x. Now define the sequence $\{c_m\}_N$ such that $mh(c_m) = 1$. Then (i) is equivalent to

$$m(1 - F(c_m \vec{x})) \to \mu(\vec{x}), \text{ say.} \tag{2.1}$$

Hence if $\{X_i, i \in N_0\} = \{(X^1_i, \cdots, X^n_i), i \in N_0\}$ is a sequence of i.i.d. random vectors with d.f. F, with $M_m := (\mathrm{Max}(X^1_1, \cdots, X^1_m), \cdots, \mathrm{Max}(X^n_1, \cdots, X^n_m))$ it follows from [4] that (2.1) holds if and only if M_m/c_m converges in distribution as $m \to \infty$. Using Cramer-Wold device [1] this is equivalent to convergence in distribution of $a \cdot M_m/c_m$ for all $a \in \boldsymbol{R}^n_+$, which in turn is equivalent to regular variation of $1 - F_{\vec{a}}(x)$. ■

3. Subordinated distributions and random sums

Let G be a d.f. on $\boldsymbol{R}^n_+$ and $\{a_k\}_N$ a probability measure on N with $a_0 < 1$. The d.f. F where

$$F(\cdot) = \sum_{k=0}^{\infty} a_k G^{*k}(\cdot) \tag{3.1}$$

is called subordinate to G with respect to $\{a_k\}_N$. It admits the following probabilistic interpretation. Let $\{X_i, i \in N_0\}$ be i.i.d. random vectors with d.f. G and let N, independent of X_1, be an integer valued r.v. with $P\{N = m\} = a_m$ ($m \in N$). With $S(0) = 0$ and $S(m) = X_1 + \cdots + X_m$, the random vector $S(N)$ has d.f. F. In Theorems 3.1 and 3.2 below we relate an asymptotic behavior of G to that of F.

In this section we will also consider the following generalization of the previous situation. Let $S(m) = (S^1(m), \cdots, S^n(m))$ be defined as before and let

$N=(N_1, \cdots, N_n)$ be an N^n-valued random vector. In Theorems 3.3 and 3.4 we will examine the asymptotic behavior of the vector $(S^1(N_1), \cdots, S^n(N_n))$. For results in dimension $n=1$ we refer to Kimbleton [7] and Stam [8]. See also Teicher [9] and Hagwood and Teicher [5].

We start with the following generalization of [8, Th. 1.4].

Theorem 3.1. *Suppose* $h\in RV_{-\alpha}$, $\alpha>1$, $\sum_{k=m+1}^{\infty} a_k=o(h(m))$ *and suppose that* G, *a d.f. on* $\boldsymbol{R}_+^n$, *and* F *are related by* (3.1). *Then for some measure* λ *and all* $\vec{x}>0$, G *satisfies*

$$\lim_{t\to\infty}\frac{G(tA^c(\vec{x}))}{h(t)}=\lambda(A^c(\vec{x})) \tag{3.2}$$

if and only if F *satisfies*

$$\lim_{t\to\infty}\frac{F(tA^c(\vec{x}))}{h(t)}=\eta\lambda(A^c(\vec{x}))\,. \tag{3.3}$$

Here $\eta=\sum_{k=0}^{\infty} k a_k$.

Proof. The d.f. G satisfies (3.2) iff $G_{\vec{a}}(x)$ satisfies, for all $x>0$,

$$\lim_{t\to\infty}\frac{1-G_{\vec{a}}(tx)}{h(t)}=\mu(x)\,, \quad \text{say} \tag{3.4}$$

Using [8, Th. 1.4] and (3.1), (3.4) holds iff (3.4) holds with $G_{\vec{a}}$ (and $\mu(x)$) replaced by $F_{\vec{a}}$ (and $\eta\mu(x)$). This in turn is equivalent to (3.3). ■

If in Th. 3.1, $a_m=1$ and $a_k=0$, $k\neq m$ it follows that (3.2) is equivalent to (3.3) with $F=G^{*m}$ and $\eta=m$. The assumption that $\alpha>1$ can also be dropped in this case.

Proposition 3.2. *Let* $h\in RV_{-\alpha}$, $\alpha>0$. *Then* G *satisfies* (3.2) *iff for all* $m\geqq 2$,

$$\lim_{t\to\infty}\frac{G^{*m}(tA^c(\vec{x}))}{h(t)}=m\lambda(A^c(\vec{x}))\,. \tag{3.5}$$

Proof. Since (3.2) holds iff (3.4) holds it follows that $1-G_{\vec{a}}(x)$ is regularly varying. This in turn is equivalent to regular variation of $1-G_{\vec{a}}^{*m}(x)$ for all $m\geqq 2$ [2, Cor. 2]. Since in this case also $1-G_{\vec{a}}^{*m}(x)\sim m(1-G_{\vec{a}}(x))$ $(x\to\infty)$, the result follows. ■

Our next theorem is devoted to the case where $\sum_{k=0}^{\infty} k a_k$ is not necessarily finite, in which case Theorem 3.1 is not applicable. We shall prove the following general result.

Theorem 3.3. *Suppose* $N=(N_1, \cdots, N_n)$ *is an* N^n*-valued random vector, belonging to the domain of attraction of a random vector* $U=(U_1, \cdots, U_n)$, *stable with indices* $(\alpha_1, \cdots, \alpha_n)$, $0<\alpha_i\leqq 1$. *Assume* $X_1=(X_1^1, \cdots, X_1^n)$ *is independent of*

N and has values in $\boldsymbol{R}_+^n$. Suppose there exist constants $\rho_i>1$, $L_i>0$ such that for $x_i>0$, $i=1, \cdots, n$,

$$x_i^{\rho_i} P\{X_1^i > x_i\} \leqq L_i < \infty . \tag{3.6}$$

Then if $X_1, X_2, \cdots$ are i.i.d. and if we set $S(0)=0$ and $S(m)=\sum_{i=1}^m X_i=(S^1(m), \cdots, S^n(m))$, if follows that $(S(N_1), \cdots, S(N_n))$ belongs to the domain of attracti on of a stable random vector V. Moreover $V \stackrel{d}{=} (\mu_1 U_1, \cdots, \mu_n U_n)$ where $\mu_i=EX_1^i$ ($i=1, \cdots, n$).

Proof. Let $Y_i=(Y_i^1, \cdots, Y_i^n)$, $i \geqq 1$ be i.i.d. random vectors with $Y_1 \stackrel{d}{=} (N_1, \cdots, N_n)$ and let $M(0)=0$, $M(m)=Y_1+\cdots+Y_m$. From the conditions of the theorem it follows that for some sequences of numbers $A^i(m)$, $B^i(m)$ with $A^i(m) \in RV_{1/\alpha_i}$,

$$\Big(\frac{M^1(m)}{A^1(m)}-B^1(m), \cdots, \frac{M^n(m)}{A^n(m)}-B^n(m)\Big) \Rightarrow U \quad (m \to \infty) . \tag{3.7}$$

Now choose θ_i such that $\max(1/2, 1/\rho_i)<\theta_i<1$, $i=1, \cdots, n$. From (3.6) it follows that $EX_1^i=\mu_i$ is finite and that

$$\frac{S^i(m)-\mu_i^m}{m^{\theta_i}} \xrightarrow{P} 0 \quad (m \to \infty) .$$

Since $M^i(m) \to \infty$ and since X_1 is independent of $M(1)$ it follows that

$$\frac{S^i(M^i(m))-\mu_i M^i(m)}{(M^i(m))^{\theta_i}} \xrightarrow{P} 0 \quad (m \to \infty) . \tag{3.8}$$

Also

$$\frac{(M^i(m))^{\theta_i}}{A^i(m)} \xrightarrow{P} 0 \quad (m \to \infty) . \tag{3.9}$$

From (3.7), (3.8), (3.9) and Cramer-Wold-device it follows that for $a \in \boldsymbol{R}^n$,

$$\sum_{i=1}^m a_i\Big(\frac{S^i(M^i(m))}{A^i(m)}-\mu_i B^i(m)\Big)$$

$$=\sum_{i=1}^m a_i\Big(\frac{S^i(M^i(m))-\mu_i M^i(m)}{(M^i(m))^{\theta_i}}\Big)\frac{(M^i(m))^{\theta_i}}{A^i(m)}+\sum_{i=1}^m a_i\mu_i\Big(\frac{M^i(m)}{A^i(m)}-B^i(m)\Big)$$

$$\Rightarrow 0+\sum_{i=1}^n a_i \mu_i U_i \quad (m \to \infty) .$$

Again using Cramer-Wold device it follows that as $m \to \infty$,

$$\Big(\frac{S^1(M^1(m))}{A^1(m)}-\mu_1 B^1(m), \cdots, \frac{S^n(M^n(m))}{A^n(m)}-\mu_n B^n(m)\Big) \Rightarrow V=(\mu_1 U_1, \cdots, \mu_n U_n)$$

which proves the theorem. ■

If in Th. 3.3, $EN_i<\infty$ ($i=1, \cdots, n$) (i.e. $M^i(m)/m \xrightarrow{P} EN_i$ ($m \to \infty$)) and if

X_1 belongs to the domain of attraction of a stable r. v. U, we shall prove the following Theorem 3.4. To state the theorem, let us assume that $X_1, X_2, \cdots$ are i. i. d. random vectors such that X_1 belongs to the domain of attraction of a stable law U i. e.

$$\left(\frac{\sum_{i=1}^{m}(X_i^1-B^1(m))}{A^1(m)}, \cdots, \frac{\sum_{i=1}^{m}(X_i^n-B^n(m))}{A^n(m)}\right)\Rightarrow U \tag{3.10}$$

for some sequences $A^i(m)$, $B^i(m)$ with $A^i(m)>0$, $i=1, \cdots, n$, where U is stable with indices $(\alpha_1, \cdots, \alpha_n)$, $0<\alpha_i\leqq 2$ $(i=1, \cdots, n)$.

Now let $\{R^i(m), m\geqq 1, i=1, \cdots, n\}$ be any n sequences of $\boldsymbol{N}$-valued r. v. for which there exist a sequence $\{c_m\}_N$ and constants r_i, $0<r_i<\infty$, $i=1, \cdots, n$, such that as $m\to\infty$,

$$\frac{R^i(m)}{c(m)}\xrightarrow{P} r_i \quad (i=1, \cdots, n).$$

Set $b^i(m)=B^i([r_i c(m)])$ and $a^i(m)=A^i([r_i c(m)])$ $(i=1, \cdots, n)$. We shall prove

Theorem 3.4. *Under the conditions stated above, we have*

$$\left(\frac{\sum_{j=1}^{R^1(m)}(X_j^1-b^1(m))}{a^1(m)}, \cdots, \frac{\sum_{j=1}^{R^n(m)}(X_j^n-b^n(m))}{a^n(m)}\right)\Rightarrow V \tag{3.11}$$

where V is an n-dimensional stable r.v., related to U and r_i $(i=1, \cdots, n)$.

To prove the theorem we first need the following result of Kimbleton [7].

Lemma 3.5. *Under the conditions of the theorem, for $i=1, \cdots, n$*

$$\frac{\sum_{j=1}^{R^i(m)}(X_j^i-b^i(m))-\sum_{j=1}^{[r_i c(m)]}(X_j^i-b^i(m))}{a^i(m)}\xrightarrow{P} 0. \quad \blacksquare \tag{3.12}$$

It follows from (3.12) and Cramer-Wold device that if all $r_i=1$ $(i=1, \cdots, n)$, then (3.11) holds with $V\stackrel{d}{=}U$. If the r_i are different for different i, we need the following lemma. For simplicity we only state and prove a result in $\boldsymbol{R}^2$.

Lemma 3.6. *Suppose* (3.10) *holds for $n=2$ and with $U=(U_1, U_2)$ stable with indices (α_1, α_2) with $0<\alpha_1, \alpha_2\leqq 2$. If $k_i=k_i(n)$ is such that $k_i(m)\sim c_i m$ $(m\to\infty$, $0<c_1, c_2<\infty)$, them*

$$\left(\frac{\sum_{i=1}^{k_1(m)}(X_j^1-B^1(m))}{A^1(m)}, \frac{\sum_{j=1}^{k_2(m)}(X_j^2-B^2(m))}{A^2(m)}\right)\Rightarrow V=(V_1, V_2), \tag{3.13}$$

with $E(e^{i\xi V_1+i\eta V_2})=\varphi(c^{1/\alpha_1}\xi, c^{1/\alpha_2}\eta)\varphi((c_1-c)^{1/\alpha_1}\xi, (c_2-c)^{1/\alpha_2}\eta)e^{i\xi p_1(c, c_1)+i\eta p_2(c, c_2)}$ where $c=\min(c_1, c_2)$, $\varphi(\xi, \eta)=E(e^{i\xi U_1+i\xi U_2})$ and $p_1(x, y)$, $p_2(x, y)$ are defined below.

Proof. From (3.10) as is well known it follow that for $i=1, 2$, $A^i(m)\in RV_{1/\alpha_i}$

and that

$$\text{(3.14)}\qquad \frac{B^i([xt])-B^i([x])}{A^i([x])/x}\to\begin{cases} K^i(t^{(1/\alpha_i)-1}-1) & \text{if } \alpha_i\neq 1\\ K^i\log t & \text{if } \alpha_i=1\end{cases}$$

for some real constants K^i.

If we set $S^i(m)=\sum_{j=1}^{m}X_j^i$, $b^i(m)=mB^i(m)$, $k(m)=\min(k_1(m), k_2(m))$ it follows that the l. h. s. of (3.13) equals

$$I=\left(\frac{S^1(k)-b^1(k)}{A^1(m)}, \frac{S^2(k)-b^2(k)}{A^2(m)}\right)+\left(\frac{\sum_{j=k+1}^{k_1}(X_j^1-B^1(k_1-k))}{A^1(m)}, \frac{\sum_{j=k+1}^{k_2}(X_j^2-B^2(k_2-k))}{A^2(m)}\right)$$

$$+\left(\frac{b^1(k_1-k)+b^1(k)-b^1(k_1)}{A^1(m)}, \frac{b^2(k_2-k)+b^2(k)-b^2(k_2)}{A^2(m)}\right)$$

$$=: I_1+I_2+I_3$$

where if e. g. $k=k_1$ the first term in I_2 and I_3 should be interpreted as zero. From (3.10) if follows that

$$I_1\Rightarrow(c^{1/\alpha_1}\tilde{U}_1, c^{1/\alpha_2}\tilde{U}_2)$$

where $c=\min(c_1, c_2)$ and $\tilde{U}=(\tilde{U}_1, \tilde{U}_2)\overset{\mathcal{D}}{=}U$.

Now suppose $c_1<c_2$ so that $k=k_1$ for large m. Then from (3.10) and independency we obtain

$$I_2\Rightarrow(0, (c_2-c)^{1/\alpha_2}U_2')$$

where $U_2'\overset{\mathcal{D}}{=}U_2$ and U_2' is independent of $\tilde{U}$. As to I_3 from (3.14) it follows that

$$I_3\to(0, p_2(c, c_2))=\begin{cases}(0, K^2((c_2-c)^{1/\alpha_2}-c_2^{1/\alpha_2}-c^{1/\alpha_2})) & \text{if } \alpha_2\neq 1\\ (0, K^2((c_2-c)\log(c_2-c)-c_2\log c_2+c\log c)) & \text{if } \alpha_2=1.\end{cases}$$

Hence

$$I\Rightarrow V$$

and the characteristic function of V has the desired form.

In case $c_2<c_1$ the result follows in a similar way. Next consider the case $c_1=c_2=c$. For those m such that $k_1<k_2$ as before we have $I=I_1+I_2+I_3$ and

$$I_1\Rightarrow(c^{1/\alpha_1}\tilde{U}_1, c^{1/\alpha_2}\tilde{U}_2).$$

As to I_2+I_3 with $\phi(\eta)=\varphi(0, \eta)$ we have

$$E\left(\exp\left\{i\eta\left[\left(\frac{\sum_{j=k+1}^{k_2}(X_j^2-B^2(k_2-k))}{A^2(m)}\right)+\frac{b^2(k_2-k)+b^2(k)-b^2(k_2)}{A^2(m)}\right]\right\}\right)$$

$$=\left(\phi^m\left(\frac{\eta}{A^2(m)}\right)e^{-i\eta(b^2(m)/A^2(m))}\right)^{(k_2-k)/m}\cdot\exp\left\{i\eta\frac{b^2(m)\frac{k_2-k}{m}+b^2(k)-b^2(k_2)}{A^2(m)}\right\}$$

$$=(\mathrm{i})\cdot(\mathrm{ii}).$$

From (3.10) and $k_2-k=o(m)$ it follows that (i)$\to 1$ $(m\to\infty)$. Finally from (3.14) we obtain also (ii)$\to 1$. This proves the lemma. ■

Remark. The functions $p_i(x, y)$ are defined as

$$p_i(x, y)=\begin{cases} 0 & x=y \\ K^i((y-x)^{1/\alpha_i}-y^{1/\alpha_i}+x^{1/\alpha_i}) & \text{if } \alpha_i\neq 1,\ x\neq y \\ K^i((y-x)\log(y-x)-y\log y+x\log x) & \text{if } \alpha_i=1,\ x\neq y\,. \end{cases}$$

Proof of Theorem 3.4. It follows from (3.12) that for all $a\in \boldsymbol{R}^n$,

$$\sum_{i=1}^{n} a_i \frac{\sum_{j=1}^{R^i(m)}(X_j^i-b^i(m))-\sum_{j=1}^{[r_i c(m)]}(X_j^i-b^i(m))}{a^i(m)} \xrightarrow{P} 0\,. \tag{3.15}$$

Also, using (3.10), lemma 3.6 and Cramer-Wold device it follows that

$$\sum_{i=1}^{n} a_i \frac{\sum_{j=1}^{[r_i c(m)]}(X_j^i-b^i(m))}{a^i(m)} \Rightarrow V\,. \tag{3.16}$$

Combining (3.15) and (3.16) proves the theorem. ■

4. A multivariate limit theorem for maxima of normed sums

In this section we prove a limit theorem for the maxima of normed sums of i. i. d. random vectors, with finite mean, that belong to the domain of attraction of a multivariate stable law U, hereby extending results of Teicher and Hagwood [10], [5] and Gut [3].

Theorem 4.1. *Let* $S_{(m)}=(S^1_{(m)}, \cdots, S^n_{(m)})=\sum_{j=1}^{m} X_j$ *where* $X_j=(X_j^1, \cdots, X_j^n)$, $j\geqq 1$ *are i. i. d. random vectors with positive mean vector* $\mu=(\mu_1, \cdots, \mu_n)$ *and such that* X_1 *belongs to the domain of attraction of a stable law* $U=(U_1, \cdots, U_n)$ *stable with indices* $\beta=(\beta_1, \cdots, \beta_n)$, $1<\beta_i\leqq 2$ $(i=1, \cdots, n)$. *Then there exist functions* $A^i(x)\in RV_{1/\beta_i}$ $(i=1, \cdots, n)$ *such that for any constant vector* $\alpha=(\alpha_1, \cdots, \alpha_n)$ *in* $[0, 1)^n$ *there holds*

$$P\left\{\bigcap_{j=1}^{n}\left[\max_{1\leq k\leq m}\frac{S^j_{(k)}}{k^{\alpha_j}}-\mu_j m^{1-\alpha_j}\leqq x_j A^j(m)m^{-\alpha_j}\right]\right\}\to P\left\{\bigcap_{j=1}^{n}[U_j\leqq -x_j]\right\}.$$

Proof. For each $j=1, \cdots, n$ and $c>0$ define the stopping rules

$$T_j(c)=\inf\{m\geqq 1: S^j(m)>cm^{\alpha_j}\}\,.$$

From [3, Th. 3.3], it then follows that with $\lambda_j=(c/\mu_j)^{1/(1-\alpha_j)}$

$$\frac{T_j(c)}{\lambda_j}\xrightarrow{\text{a.s.}} 1 \quad \text{as} \quad c\to\infty,\ 1\leqq j\leqq n \tag{4.1}$$

Also under the hypothesis of the theorem it follows that for some $A^j(x)\in RV_{1/\beta_j}$ we have

$$\left(\frac{S^1(m)-m\mu_1}{A^1(m)}, \cdots, \frac{S^n(m)-m\mu_n}{A^n(m)}\right)\Rightarrow U$$

and hence, using (4.1) and Th. 3.4 we obtain

$$\left(\frac{S^1(T_1(c_1))-\mu_1 T_1(c_1)}{A^1(\lambda_1)}, \cdots, \frac{S^n(T_n(c_n))-\mu_n T_n(c_n)}{A^n(\lambda_n)}\right)\Rightarrow V \tag{4.2}$$

as $c=\min_{1\leq j\leq n} c_j\to\infty$ such that $\lambda_j\sim\lambda_k$ $(1\leqq j<k\leqq n)$. Moreover from $\lambda_1\sim\lambda_k$ $(1\leqq k\leqq n)$, (4.1) and the remark following lemma 3.5 we have $V\overset{\mathcal{D}}{=}U$. Now for $j=1, \cdots, n$

$$0\leqq\frac{S^j(T_j(c_j))-c_j(T_j(c_j))^{\alpha_j}}{A^j(\lambda_j)}\leqq\frac{X^j_{T_j}(c_j)}{A^j(\lambda_j)}$$

which converges to zero in probability [3, Lemma 3.5] as $c\to\infty$. Hence (4.2) can be replaced by

$$P\Big\{\bigcap_{j=1}^{n}\Big[\frac{c_j(T_j(c_j))^{\alpha_j}-\mu_j T_j(c_j)}{A^j(\lambda_j)}\leqq x_j\Big]\Big\}\to P\Big\{\bigcap_{j=1}^{n}[U_j\leqq x_j]\Big\}. \tag{4.3}$$

Now using (4.1) we have

$$\begin{aligned} c_j(T_j(c_j))^{\alpha_j}-\mu_j T_j(c_j)&=\mu_j T_j(c_j)\Big\{\Big(\frac{T_j(c_j)}{j}\Big)^{\alpha_j-1}-1\Big\} \\ &=-\mu_j(1-\alpha_j)(T_j(c_j)-\lambda_j)(1+o(1)). \end{aligned} \tag{4.4}$$

Using (4.4), (4.3) and [3, Th. 3.8] it follows that

$$P\Big\{\bigcap_{j=1}^{n}\Big[\frac{-\mu_j(1-\alpha_j)(T_j(c_j)-\lambda_j)}{A^j(\lambda_j)}\leqq x_j\Big]\Big\}\to P\Big\{\bigcap_{j=1}^{n}[U_j\leqq x_j]\Big\}. \tag{4.5}$$

To conclude the proof of the theorem, for $x_1, \cdots, x_n\in \boldsymbol{R}$, define $c_i=c_i(m)=\mu_i m^{1-\alpha_j}+x_j A^j(m)m^{-\alpha_j}$. Then $\lambda_i\sim m$ $(m\to\infty)$ and

$$\frac{\lambda_i-m}{q_i}\sim x_i \quad (m\to\infty) \quad \text{where} \quad q_i=\frac{A^i(\lambda_j)}{\mu_j(1-\alpha_i)}.$$

But then from (4.5) it follows that

$$\begin{aligned} &P\Big\{\bigcap_{j=1}^{n}\Big[\max_{1\leq k\leq m}\frac{S^j(k)}{k^{\alpha_j}}-\mu_j m^{1-\alpha_j}\leqq x_j A^j(m)m^{-\alpha_j}\Big] \\ &=P\Big\{\bigcap_{j=1}^{n}\Big[\max_{1\leq k\leq m}\frac{S^j(k)}{k^{\alpha_j}}\leqq c_j\Big]\Big\}=P\Big\{\bigcap_{j=1}^{n}[T_j(c_j)>m]\Big\} \\ &=P\Big\{\bigcap_{j=1}^{n}\Big[\frac{T_j(c_j)-\lambda_j}{q_j}>\frac{m-\lambda_j}{q_j}\Big]\Big\}\to P\Big\{\bigcap_{j=1}^{n}[U_j\leqq -x_j]\Big\}. \end{aligned}$$

This proves the theorem. ■

The following corollary is interesting in its own right. Cf. [3, Th. 3.8].

Corollary 4.2. *Under the conditions of Th.* 4.1., *if* $\min_{1\leq j\leq n}(c_j)\to\infty$ *and*

$$\lambda_j := \left(\frac{j}{\mu_j}\right)^{1/(1-\alpha_j)} \sim \left(\frac{c_k}{\mu_k}\right)^{1/(1-\alpha_k)} \qquad (1\leqq j<k\leqq n)$$

then

$$P\left\{\bigcap_{j=1}^{n}\left[\frac{T_j(c_j)-\lambda_j}{A^j(\lambda_j)/\mu_j(1-\alpha_j)}\geqq -x_j\right]\right\}\to P\left\{\bigcap_{j=1}^{n}[U_j\leqq x_j]\right\}. \quad \blacksquare$$

Corollary 4.3. *If in Th.* 4.1., X_1 *belongs to the normal domain of attraction of an n-dimensional normal random vector with mean vector zero and covariance matrix* Σ, *then for any* $\alpha=(\alpha_1, \cdots, \alpha_n)$ *in* $[0, 1)^n$,

$$P\left\{\bigcap_{j=1}^{n}\left[m^{\alpha_j-(1/2)}\left(\max_{1\leq k\leq m}\frac{S_k^j}{k^{\alpha_j}}-\mu_j m^{1-\alpha_j}\right)\leqq x_j\right]\right\}\to P\left\{\bigcap_{j=1}^{n}[U_j\leqq x_j]\right\}$$

where $U=(U_1, \cdots, U_n)$ *is an n-dimensional normal random vector with mean vector zero and covariance matrix* Σ. $\blacksquare$

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