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A CHARACTERIZATION OF PUNCTURED n-SPHERES*

By

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ABSTRACT—We prove some graph-theoretical propositions and apply them to a characterization of punctured n-spheres with h boundary components, as the manifolds with vanishing regular genus and hole number equal to h.

1. Introduction.

The notions of regular genus $\mathcal{G}(\overline{M})$ and hole number $\mathcal{L}(\overline{M})$ for a PL, connected, compact n-manifold $\overline{M}$ with boundary were defined and studied in [G_4]. They extend the classical concepts of genus and hole number of a surface to dimension n. A *punctured n-sphere* $\check{S}^n_h$ (h a nonnegative integer) is the manifold with boundary obtained by taking the interiors of h disjoint n-balls out of the n-sphere S^n. The main result of this paper is the following characterization, which extends the ones given in [FG_3] for S^n and in [G_4] for D^n.

Theorem 1. *Let $\overline{M}$ be a PL, connected, compact n-manifold with (possibly empty) boundary. Then*

$$\overline{M} \overset{PL}{\cong} \check{S}^n_h \Leftrightarrow \mathcal{G}(\overline{M})=0 \quad \textit{and} \quad \mathcal{L}(\overline{M})=h\,.$$

Most of the constructions introduced in the present work (in §3) seem to find a proper place in an approach to the additivity problem for the regular genus with respect to connected sums. Our interest in this problem is mainly justified by its connection with the generalized Poincaré Conjecture in dimension 4 [FG_3, Remark 1; M, §1.1].

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2. Definitions and notations.

Throughout, we shall work in the *PL* category, for which we refer to [RS] and [Gl]; all manifolds will be compact and connected, unless otherwise stated. For graph theory see [Har].

An $(n+1)$*-coloured graph with boundary* is defined to be a pair (Γ, γ) where $\Gamma=(V(\Gamma), E(\Gamma))$ is a multigraph (no loops, but possibly multiple edges are allowed) and $\gamma: E(\Gamma)\rightarrow\Delta_n=\{i\in\mathbf{Z} \mid 0\leqq i\leqq n\}$ is an edge coloration on Γ (1). By the definition itself, the degree $d(v)$ of every vertex $v\in V(\Gamma)$ is bounded to be $\leqq n+1$. If $d(v)<n+1$, then v is called a *boundary-vertex*. For every subset $\mathcal{B}$ of the "colour set" Δ_n, $\Gamma_{\mathcal{B}}$ will denote the graph $(V(\Gamma), \gamma^{-1}(\mathcal{B}))$; for any $c\in\Delta_n$. $\hat{c}$ will stand for $\Delta_n-\{c\}$. (Γ, γ) is said to be *regular with respect to colour* c if the subgraph $\Gamma_{\hat{c}}$ is regular of degree n.

We now restrict our attention to the classes G_{n+1} of all $(n+1)$-coloured graphs with boundary, regular with respect to the last colour n. There exists a boundary operator $\partial: G_{n+1}\rightarrow G_n$, which assigns to a $(\Gamma, \gamma)\in G_{n+1}$ the graph $(\partial\Gamma, {}^{\partial}\gamma)\in G_n$ defined as follows: (a) the vertices of $\partial\Gamma$ are the boundary-vertices of Γ; (b) an edge e, such that ${}^{\partial}\gamma(e)=i\in\Delta_{n-1}$, joins two vertices v, w of $\partial\Gamma$ iff the same vertices are joined by an elementary walk in $\Gamma_{\{i,n\}}$.

(Γ, γ) is simply called an $(n+1)$*-coloured graph* if $\partial\Gamma$ is empty. It is easy to see that, for all $(\Gamma, \gamma)\in G_{n+1}$, $\partial\partial\Gamma$ is empty i.e. $(\partial\Gamma, {}^{\partial}\gamma)$ is a (possibly disconnected) n-coloured graph.

An n-dimensional pseudocomplex [HW, p. 49] $K(\Gamma)$ (2) can be uniquely associated with each $(\Gamma, \gamma)\in G_{n+1}$ by the following construction: (i) consider an n-simplex $\sigma(v)$ for each vertex v of Γ and label arbitrarily (but injectively) its 0-faces by Δ_n; (ii) if v and w are joined in Γ by an edge $e\in\gamma^{-1}(c)$, then identify the $(n-1)$-faces of $\sigma(v)$ and $\sigma(w)$, which do not contain the 0-face labelled by c, so that equally labelled 0-faces coincide. Note that the result of the construction is not just the pseudocomplex $K(\Gamma)$, but also the labelling of its 0-simplexes.

Observe that $|K(\Gamma)|$ is a pseudomanifold with (possibly empty) boundary, and that $\partial K(\Gamma)$ is a quotient of $K(\partial\Gamma)$; actually $\partial K(\Gamma)=K(\partial\Gamma)$ if the space $|K(\Gamma)|$ is a manifold. (Γ, γ) will be said to *represent* $|K(\Gamma)|$ and every homeomorphic polyhedron. Observe also that $|K(\Gamma)|$ is a manifold iff, for each $c\in\Delta_n$, each component of $\Gamma_{\hat{c}}$ represents an $(n-1)$-sphere or $(n-1)$-ball (following [F, Propositions 10, 16]).

A graph $(\Gamma, \gamma)\in G_{n+1}$ is called a *crystallization* of an n-dimensional manifold $\overline{M}$ with boundary if (A) (Γ, γ) is ∂*-contracted*, i.e. $\mathfrak{g}(\Gamma_{\hat{n}})=1$ and $\mathfrak{g}(\Gamma_{\hat{c}})=\mathfrak{g}(\partial\Gamma)$,

(1) This means that $\gamma(e)\neq\gamma(f)$ for any two adjacent edges e, f.

(2) The pseudocomplex actually depends also on the coloration γ.

for every $c \in \Delta_{n-1}$ (where $\mathfrak{g}(\Theta)$ denotes the number of connected components of the graph Θ), and (B) (Γ, γ) represents $\bar{M}$. In this case, $K(\Gamma)$ has exactly $1+n\cdot\mathfrak{g}(\partial\Gamma)$ vertices (0-simplexes). If we set, by convention, $\mathfrak{g}(\emptyset)=1$, then the above definition reduces to the usual one when $\partial\bar{M}$ is empty or connected.

There always exist such representations for closed manifolds [P_1; P_2]. Moreover, for every crystallization $(\tilde{\Gamma}, \tilde{\gamma})$ of $\partial\bar{M}$, there exists a crystallization (Γ, γ) of $\bar{M}$, such that $(\partial\Gamma, {}^{\partial}\gamma)=(\tilde{\Gamma}, \tilde{\gamma})$ [CG; G_3].

The notion of regular genus of a closed n-manifold was introduced in [G_2] by means of a particular type of 2-cell imbedding of a graph into a closed surface [Wh; G_1]. In order to define, following [G_4], a genus for manifolds with boundary we first build, for each $(\Gamma, \gamma)\in \boldsymbol{G}_{n+1}$, a graph (Γ^*, γ^*) by adding one vertex v^* for each boundary-vertex v of Γ, and an edge coloured by n between v and v^*. Let us call V^* the set of added vertices.

An imbedding $\iota: |\Gamma^*|\to\bar{F}$, where $\bar{F}$ is a surface with boundary, is called a *2-cell imbedding* if

a) $\partial\bar{F}\cap\iota(|\Gamma^*|)=\iota(V^*)$;

b) $(\mathrm{int}\,\bar{F})-\iota(|\Gamma^*|)$ has open 2-cells as connected components (called the *regions* of ι);

c) if $\mathfrak{R}$ is any such region, then either (c') $\partial\mathfrak{R}$ is the image of a cycle of Γ^* ($\mathfrak{R}$ *internal region*) or (c'') $\partial\mathfrak{R}=\alpha'(\mathfrak{R})\cup\alpha''(\mathfrak{R})$, where $\alpha'(\mathfrak{R})$ is the image of a walk of Γ^*, $\alpha''(\mathfrak{R})$ is an arc of $\partial\bar{F}$, and $\alpha'(\mathfrak{R})\cap\alpha''(\mathfrak{R})$ consists of two (possibly coincident) vertices of V^* ($\mathfrak{R}$ *boundary-region*).

Further, ι is said to be *regular* if there exists a cyclic permutation $\varepsilon=(\varepsilon_0, \cdots, \varepsilon_n)$ of Δ_n, such that for each internal (resp. boundary-) region $\mathfrak{R}$, the edges of $\partial\mathfrak{R}$ (resp. of $\alpha'(\mathfrak{R})$) are alternatively coloured by ε_i, ε_{i+1}, i being an integer mod. $n+1$.

For each colour pair $i, j\in\Delta_n$, call $\mathring{\mathfrak{g}}_{ij}$ the number of cycles of $\Gamma_{\{i,j\}}$; for $i, j\neq n$, call ${}^{\partial}\mathfrak{g}_{ij}$ the number of components of $(\partial\Gamma)_{\{i,j\}}$; finally call p the order of Γ, $\bar{p}$ the order of $\partial\Gamma$, and set $\mathring{p}=p-\bar{p}$. We report [G_4, Proposition 4, Corollary 5]:

If Γ is bipartite (resp. non-bipartite), for each cyclic permutation $\varepsilon=(\varepsilon_0, \cdots, \varepsilon_{n-1}, \varepsilon_n=n)$ of Δ_n, there exists exactly one regular imbedding $\iota: |\Gamma^*|\to\bar{F}_\varepsilon$, where $\bar{F}_\varepsilon$ is the orientable (resp. non-orientable) surface with $\lambda_\varepsilon={}^{\partial}\mathfrak{g}_{\varepsilon_0\varepsilon_{n-1}}$ holes and Euler characteristic

$$\chi_\varepsilon=\sum_{i\in \boldsymbol{Z}_{n+1}}\mathring{\mathfrak{g}}_{\varepsilon_i\varepsilon_{i+1}}+(1-n)\mathring{p}/2+(2-n)\bar{p}/2\,.$$

Moreover, (Γ^*, γ^*) cannot be regularly imbedded into any non-orientable (resp. orientable) surface.

The above formula reduces to the ones of [G_1, Propositions 19, 23] when $\partial\Gamma=\emptyset$, since $\Gamma^*=\Gamma$.

Now, if we denote the genus of $\bar{F}_\varepsilon$ by ρ_ε (which equals $1-\chi_\varepsilon/2-\lambda_\varepsilon/2$ if $\bar{F}_\varepsilon$ is orientable, and $2-\chi_\varepsilon-\lambda_\varepsilon$ if $\bar{F}_\varepsilon$ is non-orientable), then we can set $\rho(\Gamma)=\rho(\Gamma^*)=\min\{\rho_\varepsilon\}$, $\lambda(\Gamma)=\lambda(\Gamma^*)=\min\{\lambda_\varepsilon\}$. The *regular genus* $\mathcal{G}(\bar{M})$ and the *hole-number* $\mathcal{L}(\bar{M})$ of a manifold with boundary $\bar{M}$ are defined as follows:

$\mathcal{G}(\bar{M})=\min\{\rho(\Gamma) \mid (\Gamma,\gamma)$ is a crystallization of $\bar{M}\}$,
$\mathcal{L}(\bar{M})=\min\{\lambda(\Gamma) \mid (\Gamma,\gamma)$ is a crystallization of $\bar{M}\}$.

A survey on the theory of n-manifold representation by $(n+1)$-coloured graphs is given in [FGG].

3. Constructions.

This section mainly deals with the construction of some graphs with empty boundary out of graphs with boundary. Their properties and their geometrical meaning are studied for application in the proof of Theorem 1.

Identification graphs.—Let (Γ',γ'), $(\Gamma'',\gamma'')\in \boldsymbol{G}_{n+1}$ be two graphs with disjoint vertex sets and suppose that there exists an isomorphism $\phi:\partial\Gamma'\to\partial\Gamma''$ which preserves colorations (i.e. ${}^{\partial}\gamma''\phi={}^{\partial}\gamma'$). We now build a new $(n+1)$-coloured graph (with empty boundary) $(\Gamma'\cup_\phi\Gamma'',\gamma'\cup_\phi\gamma'')$ called the *identification graph of* (Γ',γ') *and* (Γ'',γ'') (*with respect to* ϕ), as follows:

a) $V(\Gamma'\cup_\phi\Gamma'')=V(\Gamma')\cup V(\Gamma'')$;

b) $E(\Gamma'\cup_\phi\Gamma'')=E(\Gamma')\cup E(\Gamma'')\cup\tilde{E}$, where $\tilde{E}=\{\tilde{e}_v \mid v$ is a boundary-vertex of $\Gamma'\}$, and $\tilde{e}_v$ joins v with $\phi(v)$;

c) $$\gamma'\cup_\phi\gamma''(e)=\begin{cases}\gamma'(e) & \text{if } e\in E(\Gamma')\\ \gamma''(e) & \text{if } e\in E(\Gamma'')\\ n & \text{if } e\in\tilde{E}\end{cases}$$

Note that if $\partial K(\Gamma')=K(\partial\Gamma')$ and $\partial K(\Gamma'')=K(\partial\Gamma'')$, then $|K(\Gamma'\cup_\phi\Gamma'')|$ is the identification space of $|K(\Gamma')|$ and $|K(\Gamma'')|$ via the homeomorphism induced by ϕ on the boundaries.

In Figures 1a and 1b both (Γ',γ') and (Γ'',γ'') are crystallizations of $\boldsymbol{S}^1\times\boldsymbol{D}^2$, and the (colour-preserving) isomorphism $\phi:\partial\Gamma'\to\partial\Gamma''$ are induced by the bijections, between their vertex sets, hinted in the drawings. The resulting identification graphs represent $\boldsymbol{S}^3$ (in Figure 1a) and $\boldsymbol{S}^1\times\boldsymbol{S}^2$ (in Figure 1b). This can be checked either by dipole eliminations [FG_1], or by realizing the Heegaard splittings related with the ϕ's.

Lemma A. *With the previous notations, and setting for each cyclic permutation* ε

$$\ddot{\rho}_\varepsilon(\Gamma)=\begin{cases}\rho_\varepsilon(\Gamma) & \textit{if } \Gamma \textit{ is bipartite}\\ \rho_\varepsilon(\Gamma)/2 & \textit{if } \Gamma \textit{ is non-bipartite,}\end{cases}$$

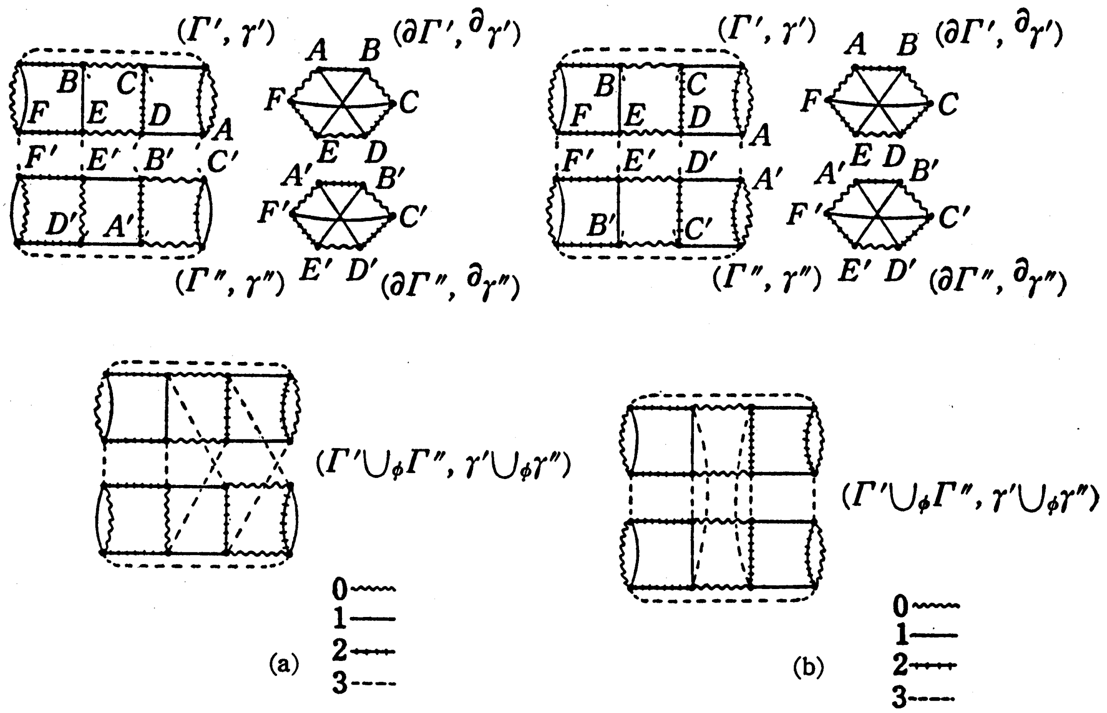


Figure 1.

we have

$$\breve{\rho}(\Gamma'\cup_\phi\Gamma'')=\breve{\rho}_\varepsilon(\Gamma')+\breve{\rho}(\Gamma'')+\lambda_\varepsilon(\Gamma')-1\,.$$

Proof. Γ'^* (resp. Γ''^*) admits a regular imbedding, associated with ε, into a surface $\bar{F}'_\varepsilon$ (resp. $\bar{F}''_\varepsilon$), orientable iff Γ' (resp. Γ'') is bipartite, of genus $\rho_\varepsilon(\Gamma')$ (resp. $\rho_\varepsilon(\Gamma'')$) and with $\lambda_\varepsilon=\lambda_\varepsilon(\Gamma')=\lambda_\varepsilon(\Gamma'')$ holes.

Therefore $\Gamma'\cup_\phi\Gamma''$ admits a regular imbedding, associated with the same ε, into the closed surface F_ε obtained by glueing together the corresponding boundary components of $\bar{F}'_\varepsilon$ and $\bar{F}''_\varepsilon$. F_ε is orientable iff both $\bar{F}'_\varepsilon$ and $\bar{F}''_\varepsilon$ are, hence $\Gamma'\cup_\phi\Gamma''$ is bipartite iff both Γ' and Γ'' are (which could have been proved by strictly combinatorial arguments).

The formula is now proved by observing that the first identification of boundary component pairs yields a surface of genus $\breve{\rho}_\varepsilon(\Gamma')+\breve{\rho}_\varepsilon(\Gamma'')$, with $2(\lambda_\varepsilon-1)$ holes, and that each further identification raises $\breve{\rho}_\varepsilon$ by one and lowers λ_ε by two. □

Note that in both Figures 1a and 1b, for any ε, $\rho_\varepsilon(\Gamma')=\rho_\varepsilon(\Gamma'')=1$, $\lambda_\varepsilon(\Gamma')=\lambda_\varepsilon(\Gamma'')=1$, and $\rho_\varepsilon(\Gamma'\cup_\phi\Gamma'')=2$, according to the formula.

Capped graphs. Given any (possibly disconnected) h-coloured graph with empty boundary (Θ, θ), the *cone over* (Θ, θ) is the $(h+1)$-coloured graph with boundary $(C\Theta, {}^C\theta)\in G_{h+1}$ where $C\Theta=\Theta$, and ${}^C\theta: E(C\Theta)\to\Delta_h$ operates as $\theta: E(\Theta)\to\Delta_{h-1}$. It is easy to see that $K(C\Theta)=CK(\Theta)$, the cone over $K(\Theta)$, if Θ is connected; if Θ is disconnected, $K(C\Theta)$ is the disjoint union of the cones over the components of $K(\Theta)$. Note that (Θ, θ) and $(C\Theta, {}^C\theta)$ are "visually" the same graph.

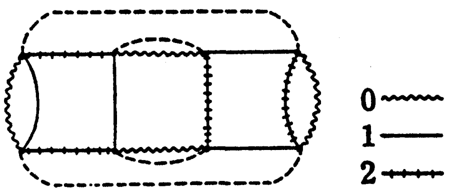
Given any $(\Gamma, \gamma)\in G_{n+1}$, representing a manifold with boundary $\overline{M}$, we define the *capped graph* $(\hat{\Gamma}, \hat{\gamma})$ to be the identification graph of (Γ, γ) and $(C(\partial\Gamma), {}^C({}^\partial\gamma))$ (the cone over its boundary-graph) with respect to the natural isomorphism between their boundaries. It is immediate to see that $(\hat{\Gamma}, \hat{\gamma})$ represents the identification space $M=\overline{M}\cup(\bigcup_i C(\partial_i\overline{M}))$, i.e. the space obtained from $\overline{M}$ by "capping off" each component $\partial_i\overline{M}$ of its boundary with a cone over it.

If (Γ', γ') is the graph of Figure 1a (or 1b), then $(\hat{\Gamma}', \hat{\gamma}')$ is obtained by joining together the vertices of Γ' and $\partial\Gamma'$ with the same labels, by edges coloured 3.

η-Sewings. For $(\Gamma, \gamma)\in G_{n+1}$, let W be a subset of the set of its boundary-vertices, and η a fixed-point-free involution on W. Then a new graph $({}^\eta\Gamma, {}^\eta\gamma)\in G_{n+1}$, the *$\eta$-sewing* of (Γ, γ), can be obtained from (Γ, γ) by adding an edge coloured n between w and $\eta(w)$ for each vertex $w\in W$.

An important particular case of η-sewing is the following. Let c be any colour in Δ_{n-1}, and consider the involution on $W=V(\partial\Gamma)$ (which will also be denoted by c), generated by the edge set ${}^\partial\gamma^{-1}(c)$. $({}^c\Gamma, {}^c\gamma)$ will then denote the $(n+1)$-coloured graph obtained from (Γ, γ) by joining two boundary-vertices with an edge coloured n iff the corresponding vertices in $(\partial\Gamma, {}^\partial\gamma)$ are joined by an edge coloured c.

For (Γ', γ') as in Figure 1a (or 1b), $({}^0\Gamma', {}^0\gamma')$ is depicted in Figure 2.



$({}^0\Gamma', {}^0\gamma')$

Figure 2.

Lemma B. *For every cyclic permutation ε of Δ_n, in which c and n are consecutive, $\rho_\varepsilon(\Gamma)=\rho_\varepsilon({}^c\Gamma)$.*

Proof. Without loss of generality, assume $\varepsilon=(0, 1, \cdots, n)$, $c=0$. Now let p, $\bar{p}$, p' be the orders of Γ, $\partial\Gamma$, ${}^0\Gamma$ respectively, and $\mathring{p}=p-\bar{p}$; with the numbers $\mathring{\mathfrak{g}}_{i,j}$, ${}^\partial\mathfrak{g}_{i,j}$ relative to (Γ, γ), defined as in § 2, and with $\mathfrak{g}'_{i,j}$ as the number of cycles of $({}^0\Gamma)_{\{i,j\}}$, we have: $p'=p$; for $k\in\Delta_{n-2}$, $\mathfrak{g}'_{k,k+1}=\mathring{\mathfrak{g}}_{k,k+1}$; $\mathfrak{g}'_{n,0}=\mathring{\mathfrak{g}}_{n,0}+\bar{p}/2$; $\mathfrak{g}'_{n-1,n}=\mathring{\mathfrak{g}}_{n-1,n}+{}^\partial\mathfrak{g}_{n-1,0}$. Thus, by [G1, Propositions 19, 23],

$$
\begin{aligned}
\chi_\varepsilon({}^0\Gamma) &= \sum_{i\in \mathbf{Z}_{n+1}} \mathfrak{g}'_{i,i+1} + (1-n)p'/2 \\
&= \sum_{k\in \Delta_{n-2}} \dot{\mathfrak{g}}_{k,k+1} + \dot{\mathfrak{g}}_{n-1,n} + {}^{\partial}\mathfrak{g}_{n-1,0} + \dot{\mathfrak{g}}_{n,0} + \bar{p}/2 + (1-n)(\mathring{p}+p)/2 \\
&= \sum_{i\in \mathbf{Z}_{n+1}} \dot{\mathfrak{g}}_{i,i+1} + (1-n)\mathring{p}/2 + (2-n)\bar{p}/2 + {}^{\partial}\mathfrak{g}_{n-1,0} \\
&= \chi_\varepsilon(\Gamma) + \lambda_\varepsilon(\Gamma)\,.
\end{aligned}
$$

Finally, the statement follows from the equalities $\ddot{\rho}_\varepsilon(\Gamma)=1-(\chi_\varepsilon(\Gamma)+\lambda_\varepsilon(\Gamma))/2$ and $\ddot{\rho}_\varepsilon({}^0\Gamma)=1-\chi_\varepsilon({}^0\Gamma)/2$, and from ${}^0\Gamma$ being bipartite iff Γ is. □

Lemma C. *Let $\check{M}$ be an n-manifold with boundary $\partial\check{M}$ and let M be the closed pseudomanifold obtained by capping off each component of $\partial\check{M}$ with a cone over it. If $(\Gamma, \gamma)\in \boldsymbol{G}_{n+1}$ represents $\check{M}$, then $({}^c\Gamma, {}^c\gamma)$ represents M for every $c\in \Delta_{n-1}$.*

Proof. One effect of the c-sewing is to identify all 0-faces coloured by c on each component S_i of $|\partial K(\Gamma)|$ to a single point a_i in $K({}^c\Gamma)$. Subdivide each n-simplex corresponding to a boundary-vertex v of Γ into two n-simplexes $\sigma_1(v)$ and $\sigma_2(v)$ so that they contain 0-faces coloured by n and c respectively. The complex K' obtained from $K({}^c\Gamma)$ by deleting all $\sigma_2(v)$ is isomorphic to $K(\Gamma)$ via a simplicial map which sends each extra 0-face to the corresponding 0-face of colour c on $\partial K(\Gamma)$. The cone structure of $\bigcup_v \sigma_2(v)$ over an $(n-1)$-face of $\partial K'$ induces that of each component of $\sigma_2(v)$ with cone points a_i in $K({}^c\Gamma)$, isomorphic to the cone CS_i. Thus, $|K({}^c\Gamma)|$ splits into $|K(\Gamma)|$ and CS_i, and is homeomorphic to M. □

As a particular case of Lemma C we have the following:

Corollary C′. *Let M be a closed n-manifold, and let $\check{M}$ be M with the interiors of a finite set of disjoint n-balls deleted. If $(\Gamma, \gamma)\in \boldsymbol{G}_{n+1}$ represents $\check{M}$, then $({}^c\Gamma, {}^c\gamma)$ represents M for every $c\in \Delta_{n-1}$.* □

Dipole nests. The next graph is a particular representation of $\mathbf{S}^n$. It will be used to produce a crystallization of $\boldsymbol{S}^{n-1}\times \boldsymbol{D}^1$.

Let (Ω^0, ω^0) be the standard crystallization of $\boldsymbol{S}^n$, consisting of two vertices X_0, Y_0 joined by $n+1$ edges of different colours; build a new graph (Ω^1, ω^1) representing $\boldsymbol{S}^n$, by adding an n-dipole Θ^1 of vertices X_1, Y_1[(3)] on the only edge of Ω^0 coloured 0. Now, from (Ω^i, ω^i) $(1\leqq i\leqq n-1)$ get a new graph $(\Omega^{i+1}, \omega^{i+1})$, always representing $\mathbf{S}^n$, by adding an n-dipole Θ^{i+1} of vertices X_i, Y_i on the only edge coloured i of Θ^i. The case $n=3$ is shown in Figure 3.

A crystallization (Ω, ω) of $\boldsymbol{S}^{n-1}\times \boldsymbol{D}^1$ is obtained from a dipole nest (Ω^n, ω^n)

(3) An *n-dipole* is a configuration of two vertices X, Y joined together by n edges (obviously of different colours), where X and Y are incident to two different edges of the residual colour. See [FG$_1$] for a general definition of dipole.

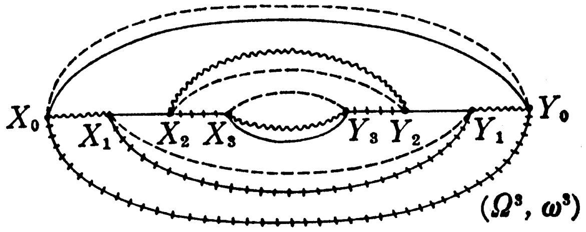


Figure 3.

by deleting the two edges coloured n, incident to X_0, Y_0 and to X_n, Y_n respectively.

4. Punctured manifolds.

We can now prove the following:

Proposition 1. *Let M be a closed n-manifold, and let $\check{M}_h$ be M with the interiors of h disjoint n-balls deleted. Then*:

$$\mathcal{G}(\check{M}_h)=\mathcal{G}(M)\,; \quad \mathcal{L}(\check{M}_h)=h\,.$$

Proof. Let (Γ, γ) be a crystallization of $\check{M}_h$ of genus g, and ε a cyclic permutation of Δ_n, such that $\rho_\varepsilon(\Gamma)=g$. By Corollary C′, $({}^{\varepsilon_0}\Gamma, {}^{\varepsilon_0}\gamma)$ represents M; moreover, Lemma B states that $\rho_\varepsilon({}^{\varepsilon_0}\Gamma)=\rho_\varepsilon(\Gamma)=g$. In general $({}^{\varepsilon_0}\Gamma, {}^{\varepsilon_0}\gamma)$ needs not be a crystallization of M (for every colour $c\neq n$, $({}^{\varepsilon_0}\Gamma)_{\hat{c}}$ actually has h components); nevertheless, the elimination of $h-1$ dipoles of type 1 [FG$_1$] for each colour $c\neq n$ yields the desired crystallization (Ξ, ξ) of M, for which $\rho_\varepsilon(\Xi)=\rho_\varepsilon({}^{\varepsilon_0}\Gamma)=g$. This proves that $\mathcal{G}(M)\leqq\mathcal{G}(\check{M}_h)$.

In order to reverse the inequality, let (Ξ, ξ) be a given crystallization of M of genus $\tilde{g}$ with a cyclic permutation ε such that $\rho_\varepsilon(\Xi)=\tilde{g}$. We shall construct a crystallization (Γ, γ) of $\check{M}_h$ for which $\rho_\varepsilon(\Gamma)=\rho_\varepsilon(\Xi)=\tilde{g}$. Moreover, $\lambda_\varepsilon(\Gamma)$ will be exactly h (the number of boundary components of $\check{M}_h$), whence also the equality $\mathcal{L}(\check{M}_h)=h$ will follow ($\mathcal{L}(\check{M}_h)$ is bounded to be $\geqq h$ by its definition). The construction of (Γ, γ) will be performed by induction on h.

(a) If $h=1$, then (Γ, γ) can be obtained from (Ξ, ξ) by deleting any edge $e\in E(\Xi)$, with $\xi(e)=n$, and by setting $\gamma=\xi|_{E(\Xi)-\{e\}}$. For such a (Γ, γ), as is simply checked, $\rho_\varepsilon(\Gamma)=\tilde{g}$, $\lambda_\varepsilon(\Gamma)=1$, and $(\partial\Gamma, {}^{\partial}\gamma)$ is the standard crystallization of S^{n-1}.

(b) While following this part, the reader may find of use to look at Figure 4, which depicts $\check{\boldsymbol{R}P}{}^3_2$, i.e. the real projective space with two spherical holes, according to our notation.

Suppose, now, that $(\tilde{\Xi}, \tilde{\xi})$ be a crystallization of $\check{M}_{h-1}$, $\rho_\varepsilon(\tilde{\Xi})=\tilde{g}$, $\lambda_\varepsilon(\tilde{\Xi})=h-1$, and $(\partial\tilde{\Xi}, {}^{\partial}\tilde{\xi})$ be the union of $h-1$ copies of the standard crystallization of S^{n-1}.

Recall the crystallization (Ω, ω) of $\mathbf{S}^{n-1}\times \boldsymbol{D}^1$ built in § 3 out of a dipole nest. Delete a boundary-vertex of $(\tilde{\Xi}, \tilde{\xi})$ and one of (Ω, ω) (both arbitrarily chosen), and paste together the equally coloured edges adjacent to them[4]; call (Γ, γ) the outcoming graph. It is easy to see that (Γ, γ) is a crystallization of $\check{M}_h=\check{M}_{h-1}\#_{\partial}(\boldsymbol{S}^{n-1}\times \boldsymbol{D}^1)$ (where $\#_{\partial}$ denotes the "boundary connected sum" of the two manifolds), and $(\partial\Gamma, {}^{\partial}\gamma)$ is the union of h copies of the standard crystallization of $\mathbf{S}^{n-1}$. This proves that $\lambda_{\varepsilon}(\Gamma)=h$. In order to show that $\rho_{\varepsilon}(\Gamma)=\tilde{g}$, observe that for any $c\in\Delta_{n-1}$, $({}^c\Gamma, {}^c\gamma)$ is obtained from $({}^c\tilde{\Xi}, {}^c\tilde{\xi})$ by cancelling $n+1$ dipoles of type n. Thus, by [FG$_2$, Lemma 1] and by Lemma B, $\rho_{\varepsilon}(\Gamma)=\rho_{\varepsilon}({}^c\Gamma)=\rho_{\varepsilon}({}^c\tilde{\Xi})=\tilde{g}$. This concludes the proof. □

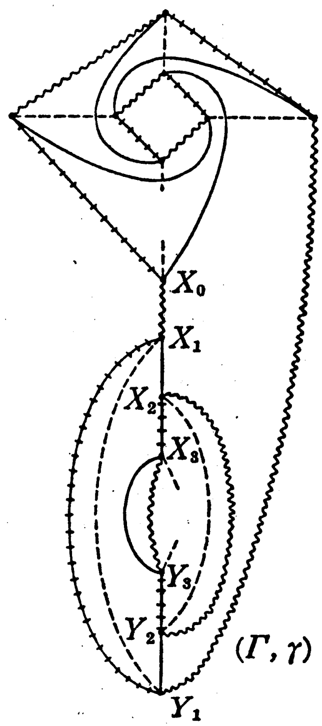


Figure 4.

We have actually proved also the following result:

Corollary 2. *For any n-manifold $\overline{M}$ with spheres as boundary components, there always exist a crystallization (Γ, γ) and a cyclic permutation ε of Δ_n, such that*

$$\rho_{\varepsilon}(\Gamma)=\mathcal{G}(\overline{M}), \quad \lambda_{\varepsilon}(\Gamma)=\mathcal{L}(\overline{M}), \quad \rho_{\varepsilon'}(\partial\Gamma)=\mathcal{G}(\partial\overline{M}),$$

where ε' is the cyclic permutation induced by ε on Δ_{n-1}. □

We do not know whether such a result holds for more general boundaries (in dimension $\geqq 4$; for dimension 3 see [G$_4$, § 4b]).

As stated in § 1, we are now able to characterize the punctured n-spheres in terms of $\mathcal{G}$ and $\mathcal{L}$.

Proof of Theorem 1. If $\overline{M}$ is a punctured sphere $\check{S}^n_h$, then Proposition 1 assures that $\mathcal{G}(\overline{M})=\mathcal{G}(S^n)=0$ and $\mathcal{L}(\overline{M})=h$. This proves one half of the statement.

On the other hand, suppose that an n-manifold with boundary $\overline{M}$, with $\mathcal{G}(\overline{M})=0$, is given. Let (Γ, γ) be a crystallization of $\overline{M}$ with $\rho_{\varepsilon}(\Gamma)=0$ (where $\varepsilon=(0, 1, \cdots, n)$). Now, by Lemma B, $\rho_{\varepsilon}({}^0\Gamma)=\rho_{\varepsilon}(\Gamma)=0$, and by Lemma C $({}^0\Gamma, {}^0\gamma)$ represents the pseudomanifold $\tilde{M}$ obtained by capping off each boundary component of $\overline{M}$ with a cone over it. But Corollary 3_n of [FG$_3$] actually assures that $\tilde{M}\cong S^n$, hence all boundary components of $\overline{M}$ must be spheres and this finally

[4] This operation has been described with some abuse of language for sake of clearness.

implies that $\overline{M}$ is a punctured n-sphere.

If we further assume that $\mathcal{L}(\overline{M})=h$, the number of the boundary components must be h by Proposition 1. This concludes the proof. □

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