YOKOHAMA MATHEMATICAL
JOURNAL VOL. 33, 1985

WEAK CONVERGENCE THEOREM FOR FUNCTIONALS OF SUMS OF REVERSED MARTINGALE ARRAYS

By

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(Received February 22, 1984)

1. Introduction.

A reversed martingale (RMG) $\{(S_n, \mathcal{F}_n), n\geqq 1\}$ is comprised of a sequence of random variables $\{S_n, n\geqq 1\}$, defined on a fixed probability space $(\Omega, \mathcal{A}, P)$ and a sequence of σ-fields $\{\mathcal{F}_n, n\geqq 1\}$ satisfying

(1) $$\mathcal{F}_{n+1}\subset\mathcal{F}_n, \quad n\geqq 1,$$

(2) S_n is a random variable measurable with respect to $\mathcal{F}_n$ and, for every $n\geqq 1$, $E|S_n|<\infty$,

(3) $$E(S_n \mid \mathcal{F}_{n+1})=S_{n+1} \quad \text{a.s., for every } n\geqq 1,$$

where "a.s." means "almost surely".

For any RMG $\{(S_n, \mathcal{F}_n), n\geqq 1\}$, S_n converges a.s. and in mean of order one to a random variable S_∞ which is measurable with respect to $\mathcal{F}_\infty=\bigcap_{n=1}^{\infty}\mathcal{F}_n$. In fact

(4) $$S_\infty=E(S_n \mid \mathcal{F}_\infty)$$

and

$$ES_n=ES_{n+1}=ES_\infty.$$

Let us put

$$X_n=S_n-S_{n+1}, \quad n\geqq 1.$$

Then we may write

(5) $$S_n-S_\infty=\sum_{k=n}^{\infty}X_k.$$

As a result of representation (5), all of the results in this paper will be formulated for infinite sums of random variables.

Let $\{X_{nk}, k\geqq n, n\geqq 1\}$ be an array of random variables and let $\{\mathcal{F}_{nk}, k\geqq n, n\geqq 1\}$ be an array of σ-fields. Assume that X_{nk} is $\mathcal{F}_{nk}$ measurable and $\mathcal{F}_{n,k+1}\subset\mathcal{F}_{nk}$ for all $k\geqq n\geqq 1$ and write

$$S_{nk}=\sum_{j=k}^{\infty}X_{nj}\,,\quad S_n=S_{nn}\,.$$

Furthermore, we also assume that $E(X_{nk}\mid \mathcal{F}_{n,k+1})=0$ for all n and k. Thus, for each $n\geqq 1$, $\{(S_{nk}, \mathcal{F}_{nk}), k\geqq n\}$ is a reversed martingale.

If the second moments of the S_{nk} are all finite (iff $ES_{n1}^2<\infty$, $n\geqq 1$) we may define for all $k\geqq n\geqq 1$

$$\sigma_{nk}^2=E(X_{nk}^2\mid \mathcal{F}_{n,k+1})\,,\quad V_{nk}^2=\sum_{j=k}^{\infty}\sigma_{nj}^2\,,\quad s_{nk}^2=EV_{nk}^2\,,\quad V_{nn}^2=V_n^2\,.$$

Further, for all $k\geqq n\geqq 1$, we have

$$s_{nk}^2=EV_{nk}^2=E\sum_{j=k}^{\infty}E(X_{nj}^2\mid \mathcal{F}_{n,j+1})=\sum_{j=k}^{\infty}EX_{nj}^2\,.$$

Let us observe that without loss of generality we may and do assume that for every $n=1$ $ES_{nn}^2=1$.

Let F be the space of functions $f(s, x)$ which are defined and have continuous first partial derivatives on $[0, 1]\times(-\infty, \infty)$. We assume that there exist positive constants Ω and α such that, for every $f\in F$,

$$|Df(t, x)|\leqq\Omega(1+|x|^{\alpha})\,,\quad (t, x)\in[0, 1]\times R\,,$$

where D denotes either the identity operator or a first partial derivative. Thus, if $f\in F$ and $f(s, x)=0$ for every $s\in[0, 1]$ and $|x|>C$ for some $C>0$, then

$$|f(s, x)-f(s_1, x_1)|\leqq K_C(|s-s_1|+|x-x_1|)\,,$$

where K_C is an absolute positive constant which depends only on C.

Assume that $f, f_n\in F$, $n\geqq 1$, and define

$$\sum_{k=n+1}^{\infty}f_n(s_{nk}^2, S_{nk})X_{n,k-1}=S(n)\,. \tag{6}$$

The results which we shall give, correspond to ones presented in [4], [5] and [2] for (ordinary) martingales. That is, we are interested in the conditions under which $S(n)$ converges weakly to the following stochastic integral

$$\int_0^1 f(t, W(t))dW(t)\,, \tag{7}$$

where $\{W(t), 0\leqq t\leqq 1\}$ is a standard Wiener process and the integral in (7) is taken in the L^2 sense.

2. Limit theorems.

The following limit theorem will be proven in Section 4. We shall adopt the notations of Section 1.

Theorem 1. *Let, for each* $n\geqq 1$, $\{(S_{nk}, \mathcal{F}_{nk}), k\geqq n\}$ *be a RMG with* $ES_{nn}^2=1$.

Assume that

(8) $$V_n^2 \xrightarrow{P} 1$$

and

(9) $$\lim_{N\to\infty} \lim_{n\to\infty} \sum_{k=0}^{\infty} EX_{n,n+k}^2 I(|X_{n,n+k}| \geqq N g_n(k))=0$$

for some double array of nonnegative numbers $\{g_n(k),\ k\geqq 0,\ n\geqq 1\}$ *such that* $\sup_{k\geq 0} g_n(k)\to 0$ *as* $n\to\infty$, *and*

(10) $$\sup_n \sum_{k=0}^{\infty} g_n^2(k)<\infty\,.$$

If $f, f_n\in F$, $n\geqq 1$, *are functions such that for each* $s\in[0, 1]$

(11) $$Df_n(s, x) \longrightarrow Df(s, x) \qquad \text{as} \quad n\to\infty$$

uniformly in x *on every finite interval, where* D *denotes either the identity operator or a first partial derivative, then*

$$\sum_{k=n+1}^{\infty} f_n(s_{nk}^2, S_{nk})X_{n,k-1} \xrightarrow{\mathcal{D}} \int_0^1 f(t, W(t))dW(t)\,.$$

Here, and in what follows, we use $\xrightarrow{P}$ *"for converges in probability to" and* $\xrightarrow{\mathcal{D}}$ *"for converges in distribution to". Similarly "*$\xrightarrow{a.s.}$*" means "converges almost surely to".*

Let $\{(S_n, \mathcal{F}_n),\ n\geqq 1\}$ be a RMG with $ES_1^2<\infty$, which converges a.s. to S_∞. Then we may define, for all $n\geqq 1$,

$$\sigma_n^2=E(X_n^2 \mid \mathcal{F}_{n+1})\,, \quad V_n^2=\sum_{i=n}^{\infty}\sigma_i^2$$

and $s_n^2=EV_n^2$ where $X_n=S_n-S_{n+1}$. Furthermore $E(S_n-S_\infty)^2=EV_n^2=s_n^2$, $n\geqq 1$.

From Theorem 1 we immediately obtain the following.

Theorem 2. *Let* $\{(S_n, \mathcal{F}_n),\ n\geqq 1\}$ *be a* RMG *with* $ES_1^2<\infty$, *which converges* a.s. *to* $S_\infty=0$, *and suppose that* $s_n^{-2}V_n^2\xrightarrow{P} 1$ *as* $n\to\infty$ *and*

$$\lim_{N\to\infty}\lim_{n\to\infty} s_n^{-2}\sum_{k=0}^{\infty} EX_{n+k}^2 I(|X_{n+k}|\geqq N s_n g_n(k))=0$$

where $\{g_n(k),\ k\geqq 0\}$ *is a double array of nonnegative numbers such that* $\sup_{n\geq 1}\sum^{\infty} g_n^2(k)<\infty$, *and* $\sup_k g_n(k)\to 0$ *as* $n\to\infty$.

If (11) *of Theorem* 1 *holds, then*

$$s_n^{-1}\sum_{k=n+1}^{\infty} f_n(s_k^2/s_n^2, S_k/s_n)X_{k-1} \xrightarrow{\mathcal{D}} \int_0^1 f(t, W(t))dW(t)$$

as $n\to\infty$.

As an application of this let Y_j, $j\geqq 0$, be independent and identically distributed random variables with mean zero and variance one. Let $\{(g(k),\ k\geqq 0\}$ be a sequence of positive numbers such that $\sum_{k=0}^{\infty} g^2(k)<\infty$ and $\sup_k g(k+n)/s_n\to 0$ as $n\to\infty$, where $s_n^2=\sum_{k=n}^{\infty} g^2(k)$. Let $X_{n,k}=g(k)Y_k/s_n$, $\mathcal{F}_{n,k}=\sigma\{Y_k, Y_{k+1}, \cdots\}$, $k\geqq 0$, $n\geqq 1$. Then it is clear that, for every $n\geqq 1$, $\{(\sum_{k=j}^{\infty} X_{nk}, \mathcal{F}_{nj}),\ j\geqq n\}$ is a *RMG* and $s_{nn}^2=V_n^2=1$. Furthermore, putting $g_n(k)=g(n+k)/s_n$, $k\geqq 0$, $n\geqq 1$, we get

$$\sum_{k=0}^{\infty} EX_{n,n+k}^2 I(|X_{n,n+k}|\geqq Ng_n(k))=EY_1^2 I(|Y_1|\geqq N)\,,$$

so that the conditions of Theorem 1 hold. Thus, under the condition (11), Theorem 1 yields

$$\sum_{k=1}^{\infty} f_n(s_{n,n+k}^2, S_{n,n+k})X_{n,n+k-1}$$

$$=s_n^{-1}\sum_{k=1}^{\infty} f_n(s_{n+k}^2/s_n^2, s_n^{-1}\sum_{i=n+k}^{\infty} g(i)Y_i)g(n+k-1)Y_{n+k-1}$$

$$\xrightarrow{\mathcal{D}} \int_0^1 f(t, W(t))dW(t) \qquad \text{as} \quad n\to\infty\,.$$

3. Auxiliary lemmas

In this Section we state and prove some lemmas which are needed for the proof of Theorem 1. All lemmas are proved under the same assumptions on the *RMG* $\{(S_{nk}, \mathcal{F}_{nk}),\ k\geqq n\}$, given in Theorem 1, therefore we do not repeat them explicitly in the formulations of lemmas.

Let, for every $t\in(0, 1]$, $m(t)=m_n(t)=\min\{i\geqq n: s_{ni}^2\leqq t\}$, and for every function $f\in F$

$$f^C(s, x)=f(s, x)I([0, 1]\times[-C, C])(s, x)\,, \tag{12}$$

where C is a positive constant and $I(A\times B)(\cdot, \cdot)$ denotes the indicator function of the set $A\times B$.

Lemma 1. *Let $\{f_n,\ n\geqq 1\}$ be a sequence of functions such that $f_n\in F$, $n\geqq 1$, and let $0=t_b<t_{b-1}<\cdots<t_1<t_0=1$ be a partition of the interval $[0, 1]$. Assume that for each n a RMG $\{(S_{nk}, \mathcal{F}_{nk}),\ k\geqq n\}$ satisfies the assumption of Theorem 1. Then for every $\varepsilon>0$ and each $C>0$*

$$\lim_{\gamma\to 0}\lim_{n\to\infty} P_1(\varepsilon, \gamma, n, C)=0\,, \tag{13}$$

where $\gamma=\max_{1\leq i\leq b}(t_{i-1}-t_i)$ and

$$P_1(\varepsilon, \gamma, n, C)=P(|\sum_{i=n+1}^{\infty} f_n^C(s_{ni}^2, S_{ni})X_{n,i-1}$$
$$-\sum_{i=1}^{b} f_n^C(t_i, S_{n,m(t_i)})(S_{n,m(t_{i-1})}-S_{n,m(t_i)})|>\varepsilon).$$

Proof. For a given $N>0$ and every i, $i\geqq 0$ we define

$$X'_{n,n+i}=X_{n,n+i}I(|X_{n,n+i}|\leqq Ng_n(i)),$$
$$Y_{n,n+i}=X'_{n,n+i}-E(X'_{n,n+i}\mid \mathcal{F}_{n,n+i+1}),$$
$$Z_{n,n+i}=X_{n,n+i}-Y_{n,n+i}$$
$$=X_{n,n+i}-X'_{n,n+i}-E([X_{n,n+i}-X'_{n,n+i}]\mid \mathcal{F}_{n,n+i+1}),$$
$$W_{ij}^{(n)}=f_n^C(s_{ni}^2, S_{ni})-f_n^C(t_j, S_{n,m(t_j)}),$$

where $\{g_n(i), i\geqq 0, n\geqq 1\}$ is a double array of nonnegative numbers satisfying (9) and (10), and $S_{n,m(0)}=0$, $n\geqq 1$.

It is easy to see that

$$P_1(\varepsilon, \gamma, n, C)\leqq P(|\sum_{i=n+1}^{\infty} f_n(s_{ni}^2, S_{ni})Y_{n,i-1}$$
$$-\sum_{j=1}^{b} f_n^C(t_j, S_{n,m(t_j)})(\sum_{i=m(t_{j-1})}^{m(t_j)-1} Y_{ni})|\geqq\varepsilon/2)$$
$$+P(|\sum_{i=n+1}^{\infty} f_n^C(s_{ni}^2, S_{ni})Z_{n,i-1}$$
$$-\sum_{j=1}^{b} f_n^C(t_j, S_{n,m(t_j)})(\sum_{i=m(t_{j-1})}^{m(t_j)-1} Z_{ni})|\geqq\varepsilon/2)$$
$$=I_1(\varepsilon, \gamma, n, C)+I_2(\varepsilon, \gamma, n, C). \tag{14}$$

Furthermore, for every i, i' such that $m(t_{j-1})\leqq i<m(t_j)$, $m(t_{j'-1})\leqq i'<m(t_{j'})$, $j<j'$, $j\geqq 1$, we have

$$EW_{ij}^{(n)}Y_{n,i-1}W_{i'j'}^{(n)}Y_{n,i'-1}=E(W_{i'j'}^{(n)}W_{ij}^{(n)}Y_{n,i-1}E[Y_{n,i'-1}\mid \mathcal{F}_{ni'}])=0,$$

and

$$EW_{ij}^{(n)}Z_{n,i-1}W_{i'j'}^{(n)}Z_{n,i'-1}=E(W_{ij}^{(n)}Z_{n,i-1}W_{i'j'}^{(n)}E[Z_{n,i'-1}\mid \mathcal{F}_{ni'}])=0.$$

Hence

$$E[\sum_{i=n+1}^{\infty} f_n^C(s_{ni}^2, S_{ni})Y_{n,i-1}-\sum_{j=1}^{b} f_n^C(t_j, S_{n,m(t_j)})\sum_{i=m(t_{j-1})}^{m(t_j)-1} Y_{n,i-1}]^2$$
$$=\sum_{j=1}^{b}\sum_{i=m(t_{j-1})}^{m(t_j)-1} E(W_{i+1,j}^{(n)}Y_{ni})^2=\sum_{j=1}^{b}\sum_{i=m(t_{j-1})}^{m(t_j)-1} E(W_{i+1,j}^{(n)}Y_{ni})^2=K$$

Furthermore, by (11), we get

$$K\leqq C_1N^2\sum_{j=1}^{b}\sum_{i=m(t_{j-1})}^{m(t_j)-1}g_n^2(i)E(W_{i+1,j}^{(n)})^2$$

$$\leqq C_1N^2\sum_{j=1}^{b}\sum_{i=m(t_{j-1})}^{m(t_j)-1}g_n^2(i)\{|s_{n,i+1}^2-t_j|^2+E(S_{n,i+1}-S_{n,m(t_j)})^2\}$$

$$\leqq 2C_1N^2\sum_{j=1}^{b}(t_{j-1}-t_j+\sup_{k\geq 0}EX_{n,n+k}^2)\sum_{i=m(t_{j-1})}^{m(t_j)-1}g_n^2(i)$$

$$\leqq 2C_1N^2(\gamma+\sup_{k\geq 0}EX_{n,n+k}^2)\sum_{i=0}^{\infty}g_n^2(i)$$

where C_1 is an absolute constant. Thus by (9) and (10)

$$\lim_{\gamma\to 0}\lim_{n\to\infty}I_1(\varepsilon,\gamma,n,C)=0$$

On the other hand

$$I_2(\varepsilon,\gamma,n,C)\leqq 4\varepsilon^{-2}\sum_{j=1}^{b}\sum_{i=m(t_{j-1})}^{m(t_j)-1}E(W_{i+1,j}^{(n)}Z_{ni})^2$$

and, by (11) and (12), there exists a positive constant C_2 such that $|W_{ij}^{(n)}|\leqq C_2$, so that

$$I_2(\varepsilon,\gamma,n,C)\leqq 4\varepsilon^{-2}C_2^2\sum_{k=n}^{\infty}EX_{nk}^2I(|X_{nk}|\geqq Ng_n(k-n)).$$

Consequently, in view of (9), we get

$$\lim_{N\to\infty}\lim_{n\to\infty}I_2(\varepsilon,\gamma,n,C)=0$$

what ends the proof of Lemma 1.

Lemma 2. *Let f, f_n, $n\geqq 1$, be functions satisfying the assumptions of Theorem 1. If the assumptions of Lemma 1 are satisfied, then for every $C>0$*

(15) $$\lim_{\gamma\to 0}\lim_{n\to\infty}P_2(\varepsilon,\gamma,n,C)=0,$$

where

$$P_2(\varepsilon,\gamma,n,C)=P(|\sum_{j=1}^{b}\{f_n^C(t_j,S_{n,m(t_j)})-f^C(t_j,S_{n,m(t_j)})\}(S_{n,m(t_{j-1})}-S_{n,m(t_j)})|>\varepsilon).$$

Proof. Let us put

$$U_{nj}(x)=f_n^C(t_j,x)-f^C(t_j,x),\quad 0\leqq j\leqq b.$$

Note that

$$E\{(S_{n,m(t_{j-1})}-S_{n,m(t_j)})^2\mid \mathcal{F}_{n,m(t_j)}\}=\sum_{i=m(t_{j-1})}^{m(t_j)-1}E(X_{ni}^2\mid \mathcal{F}_{n,m(t_j)}).$$

Thus

$$E\Big[\sum_{j=1}^{b}\{f_n^C(t_j,S_{n,m(t_j)})-f^C(t_j,S_{n,m(t_j)})\}(S_{n,m(t_{j-1})}-S_{n,m(t_j)})\Big]^2$$

$$=\sum_{j=1}^{b}E\{U_{nj}^2(S_{n,m(t_j)})\sum_{i=m(t_{j-1})}^{m(t_j)-1}E(X_{ni}^2\mid \mathcal{F}_{n,m(t_j)})\}.$$

Let $b_n=\max_{1\le j\le b}\sup_x U_{nj}^2(x)$. Then by (11) $b_n\to 0$ as $n\to\infty$. Consequently

$$\sum_{j=1}^{b}E\{U_{nj}^2(S_{n,m(t_j)})\sum_{i=m(t_{j-1})}^{m(t_j)-1}E(X_{ni}^2\mid \mathcal{F}_{n,m(t_j)})\}$$

$$\leqq b_n\sum_{j=1}^{b}\sum_{i=m(t_{j-1})}^{m(t_j)-1}EX_{ni}^2=b_n\to 0 \qquad \text{as} \quad n\to\infty,$$

so that the Chebyshev-type inequality ends the proof of Lemma 2.

Lemma 3. *Let, for each* $n\geqq 1$, $\{(S_{nk},\mathcal{F}_{nk}),\ k\geqq n\}$ *be a RMG with* $ES_{nn}^2=1$. *Assume that* (8) *holds and for every* $\varepsilon>0$

$$(16)\qquad \sum_{k=n}^{\infty}E\{X_{nk}^2I(|X_{nk}|>\varepsilon)\mid \mathcal{F}_{n,k+1}\}\xrightarrow{P}0 \qquad \text{as} \quad n\to\infty,$$

If $f\in F$, *then for every finite partition* $0=t_b<t_{b-1}<\cdots<t_0=1$ *of the internal* $[0,1]$ *and any given* $C>0$

$$(17)\qquad \sum_{j=1}^{b}f^C(t_j,S_{n,m(t_j)})(S_{n,m(t_{j-1})}-S_{n,m(t_j)})$$

$$\xrightarrow{\mathcal{D}}\sum_{j=1}^{b}f^C(t_j,W(t_j))(W(t_{j-1})-W(t_j)) \qquad \text{as} \quad n\to\infty,$$

where $\{W(t),\ 0\leqq t\leqq 1\}$ *is a standard Wiener process on* $D[0,1]$.

Proof. By the assumptions of Lemma 3 and Theorem 2 [3] $W_n\to W$ in $D[0,1]$ where W_n is a sequence of random elements defined as follows

$$W_n(0)=0,\quad W_n(1)=S_{nn},\quad W_n(t)=S_{n,k+1},\quad s_{n,k+1}^2\leqq t<s_{n,k}^2,$$

$k\geqq n$. But f^C is a continuous and bounded function so that Lemma 3 follows.
For the sake of completeness we give the following

Lemma 4. *If* $f\in F$, *then for every* $\varepsilon>0$ *and any given* $C>0$

$$(18)\qquad P\Big(\Big|\sum_{j=1}^{b}f^C(t_j,W(t_j))(W(t_{j-1})-W(t_j))-\int_0^1 f^C(t,W(t))dW(t)\Big|>\varepsilon\Big)\to 0$$

as $\gamma=\max_{1\le i\le b}(t_{i-1}-t_i)\to 0$, *where* $0=t_b<t_{b-1}<\cdots<t_1=0$ *is a partition of the interval* $[0,1]$.

Proof. See Lemma 4 [4].

4. Proof of Theorem 1.

Let $f, f_n \in F$, $n \geqq 1$, be functions satisfying (11). Let us take an arbitrary $C>0$. Then for every $\varepsilon>0$

$$P(|\sum_{k=n+1}^{\infty} f_n(s_{nk}^2, S_{nk})X_{n,k-1} - \int_0^1 f(t, W(t))dW(t)| > \varepsilon) \tag{19}$$

$$\leqq P(|\sum_{k=n+1}^{\infty} f_n^C(s_{nk}^2, S_{nk})X_{n,k-1} - \int_0^1 f^C(t, W(t))dW(t)| > \varepsilon)$$

$$+P(\sup_{k\geq n}|S_{nk}|>C)+P(\sup_{0\leq t\leq 1}|W(t)|>C).$$

Furthermore, by Theorem 2 of [3] and results given in [1] (cf. Sec's 10, 11 and Theorem 5.1) we have

$$\lim_{C\to\infty}\lim_{n\to\infty} P(\sup_{k\geq n}|S_{nk}|>C)=0$$

and

$$\lim_{C\to\infty} P(\sup_{0\leq t\leq 1}|W(t)|>C)=0.$$

Consequently Theorem 1 follows (13), (15), (17), (18) and (19).

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