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SEQUENTIALLY CLOSED GRAPHS

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1. Introduction

The purpose of this paper is to investigate various properties of sequentially closed graphs and to obtain the following characterization of sequentially compact spaces: *A T_1 topological space Y is sequentially compact if and only if for every topological space $X\in S$, each mapping of X into Y with sequentially closed graph is sequentially continuous.* In this characterization, we let S be a class of first countable topological spaces containing the class of first countable Hausdorff completely normal and fully normal spaces. This class S is more restrictive (i.e., 1°) than the class used in [2]. The one-point compactification of the positive integers N will be denoted by $\bar{N}$. We note that $\bar{N}\in S$.

2. Preliminaries

The following definitions and theorems are stated for future reference.

Definition 2.1. A subset F of a topological space X is sequentially closed if and only if sequences in F which converge in X have limits in F.

Definition 2.2. A function $f: X\rightarrow Y$ is sequentially continuous if and only if $f(x_n)\rightarrow f(x)$ whenever $x_n\rightarrow x$.

Definition 2.3. The graph of a function $f: X\rightarrow Y$, denoted by $G(f)$, is sequentially closed if and only if $G(f)$ is a sequentially closed subset of $X\times Y$.

Theorem 2.4. *A function $f: X\rightarrow Y$ is sequentially continuous if and only if $f^{-1}(F)$ is sequentially closed for every sequentially closed subset F of Y* [1, *Th.* 1.2].

Theorem 2.5. *Let $\{y_n\}$ be a sequence in Y with no convergent subsequence. Let Y be T_1. Then the set $A=\{(n, y_n)\mid n\in N\}$ is sequentially closed in $\bar{N}\times Y$* [1, *Th.* 1.3].

3. Sequentially Closed Graphs

Theorem 3.1. *Let Y be a topological space with unique sequential limits and $f: X\rightarrow Y$ a sequentially continuous function, from any space X. Then $G(f)$ is* *sequentially closed.*

Proof. Let $(x_n, f(x_n)) \to (x, y) \in X \times Y$. Since $x_n \to x$, we have by sequential continuity that $f(x_n) \to f(x)$. But Y has unique sequential limits, hence $f(x_n) \to y = f(x)$. Therefore, $(x, y) \in G(f)$ and $G(f)$ is sequentially closed.

Theorem 3.2. *Let $f: X \to Y$ be any function with sequentially closed graph. If B is a sequentially compact subset of Y, then $f^{-1}(B)$ is a sequentially closed subset of X.*

Proof. Let $\{x_n\}$ be a sequence in $f^{-1}(B)$ such that $x_n \to x$. Because B is sequentially compact, the sequence $\{f(x_n)\}$ in B contains a convergent subsequence; i.e., there exists a $y \in B$ such that $f(x_{n_k}) \to y$. We now have $(x_{n_k}, f(x_{n_k})) \to (x, y)$. With $G(f)$ sequentially closed, we conclude that $f(x) = y$, hence $x \in f^{-1}(B)$ and $f^{-1}(B)$ is sequentially closed.

Theorem 3.3. *Let $f: X \to Y$ be any function with $G(f)$ sequentially closed. If B is a sequentially compact subset of X, then $f(B)$ is a sequentially closed subset of Y.*

Proof. The proof is similar to the proof of Theorem 3.2 and is thus omitted.

Corollary 3.4. *If X is sequentially compact and $f: X \to Y$ has sequentially closed graph, then f is a sequentially closed function.*

Theorem 3.5. *If $f: X \to Y$ is sequentially continuous and Y has unique sequential limits, then $A = \{(x_1, x_2) \mid f(x_1) = f(x_2)\} = (f \times f)^{-1}(\Delta Y)$ a sequentially closed subset of $X \times X$.*

Proof. Let $\{(a_n, b_n)\} \subset A$ and suppose $(a_n, b_n) \to (a, b)$. Since $a_n \to a$ and $b_n \to b$, we have that $f(a_n) \to f(a)$ and $f(b_n) \to f(b)$ by sequential continuity. But since $f(a_n) = f(b_n)$ for every n, and additionally Y has unique sequential limits, we must have $f(a) = f(b)$. Therefore, $(a, b) \in A$ and A is sequentially closed.

Theorem 3.6. *If X is sequentially compact and $f: X \to Y$, f surjective, has sequentially closed graph, then Y has unique sequential limits.*

Proof. Let $\{y_n\}$ be any convergent sequence in Y and suppose $y_n \to y_1$ and $y_n \to y_2$. Choose $x_n \in f^{-1}(y_n)$. By sequential compactness, there exists a subsequence such that $x_{n_k} \to x \in X$. Then, we have $(x_{n_k}, f(x_{n_k})) \to (x, y_1)$ and (x, y_2). Since $G(f)$ is sequentially closed, we have $f(x) = y_1$ and $f(x) = y_2$, hence $y_1 = y_2$ and Y has unique sequential limits.

Corollary 3.7. *Let $f: X \to Y$ be a sequentially continuous surjection and X sequentially compact. Then the following are equivalent:*

(i) *$G(f)$ is sequentially closed*
(ii) *Y has unique sequential limits*
(iii) *The set $\{(y, y) | y \in Y\}$ is sequentially closed in $X \times Y$.*

Proof. The equivalence of (ii) and (iii) is trivial.

4. Characterizations of Sequential Compactness

Theorem 4.1. *If $G(f)$ is sequentially closed and Y is sequentially compact, then $f : X \to Y$ is sequentially continuous.*

Proof. Let B be a sequentially closed subset of Y. Since Y is also sequentially compact, B is sequentially compact, hence $f^{-1}(B)$ is sequentially closed by Theorem 3.2. Therefore, by Theorem 2.4, f is sequentially continuous.

Theorem 4.2. *If for every $X \in S$, each function $f : X \to Y$ with sequentially closed graph is sequentially continuous, then Y is sequentially compact if Y is T_1.*

Proof. Suppose Y is not sequentially compact. Then there exist a sequence $\{y_n\}$ with no convergent subsequence. Choose $b \in Y$. Define a function $f : \bar{N} \to Y$ by $f(n) = y_n$, $f(\infty) = b$. By Theorem 2.5, $A = \{(n, y_n)\}$ is a sequentially closed subset of $\bar{N} \times Y$. Therefore, $A \cup \{(\infty, b)\} = G(f)$ is sequentially closed. To complete the theorem, it is sufficient to show that f is not sequentially continuous. In particular, the set $\{y_n | n \in N\}$ is sequentially closed in Y but $f^{-1}\{y_n | n \in N\} = \{x_n | n \in N\}$ is not sequentially closed since $n \to \infty$. Therefore, f is not sequentially continuous and the theorem follows.

Corollary 4.3. *A T_1 topological spaace Y is sequentially compact if and only if for every first countable topological space $X \in S$, each mapping of X into Y with sequentially closed graph is sequentially continuous.*

Corollary 4.4. *For a T_1 topological space Y, the following are equivalent:*
(i) *Y is sequentially compact;*
(ii) *For every first countable topological space X, each mapping of X into Y with sequentially closed graph is sequentially continuous;*
(iii) *Each mapping of $\bar{N}$ into Y with sequentially closed graph is sequentially continuous.*

Reference

[1] Ronald Brown: *On sequentially proper maps and a sequential compactification*, Jour. London Math. Soc. (2), **7** (1973), pp. 515-522.
[2] Shouro Kasahara: *Characterizations of compactness and countable compactness*, Proc. Japan Acad., **49** (1973), pp. 522-524.

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