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APPLICATIONS OF THE POLAR DECOMPOSITION OF AN OPERATOR

By

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ABSTRACT. An operator T means a bounded linear operator on a complex Hilbert space H. In our previous paper [6], we have an equivalent condition under which an operator T_1 doubly commutes with another T_2 by an elementary method. As an application of this result, we show correlation between binormal operators and operators satisfying $[T_1{}^*T_1, T_2T_2{}^*]=0$ and moreover more precise estimation than the results of Campbell, Gupta and Bala on binormal operators. Also we show conditions on an idempotent operator implying projection and necessary and sufficient conditions under which partial isometry is direct sum of an isometry and zero.

1. Introduction

Let $N(X)$ denote the kernel of an operator X. An operator T can be decomposed into $T=UP$ where U is partial isometry and $P=|T|=(T^*T)^{1/2}$ with $N(U)=N(P)$ and this kernel condition $N(U)=N(P)$ uniquely determines U and P in the decomposition of T [8]. In this paper, $T=UP$ denotes the polar decomposition satisfying the kernel condition $N(U)=N(P)$. For two arbitrary operators A and B, $[A, B]$ denotes the commutator of A and B, that is, $[A, B]=AB-BA$. Let $T=UP$ be the polar decomposition of T, then $T^*=U^*Q$ is also the polar decompositon of T^* where $Q=|T^*|=UPU^*$ because U^* is also partial isometry satisfying $N(U^*)=N(Q)$ since

$$N(Q)=N(T^*)=R(T)^{\perp}=R(U)^{\perp}=N(U^*) .$$

We denote by (BN) the class of all operators T if $[T^*T, TT^*]=0$, and also we denote by θ the class of all operators T if $[T^*T, T+T^*]=0$. T is called dominant if there is a real number $M_\lambda\geqq 1$ such that

$$\|(T-\lambda)^*x\|\leqq M_\lambda\|(T-\lambda)x\|$$

for all x in H, and for all complex numbers λ. If there is a constant M such that $M_\lambda\leqq M$ for all λ, T is called M-hyponormal and also T is called humble M-hyponormal if there is a constant M such that $M_\lambda\leqq M$ for all real λ. T is called k-paranomal if $\|x\|^{k-1}\|T^kx\|\geqq\|Tx\|^k$ for some fixed integer $k\geqq 2$. Obviously hyponormal is k-paranormal. We denote by (WN) the class of all operators if $|T|^2 \geqq (\mathrm{Re}\, T)^2$. It is known that if T is M-hyponormal or $T \in \theta$, then T is dominant, and $\theta \subset (WN)$ and if $T \in (WN)$, then T is humble M-hyponormal. T is called quasinormal if $[T, T^*T]=0$. Obviously if T is quasinormal, then $T \in (BN)$. Moreover it is known that if T is quasinormal, then T is hyponormal and also $T \in \theta$.

In our previous paper [6], we have the following results.

Theorem A [6]. *Let $T_1=U_1P_1$ and $T_2=U_2P_2$ be the polar decompositions of T_1 and T_2 respectively. Then the following conditions* (A), (B) *and* (C) *are equivalent*:

(A) *T_1 doubly commutes with T_2 (that is, $[T_1, T_2]=0$ and $[T_1, T_2^*]=0$),*

(B) *U_1^*, U_1 and P_1 commute with U_2^*, U_2 and P_2,*

(C) *the following five equations are satisfied*

$$(1)\ [P_1, P_2]=0\,, \qquad (2)\ [U_1, P_2]=0\,, \qquad (3)\ [P_1, U_2]=0$$
$$(4)\ [U_1, U_2]=0\,, \qquad (5)\ [U_1^*, U_2]=0\,.$$

Theorem B [6]. *Let $T_1=U_1P_1$ and $T_2=U_2P_2$ be the polar decompositions of T_1 and T_2 respectively. If T_1 doubly commutes with T_2, then $T_1T_2=(U_1U_2)(P_1P_2)$ is also the polar decomposition of T_1T_2, that is, U_1U_2 is partial isometry with $N(U_1U_2)=N(P_1P_2)$ and $P_1P_2=|T_1T_2|$.*

Theorem C [6]. *Let T be normal. Then there exists unitary U such that $T=UP=PU$ and both U and P commute with V^*, V and $|A|$ of the polar decomposition $A=V|A|$ of any operator which commutes with T and T^*.*

We remark that Theorem C yields the well known and familiar result [13] of Riesz and Sz.-Nagy because Theorem C assures that U and P commutes with A. An extension of Theorem A to the intertwining case is given in [6] with respect to the Fuglede-Putnam theorem.

2. Binormal operators and operators satisfying $[T_1^*T_1, T_2T_2^*]=0$

In this section we show correlation between binormal operators and operators satisfying $[T_1^*T_1, T_2T_2^*]=0$ and we give more precise estimation than the results of Campbell [3], Gupta [7] and Bala [1] on binormal operators.

Lemma. *Let $T=UP$ be the polar decomposition of T. Then for any integer $n \geqq 1$,*

(1) *$P^n=U^*UP^n$ is the polar decomposition of P^n*

(2) *$Q^n=UU^*Q^n$ is the polar decomposition of Q^n*
where $P=|T|$ and $Q=|T^*|$.

Proof. (1) U^*UP is always valid [8] because U^*U is the initial projection of U to the range of P. Then $P^n=U^*UP^n$ holds for any integer $n\geqq 1$ and $N(U^*U)=N(P^n)$ because $N(U)=N(P)$ by the polar decomposition of T and $N(U^*U)=N(U)$ and $N(P^n)=N(P)$ are always valid, so that $P^n=(U^*U)P^n$ is the polar decomposition of P^n.

(2) Since $T^*=U^*Q$ is the polar decomposition of T^* stated in the Introduction, we have (2) by using the similar method to one stated above in (1), so the proof is complete.

Theorem 1. *Let $T_1=U_1P_1$ and $T_2=U_2P_2$ be the polar decompositions of T_1 and T_2 respectively. If $[T_1^*T_1, T_2T_2^*]=0$, then the following properties hold*:

(1) *U_1U_2 is patial isometry,*

(2) *P_1 is reduced by both $N(U_1)$ and $N(U_2^*)$*

(3) *Q_2 is reduced by both $N(U_1)$ and $N(U_2^*)$*

where $P_1=|T_1|$ and $Q_2=|T_2^*|$.

Proof. Since $T_1^*T_1=P_1^2=U_1^*U_1P_1^2$ and $T_2T_2^*=Q_2^2=U_2U_2^*Q_2^2$ by Lemma, the hypothesis $[T_1^*T_1, T_2T_2^*]=0$ is equivalent to that $U_1^*U_1P_1^2$ doubly commutes with $U_2U_2^*Q_2^2$, so that Theorem A yields

(i) $[U_1^*U_1, U_2U_2^*]=0$ (ii) $[U_2U_2^*, P_1^2]=0$ (iii) $[U_1^*U_1, Q_2^2]=0$ and
(iv) $[P_1^2, Q_2^2]=0$ (this is trivial since this is itself the hypothesis).

(i) is equivalent to that U_1U_2 is partial isometry [9]. (2) is obtained by (ii) and $[U_1^*U_1, P_1]=0$ which is always valid by Lemma. Similarly (3) is also obtained by (iii) and $[U_2U_2^*, Q_2]=0$ which is always varid by Lemma, so the proof is complete.

Corollary 1. *Let $T=UP$ be the polar decomposition of T. If $T\in(BN)$, then*

(1) *U^2 is partial isometry, that is, $U\in(BN)$,*

(2) *P is reduced by both $N(U)$ and $N(U^*)$,*

(3) *Q is reduced by both $N(U)$ and $N(U^*)$*

where $P=|T|$ and $Q=|T^*|$.

Remark 1. We remark that if $T=UP$ is the polar decomposition of binormal operator T, then U is also binormal such that $T^2=U^2|T^2|$ is the polar decomposition of T^2 and this proof will be shown in the proof of Theorem 4.

Theorem D. *Let U_1 and U_2 be partial isometry. Then the following* (1) *and* (2) *are equivalent*:

(1) U_1U_2 *is partial isometry with* $N(U_1U_2)=N(U_2)$,

(2) $U_1^*U_1 \geqq U_2U_2^*$, *that is, the initial space of* U_1 *includes the final one of* U_2.

Partial isometricity of U_1U_2 is equivalent to $[U_1^*U_1, U_2U_2^*]=0$ [9] and using this result we can easily give the proof of Theorem D and we omit it.

Corollary 2. *Let* U *be partial isometry. Then the following* (1) *and* (2) *are equivalent*:

(1) U^2 *is partial isometry with* $N(U^2)=N(U)$,

(2) U *is direct sum of an isometry and zero.*

Proof. By Theorem D, we have only to show that (2) is equivalent to $N(U)\subset N(U^*)$ and this proof is easy since $N(U)\subset N(U^*)$ if and only if $N(U)$ reduces U and U is partial isometry. We remark that (1)⇒(2) is shown in [7].

Corollary 3. *Let* $T_1=U_1P_1$ *and* $T_2=U_2P_2$ *be the polar decompositions of* T_1 *and* T_2 *respectively. If* $[T_1^*T_1, T_2T_2^*]=0$ *with* $N(U_1U_2)=N(U_2)$, *then* $U_1^*U_1\geqq U_2U_2^*$, *that is, the initial space of* U_1 *includes the final one of* U_2.

Proof. This follows by Theorem 1 and Theorem D.

Theorem 2. *Let* $T=UP$ *be the polar decomposition of* T. *If* $T\in(BN)$, *then* $T^2\in(BN)$ *if and only if the following four properties hold*:

$$(1)\ [U^{*2}U^2, U^2U^{*2}]=0 \qquad (2)\ [U^2U^{*2}, U^*PQU]=0$$
$$(3)\ [U^{*2}U^2, UPQU^*]=0 \qquad (4)\ [U^*PQU, UPQU^*]=0$$

where $P=|T|$ *and* $Q=|T^*|$.

Proof. Note that $U^*UP=P$ and $UU^*Q=Q$ are always valid by Lemma. Assume $T\in(BN)$. Then we have the following (i), (ii) and (iii)

$$(\mathrm{i})\ [U^*U, UU^*]=0 \qquad (\mathrm{ii})\ [UU^*, P]=0 \qquad (\mathrm{iii})\ [U^*U, Q]=0$$

by Corollary 1, moreover by Theorem B, $PQ=(U^*UUU^*)(PQ)$ is the polar decomposition of PQ since P doubly commutes with Q, so that we have

$$(\mathrm{iv})\quad N(PQ)=N(QP)=N(U^*UUU^*)\,.$$

Note that $T^*=U^*Q$ is the polar decomposition of T^* with $N(U^*)=N(Q)$. At first we show $|T^2|=U^*PQU$ as follows:

$$\begin{aligned}|T^2|^2=T^{*2}T^2&=(U^*QPU^*)(UPQU)=U^*PQPQU\\&=U^*P(QUU^*)PQU=(U^*PQU)^2\end{aligned}$$

so that $|T^2|=U^*PQU$ because $[P, Q]=0$ yields that U^*PQU is positive. By (ii), we have

$$T^2=UPQU=UP(UU^*Q)U=U(UU^*P)QU$$
$$=U^2(U^*PQU)=U^2|T^2| .$$

Next we show that $N(U^2)=N(|T^2|)$ as follows:

$$x\in N(|T^2|)\Leftrightarrow U^*PQUx=0\Leftrightarrow QPQUx=0 \text{ (by } N(U^*)=N(Q))$$
$$\Leftrightarrow Q^2PUx=0 \text{ (by } [P, Q]=0)\Leftrightarrow QPUx=0\Leftrightarrow U^*UUU^*Ux=0 \text{ (by (iv))}$$
$$\Leftrightarrow U^*UUx=0 \text{ (by } U=UU^*U)\Leftrightarrow UUx=0\Leftrightarrow x\in N(U^2) .$$

As U^2 is partial isometry by Corollary 1 and $N(U^2)=N(|T^2|)$ shown above, so that $T^2=U^2|T^2|=U^2(U^*PQU)$ is the polar decomposition of T^2. Next we show that $|T^{*2}|=UPQU^*$ as follows:

$$|T^{*2}|^2=T^2T^{*2}=(UPQU)(U^*QPU^*)=UPQPQU^*$$
$$=UPQ(U^*UP)QU^*=(UPQU^*)^2$$

so that $|T^{*2}|=UPQU^*$ because $[P, Q]=0$ yields that $UPQU^*$ is positive. Similarly we have $T^{*2}=U^{*2}|T^{*2}|=U^{*2}(UPQU^*)$ is the polar decomposition of T^{*2}. By Lemma we have two polar decompositions of $|T^2|^2$ and $|T^{*2}|^2$

$$|T^2|^2=T^{*2}T^2=(U^{*2}U^2)(U^*PQU)^2$$
$$|T^{*2}|^2=T^2T^{*2}=(U^2U^{*2})(UPQU^*)^2$$

so that Theorem A yields that $T^2\in(BN)$ if and only if $(U^{*2}U^2)(U^*PQU)^2$ doubly commutes with $(U^2U^{*2})(UPQU^*)^2$ if and only if the following four properties hold

$$[U^{*2}U^2, U^2U^{*2}]=0 \qquad [U^2U^{*2}, (U^*PQU)^2]=0$$
$$[U^{*2}U^2, (UPQU^*)^2]=0 \qquad [(U^*PQU)^2, (UPQU^*)^2]=0$$

these four properties are equivalent to (1), (2), (3) and (4) in Theorem 2 so the proof is complete.

Remark 2. Let $T=UP$ be the polar decomposition of T. If $T\in(BN)$, then $T^2=U^2(U^*PQU)$ is the polar decomposition of T^2 shown in the proof of Theorem 2, that is, $|T^2|=U^*PQU$ and U^2 is partial isometry and $N(U^2)=N(U^*PQU)$. Similarly $T^{*2}=U^{*2}(UPQU^*)$ is also the polar decomposition of T^{*2}, that is, $|T^{*2}|=UPQU^*$ and U^{*2} is partial isometry and $N(U^{*2})=N(UPQU^*)$. Let $T=UP$ be the polar decomposition of T. There exists an example $T=UP\in(BN)$ such that U^3 is not partial isometry as follows:

Let

$$T=\begin{pmatrix} 0 & \sqrt{3} & -1 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{then} \quad U=\frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & -1 \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix}.$$

U^2 is also partial isometry by Remark 1 but U^3 is not so. This example shows that binormal operator T is not always reduced by $N(T)$.

Corollary 4. *Let $T=UP$ be the polar decomposition of T. If $T\in(BN)$ with $N(T^2)=N(T)$, then $N(T)$ reduces T.*

Proof. By Remark 2, we have $N(T^2)=N(|T^2|)=N(U^2)$ and the hypothesis $N(T^2)=N(T)$ and $N(T)=N(U)$ is always valid, so that $N(U^2)=N(U)$ and the result follows by Corollary 3.

Corollary 5 [3]. *Let $T=UP$ be the polar decomposition of T. Assume U is unitary. If $T\in(BN)$, then $T^2\in(BN)$ if and only if $[PQ, U^2PQU^{*2}]=0$.*

Proof. As U is unitary, (1), (2) and (3) in Theorem 2 automatically hold, so that by Theorem 2 we have

$$\begin{aligned} T^2\in(BN) &\Leftrightarrow [U^*PQU, UPQU^*]=0 \\ &\Leftrightarrow [PQ, U^2PQU^{*2}]=0 \quad (\text{by } U^*=U^{-1}). \end{aligned}$$

We remark that the condition $[P^2Q^2, U^2P^2Q^2U^{*2}]=0$ is cited in [3] which is equivalent to $[PQ, U^2PQU^{*2}]=0$ in Corollary 5.

Corollary 6. *Let $T=UP$ be the polar decomposition of T. If $T\in(BN)$, then T^2 is quasinormal if and only if $[U^2, U^*PQU]=0$. In addition, assume $N(T)=N(T^*)$. If $T\in(BN)$, then T^2 is quasinormal if and only if $[U^2, PQ]=0$.*

Proof. If $T\in(BN)$, then $T^2=U^2(U^*PQU)$ is the polar decomposition of T^2 by Remark 2. Then T^2 is quasinormal if and only if $[U^2, U^*PQU]=0$. In addition, if $N(T)=N(T^*)$, equivalently $N(U)=N(U^*)$, that is, $U^*U=UU^*$, so that

$$[U^2, U^*PQU]=0\Leftrightarrow U^*SU=0 \quad \text{where} \quad S=[U^2, PQ].$$

Then U^*S is zero on $\overline{R(U)}$ and U^*S annihilates on $N(U^*)$ since $N(S)\supset N(Q)=N(U^*)=N(U)$, so that $U^*S=0$ on $H=\overline{R(U)}\oplus N(U^*)$, that is, $S^*U=0$. Similarly S^* is zero on $\overline{R(U)}$ and S^* annihilates on $N(U^*)$ since $N(S^*)\supset N(Q)=N(U^*)=N(U)$ so that $S^*=0$ on H, so the proof is complete.

3. Conditions on an idempotent operator implying projection

We show several conditions under which an idempotent operator implies projection.

Theorem 3. *If T_1 and T_2 are both idempotent operators with the same range. Then*

(i) *T_1 is projection if and only if $[T_1^*T_1, T_2T_2^*]=0$ holds.*

(ii) *T_1 is projection if $M(T_2-1)^*(T_2-1)\geqq(T_1-1)(T_1-1)^*$ holds, where M is a positive constant.*

Proof. It is known [4] that $R(T_1)=R(T_2)$ is closed and T_1 and T_2 can be decomposed into

$$T_1=\begin{pmatrix}1 & S_1\\ 0 & 0\end{pmatrix} \quad\text{and}\quad T_2=\begin{pmatrix}1 & S_2\\ 0 & 0\end{pmatrix}$$

on $R(T_1)\oplus R(T_1)^\perp$.

Proof of (i). Hypothesis of $[T_1^*T_1, T_2T_2^*]=0$ if and only if

$$\begin{pmatrix}1 & S_1\\ S_1^* & S_1^*S_1\end{pmatrix} \text{ commutes with } \begin{pmatrix}1+S_2S_2^* & 0\\ 0 & 0\end{pmatrix}\Leftrightarrow S_1^*+S_1^*S_2S_2^*=0$$
$$\Rightarrow S_1^*S_1+(S_2^*S_1)^*(S_2^*S_1)=0$$
$$\Leftrightarrow S_1=0\Leftrightarrow T_1=1\oplus 0$$

so that T_1 is projection. Conversely, if $T_1=1\oplus 0$, then $[T_1^*T_1, T_2T_2^*]=0$ is easily shown.

Proof of (ii). Hypothesis of (ii) yields

$$M\cdot\begin{pmatrix}0 & 0\\ S_2^* & -1\end{pmatrix}\begin{pmatrix}0 & S_2\\ 0 & -1\end{pmatrix}\geqq\begin{pmatrix}0 & S_1\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\ S_1^* & -1\end{pmatrix},$$

then we have

$$A\geqq 0 \quad\text{where}\quad A=\begin{pmatrix}-S_1S_1^* & S_1\\ S_1^* & M(S_2^*S_2+1)-1\end{pmatrix}.$$

Then for any $x\in R(T_1)$, then $(Ax, x)\geqq 0\Leftrightarrow(-S_1S_1^*x, x)\geqq 0\Leftrightarrow S_1=0$ if and only if $T_1=1\oplus 0$ which is the desired result.

We remark that T_2 not always turns out to be projection in Theorem 3.

Corollary 7. *An idempotent operator T is projection if and only if T satisfies any one of the following*

(1) *T is dominant* (2) *T is humble M-hyponormal* (3) *$T\in\theta$*
(4) *$T\in(WN)$* (5) *T is M-hyponomal* (6) *$T\in(BN)$.*

Proof. (1) in Theorem 3 implies (6) in Corollary 7 and also (2) in Theorem 3 implies (1) and (2) in Corollary 7. (3) and (5) are obtained by (1) and also (4) follows by (2). (1) is shown in [12] and (3) and (6) are shown in [1] and [2].

Other extensions are in [4].

4. Necessary and sufficient conditions under which partial isometry is the direct sum of an isometry and zero

We show several equivalent conditions under which partial isometry is the direct sum of an isometry and zero.

Theorem 4. *Let T be partial isometry. Then T is quasinormal if and only if T satisfies any one of the following*

(1) *T is k-paranormal,* (2) $T\in\theta$, (3) $T\in(WN)$, (4) *T is dominant,* (5) *T is humble M-hyponormal,* (6) *T is M-hyponormal,* (7) $N(T)\subset N(T^*)$, *that is,* $N(T)$ *reduces T.*

Proof. It is shown that "T is partial isometry and quasinormal"$\Leftrightarrow$"Partial isometry T is reduced by $N(T)$", so that we have (7). It is easily shown that all operators T in (4) and (5) are reduced by $N(T)$, so that the proofs of "if" of (4) and (5) are obtained by (7) and the proofs of "only if" of (4) and (5) are trivial. (4) implies (6) and (5) implies (3) and (2) is derived from (4) or (5).

(1) is shown in [5], (2) is shown in [10] and also (3) is shown in [11]. With respect to Theorem 4, we remark that there exists an example U of partial isometry and binormal such that U is not quasinormal and this example is cited in Remark 2.

Addendum

As stated in §1, let $T=U|T|$ be the polar decomposition of T, then $T^*=U^*|T^*|$ is also the polar decomposition of T^* and we would like to cite an elementary proof of this as follows; $TT^*=U|T|\cdot|T|U^*=U|T|U^*\cdot U|T|U^*=(U|T|U^*)^2$, so $|T^*|=U|T|U^*$ and $T^*=|T|U^*=U^*U|T|U^*=U^*|T^*|$ and $N(T^*)=N(|T^*|)=N(U^*)$ because $T^*x=0\leftrightarrow|T|U^*x=0\leftrightarrow UU^*x=0\leftrightarrow U^*x=0$, that is, $T^*=U^*|T^*|$ is also the polar decomposition of T^* since U^* is also partial isometry.

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