YOKOHAMA MATHEMATICAL
JOURNAL VOL. 32, 1984

THE SEMI-FOURIER TRANSFORM

By

TAKUICHI HASEGAWA

(Received June 14, 1984)

The multiple Fourier transform is defined by the use of the inner product $x\cdot y\equiv x_1y_1+\cdots+x_ny_n$, $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)$ of $\boldsymbol{R}^n$. A similar transform can also be defined by using the indefinite inner product $\langle x, y\rangle\equiv -x_1y_1-\cdots-x_jy_j+x_{j+1}y_{j+1}+\cdots+x_ny_n$ of $\boldsymbol{R}_j^n$. For the sake of simplicity we assume $j=1$, and call such a transform the semi-Fourier transform. It is a mixture of one dimensional Fourier transforms and its inverse, and has the almost same properties with the usual Fourier transform except the coefficient (-1). It might be useful in the analytic study of Lorentz manifolds. In this note we state some properties of the semi-Fourier transform and the analogy of Poisson summation formula.

Now we list up some properties of the semi-Fourier transform. Functions are assumed, unless otherwise stated, to be rapidly decreasing.

$$\mathscr{F}_s f(\xi)\equiv\int_{R^n} e^{-2\pi i\langle x,\xi\rangle}f(x)dx=\int_{R^n} e^{-2\pi i(-x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n)}f(x)dx\ ,$$

$$\bar{\mathscr{F}}_s g(x)\equiv\int_{R^n} e^{2\pi i\langle x,\xi\rangle}g(\xi)d\xi\ .$$

(1) $\mathscr{F}_s f(\xi)=\mathscr{F}f(-\xi_1,\xi_2,\cdots,\xi_n)$, $\bar{\mathscr{F}}_s g(x)=\bar{\mathscr{F}}g(-x_1\, x_2,\cdots,x_n)$.

(2) If $f(x)=f_1(x_1)f_2(x_2)\cdots f_n(x_n)$, $f_i\in L^1$, then $\mathscr{F}_s f(\xi)=\check{f}_1(\xi_1)\hat{f}_2(\xi_2)\cdots\hat{f}_n(\xi_n)$, where $\check{f}_1(\xi_1)\equiv\bar{\mathscr{F}}f_1(\xi_1)$.

(3) Let α be a multi index. $\mathscr{F}_s[D^\alpha f](\xi)=(-1)^{\alpha_1}(2\pi i\xi)^\alpha\mathscr{F}_s[f](\xi)$, $|(2\pi\xi)^\alpha\mathscr{F}_s[f](\xi)|\leqq\|D^\alpha f\|_{L^1}$.

(4) $D_\xi^\alpha\mathscr{F}_s[f]=(-1)^{\alpha_1}\mathscr{F}_s[(-2\pi ix)^\alpha f(x)]$, $|D_\xi^\alpha\mathscr{F}_s[f]|\leqq\|(2\pi x)^\alpha f(x)\|_{L^1}$.

(5) $\mathscr{F}_s[f(x-h)](\xi)=e^{-2\pi i\langle h,\xi\rangle}\mathscr{F}_s[f](\xi)$.

(6) $\mathscr{F}_s\bar{\mathscr{F}}_s f=f$, $\mathscr{F}_s\bar{\mathscr{F}}_s g=g$.

(7) $(f,g)_{L^2}=(\mathscr{F}_s f,\mathscr{F}_s g)_{L^2}$, $\|\mathscr{F}_s f\|_{L^2}=\|\bar{\mathscr{F}}_s f\|_{L^2}=\|f\|_{L^2}$.

(8) $\mathscr{F}_s[f*g]=\mathscr{F}_s[f]\mathscr{F}_s[g]$.

(1) is fundamental. Almost the rest is derived from it and the formulae for the usual Fourier transform. Every proof is easy, so we omit it.

Next we treat the analogy of Poisson summation formula. Let Γ be a lattice of $\boldsymbol{R}^n$, $f_1,\cdots,f_n$ be a basis of Γ. Let $\tilde{f}_1,\cdots,\tilde{f}_n$ be the dual basis of Γ with respect to $\langle\ ,\ \rangle$, i.e., $\langle\tilde{f}_1, f_1\rangle=-1$, $\langle f_i, f_j\rangle=0$ $(i\neq j)$, $\langle\tilde{f}_i, f_i\rangle=1$ $(2\leqq i\leqq n)$. Let Γ^* be defined by $\{x\in\boldsymbol{R}^n\,|\,\langle x, y\rangle\in\boldsymbol{Z},\ \forall y\in\Gamma\}$, which is called the dual lattice of Γ (w.r.t. to $\langle\ ,\ \rangle$). $\tilde{f}_1,\cdots,\tilde{f}_n$ are a basis of Γ^*. Let Φ be the linear isomorphism of $\boldsymbol{R}^n$ such that $\Phi(e_i)=f_i$, $1\leqq i\leqq n$, and $\tilde{\Phi}$ be a diffeomorphism from $T_0\equiv\boldsymbol{R}_1^n/\Gamma_0$ to $T\equiv\boldsymbol{R}^n/\Gamma$ such that $\tilde{\Phi}([x])\equiv[\Phi(x)[$, where $e_1,\cdots,e_n$ are the natural basis of $\boldsymbol{R}^n$ and Γ_0 is the lattice defied by $e_1,\cdots,e_n$.

If g is a function on T, then

$$\int_T g(x)v_T(x)=\mathrm{vol}(\Gamma)\int_{T_0} g(\tilde{\Phi}(y))v_{T_0}(y)$$

Thus,

$$\begin{aligned}\int_T g(x)e^{-2\pi i\langle m,x\rangle}v_T(x)&=\mathrm{vol}(\Gamma)\int_{T_0} g(\tilde{\Phi}(y))e^{-2\pi i\langle m,\tilde{\Phi}(y)\rangle}v_{T_0}(y)\\ &=\mathrm{vol}(\Gamma)\int_{[0,1]^n} g(\tilde{\Phi}(y_1,\cdots,y_n))e^{-2\pi i(-m_1y_1+m_2y_2+\cdots+m_ny_n)}dy_1\cdots dy_n\\ &=\mathrm{vol}(\Gamma)\int_{[0,1]^n} g(\tilde{\Phi}(y))e^{-2\pi i\langle\Psi(m),y\rangle}dy\ ,\end{aligned}$$

where $m=m_1\tilde{f}_1+\cdots+m_n\tilde{f}_n$, $\Psi\colon \boldsymbol{R}^n\overset{\simeq}{\rightarrow}\boldsymbol{R}^n$, $\Psi(\tilde{f}_i)\equiv e_i$. From this we obtain that

$$\left\{\frac{1}{\sqrt{\mathrm{vol}(\Gamma)}}e^{2\pi i\langle m,x\rangle}\,|\,m\in\Gamma^*\right\}\ \text{is an orthonormal system in } L^2(T_x)\ .$$

And, on the other hand, we know that

$$\{e^{2\pi i\langle m,x\rangle}\,|\,m\in\boldsymbol{Z}^n\}\ \text{is a complete orthonormal system in } L^2(T_{0,x})\ .$$

Thus we can conclude that

$$\left\{\frac{1}{\sqrt{\mathrm{vol}(\Gamma)}}e^{2\pi i\langle m,x\rangle}\,|\,m\in\Gamma^*\right\}\ \text{is complete in } L^2(T_x)\ .$$

Theorem

Let f be a rapidly decreasing function on $\boldsymbol{R}^n$ and Γ be an arbitrary lattice of $\boldsymbol{R}^n$, then

$$\sum_{k\in\Gamma} f(k)=\frac{1}{\mathrm{vol}(\Gamma)}\sum_{m\in\Gamma^*}\mathscr{F}_{\iota}f(m)$$

The proof goes in the same way as in [1], p. 157, but, where we must replace the inner product (1) by the indefinite inner product $\langle\ ,\ \rangle$ and must use the above Fourier expansion on a Lorentzian flat Torus $\boldsymbol{R}_1^n/\Gamma=T$ instead of the usual Fourier expansion.

Reference

[1] M. Berger, P. Gauduchon et E. Mazet, *Le Spectre d'une Variété riemannienne*, Lect. Notes in Math. Springer-Verlag, 194.

498 Ueshimo-cho
Sano-shi Tochigi
327 Japan