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SPECTRA OF GEODESICS ON A LOCALLY SYMMETRIC SPACE

By

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Let M be an n-dimensional locally symmetric space, p a point of M. Let $\gamma: [0, l]\rightarrow M$, be a geodesic parametrized by arc length, and $\gamma(0)=p$, $\dot{\gamma}(t):=V(t)$. Let J be a Jacobi operator on γ acting on a normal vector field W along γ, vanishing at $\gamma(0)$ and $\gamma(1)$.

$$J=-\nabla_V\nabla_V-K_V\ ,$$

where $K_V(W):=R(V,W)V$, R is the curvature tensor of M. Let

$$\mathrm{Spec}(\gamma):=\{\lambda\mid JW=\lambda W,\ \exists W\neq 0,\ W(0)=0,\ W(l)=0\}$$

and it is called the spectrum of γ. Let $e_1, \cdots, e_n$ be the eigenvalues of the symmetric endomorphism $K_{V(0)}$ of TM_p. Then, just as the conjugate points of p along γ are determined by $e_1, \cdots, e_n$ ([1], §20), the spectrum of γ is also determined by them.

Theorem.

$$\mathrm{Spec}(\gamma)=\left\{\left(\frac{k\pi}{l}\right)^2-e_i\mid k\in N,\ 1\leqq i\leqq n\right\}$$

From this Theorem we know that in a space of constant curvature, geodesics with the same length have the same spectrum.

Proof. Let $U_1, \cdots, U_n$ be an orthonormal basis of TM_p such that $K_{V(0)}U_i=e_iU_i$, $1\leqq i\leqq n$. Extend U_i along γ as a parallel vector field. Then, since M is locally symmetric,

$$R(V(t),U_i(t))V(t)=e_iU_i(t)\ ,\quad 0\leqq t\leqq l\ .\quad 1\leqq i\leqq n$$

holds. Any normal vector field W along γ is expressed uniquely as

$$W(t)=\sum_{i=1}^{n} w_i(t)U_i(t)\ .$$

Then, $JW=\lambda W$, $W(0)=0$, $W(l)=0$ is equivalent to

$$\frac{d^2w_i}{dt^2}(t)+(\lambda+e_i)w_i(t)=0\ ,\quad w_i(0)=0\ ,\quad w_i(l)=0\ ,\quad 0\leqq t\leqq l\ ,\quad 1\leqq i\leqq n\ .$$

The solutions to these equations are

$$w_i(t)=c_i \sin\left(\frac{k\pi}{l}t\right), \quad \lambda=\left(\frac{k\pi}{l}\right)^2-e_i, \quad k\in N.$$

Thus the Theorem is proved.

Reference

[1] J. Milnor, *Morse Theory*, Princeton University Press, 1963.

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