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ON THE POSITION OF A CONJUGATE POINT OF A REFLECTED GEODESIC IN E^2 AND E^3

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In the Euclidean two plane E^2, geodesics are straight lines and they have no conjugate points along themselves. However, if geodesics are reflected against some boundary curve, they might have some conjugate point. The typical example of this phenomenon is given by the case that the boundary curve is an ellipse and the starting point of the geodesic is its one focus. Then the other focus is a conjugate point of the reflected geodesic. In this note we search the position (parameter value) of a conjugate point of a reflected geodesic in the Euclidean two and three planes E^2 and E^3.

First we treat the E^2 case.

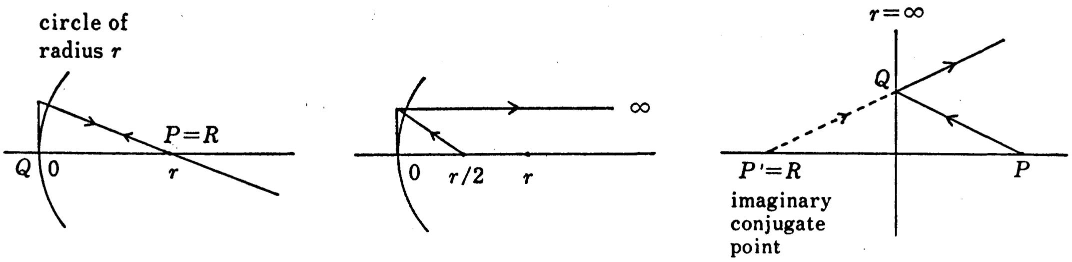
Let P be the starting point of a geodesic c, Q be its reflecting point against the boundary curve f, and we denote the reflected part of the geodesic by d. The c, d, and f are parametrized by arc length. Let $P=c(0)$ and $Q=c(a)=d(a)$. Let θ be the angle at Q between f and c $(0\leqq\theta<\pi/2)$, and k be the curvature of f at Q. If s_0 denote the parameter value of the conjugate point of P along c and d, then

Theorem 1. $s_0=2a^2k/\cos\theta+2ak.$

$s_0-a=-a\cos\theta/\cos\theta+2ak$. Let r be the radius of osculating circle of f at Q, then $k=-1/r$. And $r<2a/\cos\theta\Rightarrow s_0-a>0$, $r=2a/\cos\theta\Rightarrow$ without conjugate point, $r>2a/\cos\theta\Rightarrow$ the conjugate point is imaginary.

A conjugate point R of P along c and d is defined by the one where a non-zero admissible normal Jacobi field along c and d vanishes at P and R. And an admissible normal Jacobi field V is defined by the condition that it is a usual Jacobi field along c and d which is normal to c and d, and at the boundary point Q, $V'(a-)-V'(a+)+S_{T(a-)-T(a+)}\tilde{V}(a)$ is normal to f where S is the second fundamental form of f and $\tilde{V}(a)$ is such a unique vector tangent to f that its orthogonal projection to $\perp T(a-)$ is equal to $V(a-)$ ($\perp T(a-)$ is the orthogonal complement of $T(a-)$, and $T(a-)\equiv\dot{c}(a-)$, $T(a+)\equiv\dot{d}(a+)$).

Let us figure some cases when $\theta=0$.

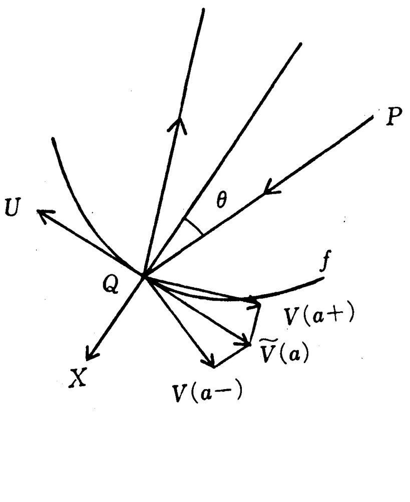


Proof. Let U be the velocity vector of f at Q, X be the unit vector normal to f such that $T(a-)-T(a+)=2\cos\theta\cdot X$. Let 1 be the length of $V(a-)$. Then $\langle V(a-), U\rangle=-1\cos\theta$, $\langle V(a-), X\rangle=1\sin\theta$, thus we have $V(a-)=-1\cos\theta\cdot U+1\sin\theta\cdot X$, $V(a+)=V(a-)-1\tan\theta(T(a-)-T(a+))=-1\cos\theta\cdot U-1\cos\theta\cdot X$. Let $\bar{\nabla}$ be the covariant differentiation of E^2, and e_1 and e_2 be the Frenet frame of f around Q; $e_1(0)=U$, $e_2(0)=X$. Then, $\tilde{V}(a)=-(l/\cos\theta)U$, $\bar{\nabla}_{\tilde{V}(a)}(T(a-)-T(a+))=\bar{\nabla}_{\tilde{V}(a)}(2\cos\theta e_2)=-21\bar{\nabla}_{e_1(0)}e_2=2k1U$, $S_{T(a-)-T(a+)}\tilde{V}(a)=-2k1U$. If we denote by V^t the tangential part to f of the vector V, then, since $V'(a-)-V'(a+)+S_{T(a-)-T(a+)}\tilde{V}(a)$ is normal to f, $V'(a+)^t=V'(a-)^t+S_{T(a-)-T(a+)}\tilde{V}(a)$. Thus,

$$V'(a+)^t=V'(a-)^t+\langle V'(a-), T(a+)\rangle\frac{T(a)-T(a+)}{2(1+\cos 2\theta)}+S_{T(a-)-T(a+)}\tilde{V}(a)\,,$$

$$\begin{aligned}V'(a+)&=V'(a+)^t+\frac{\langle V'(a+)^t, T(a+)\rangle}{1+\cos 2\theta}(T(a-)-T(a+))\\&=V'(a-)+S_{T(a-)-T(a+)}\tilde{V}(a)\\&\quad+\langle V'(a-)+S_{T(a-)-T(a+)}\tilde{V}(a),\ T(a+)\rangle\frac{T(a-)-T(a+)}{1+\cos 2\theta}\,.\end{aligned}$$

In this way, $V'(a+)$ is determined by $V(a-)$ and $V'(a-)$. If $W(s)$ $(s\geqq 0)$ denotes the reflected part of the Jacobi field V, then $W(s)=V(a+)+(s-a)V'(a+)$. $W(s)=-1\{\cos\theta+(s-a)(1/a\cos\theta+2k)\}U-1\sin\theta\{1+(s-a)(1/a+2k(1/\cos\theta))\}X$. Thus



$W(s_0)=0$ for some $s_0 \geqq 0 \Leftrightarrow$

$$\cos\theta+(s_0-a)\left(\frac{1}{a}\cos\theta+2k\right)=0 \quad \text{and} \quad 1+(s_0-a)\left(\frac{1}{a}+\frac{2k}{\cos\theta}\right)=0$$

$$\Leftrightarrow s_0=\frac{2a^2k}{2ak+\cos\theta}\,. \qquad \square$$

Now we treat the E^3 case.

Let P be the starting point of a geodesic c in E^3, Q be its reflecting point against the boundary surface M, and we denote the reflected part of the geodesic by d. The c and d are parametrized by arc length. Let $P=c(0)$ and $Q=c(a)=d(a)$. Let θ be the angle at Q between c and M. M is parametrized by a map $\boldsymbol{P}$ around Q, its parameter u and v. And we assume $\langle \boldsymbol{P}_u, \boldsymbol{P}_u\rangle=1$, $\langle \boldsymbol{P}_u, \boldsymbol{P}_v\rangle=0$, $\langle \boldsymbol{P}_v, \boldsymbol{P}_v\rangle=1$ at Q, where $\langle\ ,\ \rangle$ is the inner product of E^3. $\boldsymbol{P}_u$ and $\boldsymbol{P}_v$ can be taken to be principal vectors with principal curvature k_1 and k_2. Set $\boldsymbol{e}\equiv\boldsymbol{P}_u\times\boldsymbol{P}_v$. Then we know that $\langle \boldsymbol{P}_u, \boldsymbol{e}_u\rangle=-k_1$, $\langle \boldsymbol{P}_u, \boldsymbol{e}_v\rangle=\langle \boldsymbol{F}_v, \boldsymbol{e}_u\rangle=0$, $\langle \boldsymbol{P}_v, \boldsymbol{e}_v\rangle=-k_2$, $\boldsymbol{e}_u=-k_1\boldsymbol{P}_u$, and $\boldsymbol{e}_v=-k_2\boldsymbol{P}_v$. Let V be a non-zero admissible normal Jacobi field along c vanishing at P and W be its reflection along d. Let 1 be the length of a vector $V(a-)$, and η be the angle between $V(a-)$ and $\tilde{V}(a)$. We set $V(a-)^t\equiv a(b\boldsymbol{P}_u+c\boldsymbol{P}_v)$. And we say that $\tilde{V}(a)$ is a *conjugate direction* if $W(s_0)=0$ for some $s_0\geqq 0$. Note that if $0\leqq s_0<a$, then the conjugate point is imaginary·

Theorem 2. *In case $\theta=0$, a conjugate direction coincides with a principal direction. The parameter value s_0 from P of the conjugate point is given by $s_0=2a^2k/1+2ak$ where k is the principal curvature of the direction.*

Proof. $\eta=0$ and $V(a-)^t=V(a-)=\tilde{V}(a)=W(a+)=ab\boldsymbol{P}_u+ac\boldsymbol{P}_v$. Since $V'(a-)-W'(a+)+S_{T(a-)-T(a+)}\tilde{V}(a)$ is normal to M, $V'(a-)-W'(a+)+S_{T(a-)-T(a+)}\tilde{V}(a)=0$, $W'(a+)=V'(a-)+S_{T(a-)-T(a+)}\tilde{V}(a)$. And since V is a Jacobi field in E^3, $V(t)=(t/a)V(a-)$. Thus $V'(t)=(1/a)V(a-)=(1/a)V(a-)^t=b\boldsymbol{P}_u+c\boldsymbol{P}_v$. On the other hand, from $T(a-)-T(a+)=2e$, we obtain $\bar{\nabla}_{\tilde{V}(a)}(T(a-)-T(a+))=2\bar{\nabla}_{ab\boldsymbol{P}_u+ac\boldsymbol{P}_v}\boldsymbol{e}=2a(b\boldsymbol{e}_u+c\boldsymbol{e}_v)=-2a(bk_1\boldsymbol{P}_u+ck_2\boldsymbol{P}_v)$. Consequently, $S_{T(a-)-T(a+)}\tilde{V}(a)=2a(bk_1\boldsymbol{P}_u+ck_2\boldsymbol{P}_v)$, and $W'(a+)=b(1+2ak_1)\boldsymbol{P}_u+c(1+2ak_2)\boldsymbol{P}_v$. $W(s)=W(a+)+(s-a)W'(a+)=b\{a+(s-a)(1+2ak_1)\}\boldsymbol{P}_u+c\{a+(s-a)(1+2ak_2)\}\boldsymbol{P}_v$. $W(s_0)=0\Leftrightarrow 0=b\{a+(s_0-a)(1+2ak_1)\}$ and $0=c\{a+(s_0-a)(1+2ak_2)\}$.

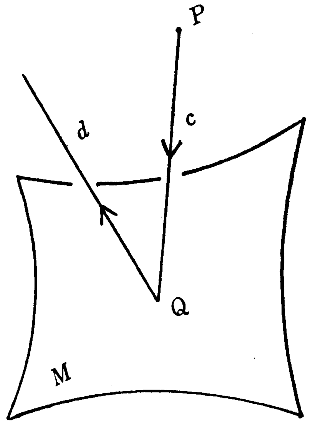
(1) Case $b=0$; $c\neq 0$, thus $s_0=a-a/1+2ak_2=2a^2k_2/1+2ak_2$. $W(s)=ca(s_0-s/s_0-a)\boldsymbol{P}_v$, $\tilde{V}(a)=ac\boldsymbol{P}_v$. Thus the principal vector $\boldsymbol{P}_v$ is a conjugate direction, and its parameter value s_0 of the conjugate point is given by $s_0=2a^2k_2/1+2ak_2$.

(2) Case $c=0$; $b\neq 0$, and the principal vector $\boldsymbol{P}_u$ is a conjugate direction and its parameter value s_0 of the conjugate point is given by $s_0=2a^2k_1/1+2ak_1$.

(3) Case $b \neq 0$ and $c \neq 0$; then $k_1 = k_2 \equiv k$, $s_0 = 2a^2k/1+2ak$, $W(s) = (s_0 - s/s_0 - a)\tilde{V}(a)$. Conversely in case $k_1 = k_2 \equiv k$, setting $s_0 \equiv 2a^2k/1+2ak$, we obtain $W(s) = (s_0 - s/s_0 - a)\tilde{V}(a)$ for any b and c. Thus in this case the conjugate point occurs if and only if the point Q is umbilic, and any direction is conjugate.

(1), (2), and (3) are summerized as in the statement of the Theorem. □

Theorem 3. *In case $\theta \neq 0$ and Q is umbilical with the principal curvature $k \neq 0$, there are two conjugate directions. Namely the one is the same direction with c and d, in which case the parameter value of the conjugate point is $s_0 = 2a^2k/\cos\theta + 2ak$, and the other is perpendicular to c and d, in that case the parameter value is $s_0 = 2a^2k\cos\theta/1+2ak\cos\theta$.*



Proof. We take $\boldsymbol{P}_u$ to be normal to $T(a-)$ and $T(a+)$, i.e., $\langle T(a-), \boldsymbol{P}_u\rangle = 0$ and $\langle T(a-), \boldsymbol{P}_v\rangle = \sin\theta$. Since $T(a-) = \sin\theta\cdot\boldsymbol{P}_v + \cos\theta\cdot\boldsymbol{e}$ and $V(a-) = ab\boldsymbol{P}_u + ac\boldsymbol{P}_v + 1\tan\eta\cdot\cos\theta\cdot\boldsymbol{e}$, we have $\tilde{V}(a) = V(a-) - 1\tan\eta\cdot T(a-) = ab\boldsymbol{P}_u + (ac - 1\tan\eta\cdot\sin\theta)\boldsymbol{P}_v$. And $T(a-) - T(a+) = 2\cos\theta\cdot\boldsymbol{e}$, thus $\bar{\nabla}_{\tilde{V}(a)}(T(a-) - T(a+)) = 2\cos\theta\cdot\bar{\nabla}_{ab\boldsymbol{P}_u + (ac - l\tan\eta\sin\theta)\boldsymbol{P}_v}\boldsymbol{e} = 2\cos\theta\{ab\boldsymbol{e}_u + (ac - 1\tan\eta\cdot\sin\theta)\boldsymbol{e}_v\} = -2\cos\theta\cdot\{abk_1\boldsymbol{P}_u + (ac - 1\tan\eta\cdot\sin\theta)k_2\boldsymbol{P}_v\}$, $S_{T(a-)-T(a+)}\tilde{V}(a) = 2\cos\theta\cdot\{abk_1\boldsymbol{P}_u + (ac - 1\tan\eta\cdot\sin\theta)k_2\boldsymbol{P}_v\}$. Since $V(a-) - W(a+) + S_{T(a-)-T(a+)}\tilde{V}(a)$ is normal to M, we have $V(a-)^t - W(a+)^t + S_{T(a-)-T(a+)}\tilde{V}(a) = 0$, $W(a+)^t = V(a-)^t + S_{T(a-)-T(a+)}\tilde{V}(a)$. On the other hand, $V(t) = (t/a)V(a-)$, $V'(a-) = (1/a)V(a-)$, $V'(a-)^t = (1/a)V(a-)^t = (1/a)V(a-) - l/a\tan\eta\cdot\cos\theta\cdot\boldsymbol{e}$. Thus, $W(a+)^t = (1/a)V(a-) - l/a\tan\eta\cdot\cos\theta\cdot\boldsymbol{e} + S_{T(a-)-T(a+)}\tilde{V}(a) = \{b + 2abk_1\cos\theta\}\boldsymbol{P}_u + \{c + 2k_2(ac - 1\tan\eta\cdot\sin\theta)\cos\theta\}\boldsymbol{P}_v$. Let η' be the angle between $W'(a+)$ and $W'(a+)^t$, and $1'$ be the length of the vector $W'(a+)^t$, then $W'(a+) = W'(a+)^t - 1'\tan\eta'\cdot\boldsymbol{e}$. Since $W(a+) = ab\boldsymbol{P}_u + ac\boldsymbol{P}_v - 1\tan\eta\cos\theta\cdot\boldsymbol{e}$, we have $W(s) = W(a+) + (s-a)W'(a+) = \{ab + (s-a)(b + 2abk\cos\theta)\}\boldsymbol{P}_u + \{ac + (s-a)[c + 2k\cos\theta(ac - 1\sin\theta\tan\eta)]\}\boldsymbol{P}_v - (1\tan\eta\cos\theta + (s-a)1'\tan\eta')\boldsymbol{e}$. Consequently

$$W(s_0)=0\Leftrightarrow\begin{cases}① & 0=b\{a+(s_0-a)(1+2ak\cos\theta)\}\\ ② & 0=ac+(s_0-a)[c+2k\cos\theta(ac-1\tan\eta\sin\theta)]\\ ③ & 0=1\tan\eta\cos\theta+(s_0-a)1'\tan\eta'\ .\end{cases}$$

(1) Case $b=0$; $V(a-)^t=ac\boldsymbol{P}_v$, $c<0$, $\eta=\theta$, $1=-ac/\cos\theta$. Thus, from ②, we obtain $s_0=2a^2k/\cos\theta+2ak$.

(2) Case $c=0$; $V(a-)^t=ab\boldsymbol{P}_u$, $b\neq 0$, $\eta=0$. From ①, we obtain $s_0=2a^2k\cos\theta/1+2ak\cos\theta$.

(3) Case $b\neq 0$, *and* $c\neq 0$; From ① and ②, $s=2ak\cos\theta/c+2k\cos\theta=2a^2k\cos\theta/1+2ak\cos\theta$. Thus $k\tan\eta=0$. Since $k\neq 0$, it follows that $\eta=0$, which contradicts $c\neq 0$. Therefore in this case there are no conjugate directions.

(1), (2), and (3) are summerized as in the statement of the Theorem. □

Theorem 4. *In case $k_1=k_2=0$, any direction is a conjugate direction and its parameter value of the conjugate point is $s_0=0$.*

Proof. The Proof of Theorem 3 can be applied except for a little modification of (3). □

Reference

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