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SPECTRAL GEOMETRY OF COMPACT RIEMANNIAN MANIFOLDS WITH BOUNDARY

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This paper is concerned with spectral geometry of compact Riemannian manifolds with boundary under Dirichlet and Neumann boundary conditions. Under these conditions we can take out constant curvature property of the interior and totally umbilical property of the boundary, and using these two properties we can characterize some canonical domains in simply connected space forms by their Dirichlet and Neumann spectra. In §1 we obtain some spectral invariants and spectral properties. §2 is devoted to the characterizations of some canonical domains. §3 treats the Kaehlerian case.

§1. Some Spectral Invariants and Spectral Properties

Let M be an m-dimensional compact Riemannian manifold with smooth boundary ∂M. Let $\tilde{M}$ be a closed double of M. Let Δ and $\tilde{\Delta}$ be the Laplacians of M and $\tilde{M}$ acting on functions. Let $\{\lambda_1^- \leqq \lambda_2^- \leqq \cdots\}$ and $\{\lambda_1^+ \leqq \lambda_2^+ \leqq \cdots\}$ be the spectra of Δ under Dirichlet and Neumann boundary conditions respectively. If e denotes the fundamental solution of the heat operator $\partial/\partial t+\tilde{\Delta}$, and e^- and e^+ denote the fundamental solution of $\partial/\partial t+\Delta$ under Dirichlet and Neumann conditions, then ([4])

$$e^{\pm}(t, x, x)=e(t, x, x)\pm e(t, x, x^*)\ , \quad x\in M$$

where x^* being the double point of x. If we set $Z^{\pm}(t)=\sum_{i=1}^{\infty} e^{-\lambda_i^{\pm} t}$, then

$$Z^{\pm}(t)=\int_M e^{\pm}(t, x, x)=\int_M e(t, x, x)\pm\int_M e(t, x, x^*)$$

$\int_M e(t, x, x)$ and $\int_M e(t, x, x^*)$ admit the following asymptotic expansions for $t\downarrow 0$:

$$\int_M e(t, x, x)\sim(4\pi t)^{-m/2}(a_0+a_2t+a_4t^2+\cdots)\ ,$$

$$\int_M e(t, x, x^*)\sim(4\pi t)^{-m/2}(b_1t^{1/2}+b_2t+b_3t^{3/2}+b_4t^2+\cdots)\ .$$

Thus we obtain

$$Z^{\pm}(t)\underset{t\downarrow 0}{\sim}(4\pi t)^{-m/2}(a_0\pm b_1t^{1/2}+(a_2\pm b_2)t\pm b_3t^{3/2}+(a_4\pm b_4)t^2+\cdots)\,.$$

The geometrical meanings of some coefficients are already known ([2], [3]) as follows:

$$a_0=\mathrm{vol}(M)\,,\quad b_1=\frac{\sqrt{\pi}}{2}\mathrm{area}(\partial M)\,,\quad a_2\frac{1}{6}\int_M\bar{\tau}\,,\quad b_2=-\frac{1}{3}\int_{\partial M}H\,,$$

$$b_3=-\int_{\partial M}\sqrt{\pi}\left(-\frac{1}{12}\tau-\frac{1}{8}\tilde{\tau}+\frac{13}{192}H^2-\frac{5}{96}S\right),$$

$$a_4=\frac{1}{360}\int_M(2|\bar{R}|^2-2|\bar{\rho}|^2+5\bar{\tau}^2+12\Delta\bar{\tau})\,,$$

$$=\frac{1}{360}\int_M\left(2|\bar{C}|^2+\frac{2(6-m)}{m-2}|\bar{G}|^2+\frac{5m^2-7m+6}{m(m-1)}\bar{\tau}^2+12\Delta\bar{\tau}\right).$$

Here we explain notations used above. $\bar{R}, \bar{\rho}, \bar{\tau}, \bar{C}$, and $\bar{G}$ are curvature tensor, Ricci tensor, scalar curvature, Weyl's conformal curvature tensor, and Einstein tensor of M respectively. And H, τ, S are the mean curvature, scalar curvature, square norm of the second fundamental tensor of M respectively. $\tilde{\tau}$ is defined by $\sum_{i=1}^{m}\bar{R}_{imim}$.

Let D-Spec(M) and N-Spec(M) denote the spectra of Δ under Dirichlet and Neumann boundary conditions respectively. Then all the a_{2i} and b_j are spectral invariants of D-Spec(M) and N-Spec(M), i.e. if D-Spec(M)=D-Spec(M') and N-Spec(M)=N-Spec(M') hold, then $a_{2i}=a_{2i}'$ and $b_j=b_j'$ hold.

In the same way as in [5] and [3], we can take out constant curvature property of the interior by the use of a_0, a_2 and a_4, and totally umbilical property of the boundary by the use of b_1, b_2 and b_3, as the following propositions will show. We shall omit their proofs since they are easy. ν will denote outer unit normal vector field of ∂M, and we note that in their proofs Green's theorem is used.

Proposition 1.1. *Suppose that* D-Spec(M)=D-Spec(M') *and* N-Spec(M)=N-Spec(M') *hold. If* $m=3$, *M has a constant curvature k, ∂M is totally umbilic, and* $\int_{\partial M'}d\tilde{\tau}'/d\nu'\leqq 0$ *holds, then M' has also the constant curvature k and $\partial M'$ is totally umbilic.*

Proposition 1.2. *Suppose that* D-Spec(M)=D-Spec(M') *and* N-Spec(M)=N-Spec(M') *hold. If* $m=4$ *or* 5, *M has a constant curvature k, H' is constant and* $\int_{\partial M'}d\tilde{\tau}'/d\nu'\leqq 0$ *holds, then M' has also the constant curvature k, $\partial M'$ is totally umbilic, and $H=H'$ holds.*

Proposition 1.3. *Suppose that* D-Spec(M)=D-Spec(M') *and* N-Spec(M)=N-Spec(M') *hold. If* $3\leqq m\leqq 5$, *M has a constant curvature k, ∂M is totally geodesic, $H'\geqq 0$ and* $\int_{\partial M'} d\bar{\tau}/d\nu'\leqq 0$ *hold, then M' has also the constant curvature k and $\partial M'$ is totally geodesic.*

Proposition 1.4. *Suppose that* D-Spec(M)=D-Spec(M') *and* N-Spec(M)=N-Spec(M') *hold. If, for an arbitrary m, M has a constant curvasure k, ∂M is totally umbilic, M' is Einstein, and H' is constant, then M' has also the constant curvature k $\partial M'$ is totally umbilic.*

§2. Characterizations of Some Canonical Domains

Let D^{2m} be an m- dimensional unit disk in R^m, and D_r^m denotes a circular domain of radius r in $S^m(0\leqq r<\pi)$. Let M be an abstract compact Riemannian manifold with smooth boundary. Note that M needs not to be a campact domain in R^m.

Theorem 2.1. *Suppose that* D-Spec(M) = D-Spec(D^3) *and* N-Spec(M) = N-Spec(D^3) *hold. If* $\int_{\partial M} d\bar{\tau}/d\nu\leqq 0$, *then* $M=D^3$.

Proof. By Proposition 1.1 M is flat and ∂M is totally umbilic. Let $\tilde{M}$ be a universal covering manifold of M. Then by the metric induced from that of M, $\tilde{M}$ becomes flat and $\partial\tilde{M}$ is totally umbilic with the same umbilicity with D^3. Then by a theorem of Alexander ([1]), $\bar{M}=D^3$. On the other hand vol(M)=vol(D^3), thus vol(M)=vol($\bar{M}$) and the covering degree is 1, i.e. M is simply connected. Then by the same argument as applied for $\bar{M}$, $M=D^3$ □

In the same reasoning as in the proof of Theorem 2.1, using one of Propositions in §1, we get the following spectral characterizations of D^m and D_r^m.

Theorem 2.2. *Suppose that* D-Spec(M) = D-Spec(D^m) *and* N-Spec(M) = N-Spec(D^m) *hold. If $m=4$ or 5, H is constant, and* $\int_{\partial M} d\bar{\tau}/d\nu\leqq 0$ *holds, then* $M=D^m$.

Theorem 2.3. *Suppose that* D-Spec(M) = D-Spec(D^m) *and* N-Spec(M) = N-Spec(D^m) *hold. If $m\geqq 3$, M is Einstein, and H is constant, then,* $M=D^m$.

Theorem 2.4. *If* D-Spec(M) = D-Spec(D_r^3), N-Spec(M) = N-Spec(D_r^3), and $\int_{\partial M} d\bar{\tau}/d\nu\leqq 0$ *hold, then* $M=D_r^3$.

Theorem 2.5. *Suppose that* D-Spec(M) = D-Spec(D_r^m) *and* N-Spec(M) = N-Spec(D_r^m) *hold. If $m=4$, or 5, H is constant, and* $\int_{\partial M} d\bar{\tau}/d\nu\leqq 0$ *holds, then* $M=D_r^m$.

Theorem 2.6. *Suppose that* D-Spec$(M)=$D-Spec$(D^m_{\pi/2})$ *and* N-Spec$(M)=$N-Spec$(D^m_{\pi/2})$ *hold. If* $3\leqq m\leqq 5$, $H\geqq 0$, *and* $\int_{\partial M} d\bar{\tau}/d\nu\leqq 0$ *holds, then* $M=D^m_{\pi/2}$.

Theorem 2.7. *Suppose that* D-Spec$(M)=$D-Spec$(D_r{}^m)$ *and* N-Spec$(M)=$N-Spec$(D_r{}^m)$ *hold. If* $3\leqq m$, *M is Einstein, and H is constant, then* $M=D_r{}^m$.

Theorem 2.8. *Suppose that* D-Spec$(M)=$D-Spec$(D^m_{2/\pi})$ *and* N-Spec$(M)=$N-Spec$(D^m_{2/\pi})$ *hold. If M is Einstein and* $H\geqq 0$, *then* $M={}^m_{\pi/2}$.

Finally we note that we can give similar spectral characterizations of the canonical umbilical domain in a Hyperbolic Space.

§3. Kaehlerian Case

Let M be a complex m-dimensional compact Kaehler manifold with smooth boundary. Let B be the Bochner curvature tensor of M, then $|\bar{R}|^2=|\bar{B}|^2+(8/m+2)|\bar{G}|^2+(2/m(m+2))\bar{\tau}^2$ and a_4 becomes

$$a_4=\frac{1}{360}\int_M\left(2|\bar{B}|^2+\frac{2(6-m)}{m+2}|\bar{G}|^2+\frac{5m^2+4m+3}{m(m+1)}\bar{\tau}^2+12\Delta\bar{\tau}\right).$$

Then as in §1 we obtain the following Propositions.

Proposition 3.1. *Suppose that* D-Spec$(M)=$D-Spec(M') *and* N-Spec$(M)=$N-Spec(M') *hold. If* $2\leqq m\leqq 5$, *M has a constant holomorphic curvature k,* ∂M *is totally umbilic,* H' *is constant, and* $\int_{\partial M'} d\bar{\tau}'/d\nu'\leqq 0$ *holds, then* M' *has also the constant holomorphic curvature k and* $\partial M'$ *is totally umbilic.*

Proposition 3.2. *Suppose that* D-Spec$(M)=$D-Spec(M') *and* N-Spec$(M)=$N-Spec(M') *hold. If* $2\leqq m\leqq 5$, *M has a constant holomorphic curvature k,* ∂M *is totally geodesic,* $H'\geqq 0$, *and* $\int_{\partial M'} d\bar{\tau}'/d\nu'\leqq 0$ *hold, then* M' *has also the constant holomorphic curvature k and* $\partial M'$ *is totally geodesic.*

Proposition 3.3. *Suppose that* D-Spec$(M)=$D-Spec(M') *and* N-Spec$(M)=$N-Spec(M') *hold. If, for an arbitrary m, M has a constant holomorphic curvature k,* ∂M *is totally umbilic,* M' *is Einstein, and* H' *is constant, then* M' *has also the constant holomorphic curvature k,* $\partial M'$ *is totally umbilic.*

We can regard D^{2m} as a compact Kaehler manifold of constant holomorphic curvature 0 with totally umbilic boundary. Then, as in §2, using the above Propositions, we can easily prove the following Theorems.

Theorem 3.4. *Suppose that* D-Spec$(M)=$D-Spec(D^{2m}) *and* N-Spec$(M)=$N-Spec(D^{2m}) *hold. If* $2\leqq m\leqq 5$, *H is constant, and* $\int_{\partial M} d\bar{\tau}/d\nu\leqq 0$ *hold, then* $M=D^{2m}$.

Theorem 3.5. *Suppose that* D-Spec(M) = D-Spec(D^{2m}) *and* N-Spec(M) = N-Spec(D^{2m}) *hold. If* $2 \leqq m$*, M is Einstein, and H is constant, then* $M = D^{2m}$.

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