YOKOHAMA MATHEMATICAL
JOURNAL VOL. 32, 1984

A COUNTER EXAMPLE TO A RESULT ON APPROXIMATE POINT SPECTRA OF POLAR FACTORS OF HYPONORMAL OPERATORS

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(Received June 14, 1984)

ABSTRACT. We show a counter example to Rai [2] on approximate point spectra of polar factors of hyponormal operators.

1. An operator T means a bounded linear operator on a separable Hilbert space H. Let $\sigma(T)$ denote the spectrum of T, $A_{\sigma}(T)$ the approximate point spectrum of T and $\Gamma(T)$ the compression spectrum of T respectively. An operator T is said to be hyponormal if $T^{*}T\geqq TT^{*}$. Using an elegant technique of the Cayley transform of a self adjoint operator, Putnam [1] shows Theorem A.

Theorem A [1]. *Let T be hyponormal and suppose that*

$$r\in\sigma(T^{*}T) \quad (\textit{hence } r\geqq 0)\ .$$

Then there exists a $z\in\sigma(T)$ for which $|z|=r^{1/2}$.

In order to attempt a precise estimation of Theorem A, Rai [2] states the following Theorem B.

Theorem B [2]. *Let T be hyponormal and suppose*

$$r\in A_{\sigma}(T^{*}T) \quad (\textit{hence } r>0)\ .$$

Then there exists a $z\in A_{\sigma}(T)$ for which $|z|=r^{1/2}$.

However we show a counter example to Theorem B [2] as follows.

Example. Put $H=l^{2}$. Let T denote a weighted right shift in l^{2} defined by $Te_{1}=(1/2)e_{2}$ and $Te_{j}=e_{j+1}$ $(j=2,3,\cdots)$ where (e_{j}) is an orthonormal basis of l^{2}. Then $\sigma(T)=D$, $A_{\sigma}(T)=C$ and $\Gamma(T)=D-C$ where D denote the closed unit disk and also C denotes the unit circle.

Proof. (i) $|\lambda|<1$. Put $x=(1,2\lambda,2\lambda^{2},2\lambda^{3},\cdots)$, then $T^{*}x=\lambda x$, so that $\bar{\lambda}\in\overline{P_{\sigma}(T^{*})}=\Gamma(T)$ namely $\Gamma(T)\supset D-C$. Clearly $||T||=1$ and the boundary of $\sigma(T)$ is included in $A_{\sigma}(T)$, that is, $A_{\sigma}(T)\supset C$.

(ii) $0\leqq|\lambda|<1/2$. Then we have $\|Tx-\lambda x\|\geqq|\,\|Tx\|-\|\lambda x\|\,|\geqq|1/2-|\lambda|\,|\,\|x\|$ for any x, that is, $\lambda \notin A_{\sigma}(T)$ for $0 \leqq |\lambda| < 1/2$.

(iii) $1/2 \leqq |\lambda| < 1$. For any unit vector $x = (a_1, a_2, \cdots)$, by Schwarz's inequality we have

$$\begin{aligned} 4\|Tx - \lambda x\|^2 &= 4|\lambda|^2 + 4 - 3|a_1|^2 - 4\,\mathrm{Re}(\lambda \bar{a}_1 a_2) - 8\,\mathrm{Re}[\lambda(\bar{a}_2 a_3 + \bar{a}_3 a_4 + \cdots)] \\ &\geqq 4|\lambda|^2 + 4 - \{3|a_1|^2 + 4|\lambda|\,|a_1|\,|a_2| + 8|\lambda|\sqrt{1-|a_1|^2}\cdot\sqrt{1-|a_1|^2-|a_2|^2}\} \\ &\geqq 4|\lambda|^2 + 4 - 8|\lambda| = 4(1-|\lambda|)^2 > 0 \end{aligned}$$

because the maximum of { } in the inequality above turns out to be $8|\lambda|$ by a differential calculus on two variables $|a_1|$ and $|a_2|$, so that $\lambda \notin A_{\sigma}(T)$ for $1/2 \leqq |\lambda| < 1$. Combining (i), (ii) and (iii), we have $\sigma(T) = D$, and $A_{\sigma}(T) = C$ and $\Gamma(T) = D - C$, so the proof is complete.

In fact it is known that $A_{\sigma}(T) = C$ and $\Gamma(T) = D - C$ in the example, but we give an elementary proof if this fact.

Remark. As easily seen, T in the example is hyponormal and $1/4 \in A_{\sigma}(T^*T)$, but $z \notin A_{\sigma}(T)$ for which $|z| = 1/2$ by the example, so that this example, shows a counter example to Theorem B.

References

[1] C. R. Putnam, *Spectra of polar factors of hyponormal operators*, Trans. Amer. Math. Soc., 188 (1974), 419–428.

[2] S. N. Rai, *Approximate point spectra of polar factors of hyponormal operators*, Indian J. Pure Appl. Math., 8 (1977), 1083–1088.

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