YOKOHAMA MATHEMATICAL
JOURNAL VOL. 32, 1984

DERIVATIONS WITH A DOMAIN CONDITION

By

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(Received May 31, 1984)

There are certain dense *-subalgebras associated with the generator of a strongly continuous one-parameter automorphism group α of a C^*-algebra. Among the typical ones so far considered are the domain of the generator, the algebra of C^∞-elements with respect to α, and the algebra of elements of compact α-spectra. Any derivation defined on any of them seems to behave very well and is likely to be closely connected to the generator itself [1, 2, 3, 7]. In this note we continue to study this problem and give a result which generalizes Theorem 3.1 in [1], at least when the C^*-algebra is separable. In particular we can show the following:

Let A be a unital simple C^*-algebra and α a strongly continuous one-parameter automorphism group of A with δ_α its generator. Let $A_\infty=\bigcap D(\delta_\alpha{}^n)$, i.e., the set of C^∞-elements of A with respect to α, and let δ be a *-derivation of A_∞ into A. When α is approximately inner, δ is closable and its closure generates a one-parameter automorphism group of A.

The assumption that α is approximately inner was made only to ensure that there is an α-covariant irreducible representation of A, which follows, in this case, from the existence of ground states [9]. Hence we may have replaced this assumption by the following when A is separable: The crossed product of A by α is not simple (or in particular α is periodic) [6]. But we do not know if this kind of assumption is really necessary in the above assertion.

The assumtion that the domain $D(\delta)$ is A_∞ can be replaced by the one that $D(\delta)$ is algebraically isomorphic to A_∞ because then the isomorphism is easily shown to be isometric using that A_∞ is closed under the C^∞-calculus.

Now we state the main result:

Theorem. *Let A be a C^*-algebla, G a locally compact abelian group, and α a continuous action of G on A. Let $A_F=\bigcup A^\alpha(K)$, where K runs over the compact subsets of the dual $\hat{G}$ of G, and $A^\alpha(K)$ denotes the α-spectral subspace corresponding to K, and let δ be a *-derivation of A_F into A.*

Suppose that $\delta\,|\,A^\alpha(K)$ is bounded for any compact subset K of $\hat{G}$ and that there exists a faithful family of α-covariant irreducible representations of A.

Then δ is closable and the closure $\bar{\delta}$ of δ is a generator, and there exists a constant $c \geqq 0$ such that for any $f \in L^1(G)$ with $\hat{f}(0)=0$, the linear map δ_f on A_F defined by

$$\delta_f(x)=\int_G f(t)\alpha_t \circ \delta \circ \alpha_{-t}(x)dt\,, \quad x \in A_F$$

is bounded by $c\|f\|_1$.

We first indicate how to prove the assertion made before the theorem. The assumption $D(\delta)=A_\infty$ implies that δ is $\{\delta_\alpha{}^k\colon k=1, \cdots, n\}$-relatively bounded for some $n=1, 2, \cdots$ [2]. Hence $\delta \mid A^\alpha(K)$ is bounded for any compact subset K of $\hat{G}$, which is the only assumption left to prove. Thus by the theorem the closure δ_1 of $\delta \mid A_F$ is a generator. By the relative boundedness of δ, $D(\delta_1)$ contains A_∞ and δ_1 extends δ.

In the above theorem with $G=\boldsymbol{R}$ one cannot conclude that $D(\bar{\delta}) \supset A_\infty$ in general. For example consider the example constructed in [7], i.e., the infinite tensor product $A=A_1 \otimes A_2 \otimes \cdots$ of copies of the 2×2 matrices with a one-parameter automorphism group α of product type: $\alpha_t = AdU_t{}^{(1)} \otimes AdU_t{}^{(2)} \otimes \cdots$ where

$$U_t{}^{(n)}=\begin{pmatrix} 1 & 0 \\ 0 & \exp(-i\lambda_n t) \end{pmatrix}.$$

Note that α is determined by the sequence $\{\lambda_1, \lambda_2, \cdots\}$ of real numbers. If $\{\lambda_n\}$ increases sfficiently rapidly, the Arveson spectrum of α is discrete and hence A_F is the algebra generated by the C^*-subalgebra of diagonal matrices and $A_1, A_2, \cdots$ where A_n are regarded as subalgebras of A. Define $\sigma_n \in A_n$ by

$$\sigma_n=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $x \equiv \sum e^{-\lambda_n}\sigma_n$ is analytic for α. Let δ be the generator of the automorphism group corresponding to the sequence $\{e^{2\lambda_1}, e^{2\lambda_2}, \cdots\}$. Then x is not in the domain of the generator δ, but apparently A_F is a core for δ.

Before going into the proof of the theorem, we show the following as an application of the theorem and results in [1, 7].

Corollary. *Let A be the CAR algebra and α a strongly continuous one-parameter group of quasi-free automorphisms of A. Let A_∞ be the algebra of C^∞-elements, and let δ be a *-derivation of A_∞ into A.*

*Then there are a pregenerator δ_1 (i.e., its closure is a generator) which commutes with α and a bounded *-derivation δ_b such that $\delta=\delta_1+\delta_b$. Furthermore if* *the Connes spectrum of* α *is not trivial, then* δ_1 *can be chosen as* $\lambda\delta_\alpha | A_\infty$ *for some* $\lambda \in \boldsymbol{R}$ *where* δ_α *is the generator of* α.

Proof. First note that α is approximately inner, or the Fock representation is α-covariant and irrducible. Hence we can apply the theorem to this case.

Let H be the generator of the one-parameter unitary group, on the one-particle Hilbert space, which corresponds to α. If H has a pure point spectrum, then α is almost periodic. Thus

$$\bar{\delta}(x) = \lim \frac{1}{2T} \int_{-T}^{T} \alpha_t \circ \delta \circ \alpha_{-t}(x)\, dt$$

exists for any $x \in A_\infty$, and $\bar{\delta}$ commutes with α. The boundedness of $\delta - \bar{\delta}$ follows from the last statement of the theorem. If the Connes spectrum of α is not trivial, $\bar{\delta}$ equals $\lambda\delta_\alpha + \delta'$ with δ' a bounded *-derivation by Corollary 2.2 in [7].

If H has a continuous part, a KMS state of this system gives a type III_1 factorial covariant representation. Hence the conclusion follows from Theorem 3.6 in [1] with slight modifications.

Now we come to the proof of the theorem.

Let (π, u) be an α-covariant irreducible representation, and let $M = \pi(A)'' = B(H_\pi)$ and $\bar{\alpha}$ the automorphism group of M with $\bar{\alpha}_t \circ \pi = \pi \circ \alpha_t$, $t \in G$. Our first purpose is to prove that the map $\pi(x) \mapsto \pi \circ \delta(x)$ extends to a generator on M.

In the following let K be a compact subset of $\hat{G}$ and let γ_K be the infimum of $\|\delta | A^\alpha(K+\Omega)\|$ where Ω runs over the compact neighbourhoods of $0 \in \hat{G}$.

Lemma 1. *For any* $x \in A^\alpha(K)$, $\|\pi \circ \delta(x)\| \leqq \gamma_K \|\pi(x)\|$.

Proof. Let I be the kernel of π. Then I is an α-invariant closed ideal of A. If $I = (0)$, there is nothing to prove. So suppose that $I \neq (0)$.

Let (e_λ) be an approximate identity for I, and let Ω be a compact neighbourhood of $0 \in \hat{G}$. By considering $\int f(t)\alpha_t(e_\lambda)dt$ with a suitable non-negative $f \in L^1(G)$ instead of e_λ, we may suppose that $e_\lambda \in A^\alpha(\Omega) \cap I$. Then for $x \in I$ with compact α-spectrum, $e_\lambda x$ converges to x, and the spectra of $e_\lambda x$ are contained in a compact subset of $\hat{G}$. Hence $\delta(e_\lambda x)$ converges to $\delta(x)$. Since

$$\delta(e_\lambda x) = e_\lambda \delta(x) + \delta(e_\lambda) x \in I\,,$$

it follows that $\delta(x) \in I$. Since $\delta(e_\lambda)$ is bounded, $\delta(e_\lambda)x$ converges to zero for any $x \in I$. Thus, as $\delta(e_\lambda) \in I$, $\delta(e_\lambda)$ converges to zero in the weak* topology. Hence by choosing a suitable net from the convex hull of e_λ, we may further suppose that $\delta(e_\lambda)$ converges to zero in norm.

Now for $x \in A^{\alpha}(K)$,

$$\begin{aligned}\|\pi \circ \delta(x)\| &= \lim \|(1-e_\lambda)\delta(x)\| = \lim \|\delta(x-e_\lambda x)+\delta(e_\lambda)x\| = \lim \|\delta(x-e_\lambda x)\| \\ &\leqq \|\delta \mid A^{\alpha}(K+\Omega)\| \lim \|x-e_\lambda x\| = \|\delta \mid A^{\alpha}(K+\Omega)\| \, \|\pi(x)\|\end{aligned}$$

where the first and last equalities follow from e.g., 1.5.4 in [8]. Since Ω is arbitrary this completes the proof.

Lemma 2. *The linear map $\pi(x) \mapsto \pi \circ \delta(x)$ from $\pi(A^{\alpha}(K))$ into $\pi(A)$ is continuous in the σ-weak topology on each bounded set.*

Proof. Let Ω be a compact neighbourhood of $0 \in \hat{G}$, and let $a, b \in A^{\alpha}(\Omega)$ with $\|a\|, \|b\| \leqq 1$. Then for $x \in A^{\alpha}(K)$,

$$\|\pi(a\delta(x)b)\| = \|\pi(\delta(axb)-\delta(a)xb-ax\delta(b))\| \leqq \gamma'(\|\pi(axb)\|+\|\pi(xb)\|+\|\pi(ax)\|)$$

where $\gamma' = \max(\|\delta \mid A^{\alpha}(K+\Omega+\Omega)\|, \|\delta \mid A^{\alpha}(\Omega)\|)$. Hence with $\gamma = 3\gamma'/2$,

$$(*) \qquad \|\pi(a\delta(x)b)\| \leqq \gamma(\|\pi(ax)\|+\|\pi(xb)\|) .$$

This is the basic inequality we use below.

Let p be a one-dimensional projection in M and φ the pure state of A defined by $p\pi(x)p = \varphi(x)p$, $x \in A$. Let e_λ be an approximate identity of the hereditary C^*-subalgebra $\{x \in A : \varphi(x^*x) = \varphi(xx^*) = 0\}$, and let $a_\lambda = 1-e_\lambda$. Then a_λ decreases to p in A^{**} where p is regarded as a minimal projection of A^{**}.

Let f be a non-negative function in $L^1(G)$ with $\operatorname{supp} \hat{f} \subset \Omega$ and $\hat{f}(0) = 1$. Define

$$a_\lambda(f) = \int \alpha_t(a_\lambda) f(t) dt$$

$$p(f) = \int \bar{\alpha}_t(p) f(t) dt .$$

Then we assert that $a_\lambda(f)$ decreases to $p(f)$ in A^{**}. Since $\pi(a_\lambda(f))$ converges to $p(f)$ on H_π, it suffices to show that for any state ψ of A which is singular in π, $\psi(a_\lambda(f))$ converges to zero. Since $\pi_\psi \circ \alpha_t$ is disjoint from π for any $t \in G$, $\psi(\alpha_t(a_\lambda))$ converges to zero. Thus

$$\psi(a_\lambda(f)) = \int \psi(\alpha_t(a_\lambda)) f(t) dt$$

converges to zero by the Lebesgue dominated convergence theorem.

Then one obtains that

$$\lim \|x^* a_\lambda(f) x\| = \|x^* p(f) x\| = \|\pi(x^*) p(f) \pi(x)\|$$

and so

$$\limsup \|\pi(x^*)\pi(a_\lambda(f)^2)\pi(x)\| \leqq \lim \|\pi(x^*)\pi(a_\lambda(f))\pi(x)\| = \|\pi(x^*)p(f)\pi(x)\| .$$

Hence

$$\limsup \|\pi(a_\lambda(f))\pi(x)\| \leqq \|p(f)^{1/2}\pi(x)\| .$$

On the other hand

$$\liminf \|\pi(a_\lambda(f))\pi(x)\| \geqq \|p(f)\pi(x)\| .$$

Now we substitute $a_\lambda(f)$ into a in (*). This is justified even if A does not have an identity because in this case $1-a_\lambda(f)$ belongs to A and $\delta(a_\lambda(f))$ can read $-\delta(1-a_\lambda(f))$ which satisfies that $\|\delta(1-a_\lambda(f))\| \leqq \gamma'$. Thus we obtain

$$\|p(f)\pi\circ\delta(x)\pi(b)\| \leqq \gamma(\|p(f)^{1/2}\pi(x)\| + \|\pi(x)\pi(b)\|) .$$

By a similar procedure for b with a one-dimensional projection q in M and a non-negative $g\in L^1(G)$ with supp $\hat{g}\subset\Omega$ and $\hat{g}(0)=1$,

$$\|p(f)\pi\circ\delta(x)q(g)\| \leqq \gamma(\|p(f)^{1/2}\pi(x)\| + \|\pi(x)q(g)^{1/2}\|) .$$

Suppose that there is a bounded net x_λ in $A^\alpha(K)$ such that $\pi(x_\lambda)$ converges to zero and $\pi\circ\delta(x_\lambda)$ converges to Q weakly. By choosing a net from the convex hull of x_λ, we suppose that $\pi(x_\lambda)$ (resp. $\pi\circ\delta(x_\lambda)$) converges strongly* to 0 (resp. Q) ([4], 2.4.7). Since $p(f)$ and $q(g)$ are compact, we must have that $\|p(f)^{1/2}\pi(x_\lambda)\|\to 0$ etc. Hence

$$p(f)Qq(g)=0 .$$

Since this does not depend on Ω, it follows that $pQq=0$. Thus $Q=0$ since p and q are arbitrary one-dimensional projections.

Let x_λ be a net in $A^\alpha(K)$ such that $\pi(x_\lambda)$ is bounded and converges to zero. Since $\pi\circ\delta(x_\lambda)$ forms a bounded set in $\pi(A)$, there is a subset y_λ of x_λ such that $\pi\circ\delta(y_\lambda)$ converges in the weak topology. Then by the above argument the limit must be zero. This shows the continuity of the map $\pi(x)\mapsto\pi\circ\delta(x)$, $x\in A^\alpha(K)$.

Lemma 3. *Let $M_F=\bigcup M^{\bar{\alpha}}(K)$ where K runs over the compact subsets of $\hat{G}$. Then there is a *-derivation Δ of M_F into M such that*

$$\Delta\circ\pi(x)=\pi\circ\delta(x) , \quad x\in A_F ,$$

$\Delta\,|\,M^{\bar{\alpha}}(K)$ is σ-weakly continuous on each bounded set, and $\|\Delta\,|\,M^{\bar{\alpha}}(K)\|\leqq\gamma_K$.

Proof. The weak closure of $\pi(A^\alpha(K))$ is contained in $M^{\bar{\alpha}}(K)$. Let $Q\in M^{\bar{\alpha}}(K)$. Then by Kaplansky's density theorem there is a bounded net $\pi(x_\lambda)$ in $\pi(A)$ such that $\pi(x_\lambda)$ converges to Q. For a compact neighbourhood Ω of $0\in\hat{G}$, let f be a function in $L^1(G)$ such that $\hat{f}(p)=1$, $p \in K$ and $\operatorname{supp}\hat{f} \subset K+\Omega$. By replacing x_λ by $\int f(t)\alpha_t(x_\lambda)dt$, we suppose that $x_\lambda \in A^\alpha(K+\Omega)$. Thus $M^{\bar{\alpha}}(K)$ is contained in the weak closure of $\pi(A^\alpha(K+\Omega))$ and the map $\pi(x) \mapsto \pi \circ \delta(x)$ on $\pi(A_F)$ extends to a linear map from M_F into M by the continuity shown in Lemma 2. The rest is easy.

Lemma 4. *There exists a self-adjoint k in M such that $\|k\| \leqq \gamma_{\{0\}} \equiv \gamma_0$ and $\Delta_1 = \Delta - [ik, \cdot]$ commutes with $\bar{\alpha}$ i.e., $\bar{\alpha}_t \circ \Delta_1 \circ \bar{\alpha}_{-t} = \Delta_1$, $t \in G$. Furthermore if there is a compact neighbourhood Ω of $0 \in \hat{G}$ such that $\delta(A^\alpha(K)) \subset A^\alpha(K+\Omega)$ for any compact K, then k can be chosen from $M^{\bar{\alpha}}(\Omega)$.*

Proof (Lemma 3.5 in [1]). Let C be the von Neumann algebra generated by u_t, $t \in G$. Then $C \subset M^{\bar{\alpha}}(\{0\})$ and $\|\Delta \mid C\| \leqq \gamma_0$. Then there exists a $k=k^* \in M$ with $\|k\| \leqq \gamma_0$ such that $\Delta(Q)=[ik, Q]$, $Q \in C$. This k satisfies the above conditions.

If δ satisfies the additional assumption, Δ also satisfies that

$$\Delta(M^{\bar{\alpha}}(K)) \subset M^{\bar{\alpha}}(K+\Omega) .$$

From this the second assertion easily follows.

Lemma 5. *Δ_1 is σ-weakly closable and its closure generates a one-parameter automorphism group of M which commutes with $\bar{\alpha}$.*

Proof. Let M_0 be the norm-closure of M_F. Then Δ_1 extends to the generator of a one-parameter automorphism group of M_0 (cf. [5, 7]). Denoting by β this automorphism group, we show that β_s is continuous in the σ-weak topology.

Suppose that a bounded net Q_λ in M_0 converges to zero and $\beta_s(Q_\lambda)$ converges to Q σ-weakly. Then for any $f \in L^1(G)$ with $\operatorname{supp}\hat{f}$ compact,

$$\int \bar{\alpha}_t(\beta_s(Q_\lambda))f(t)dt = \beta_s\Big(\int \bar{\alpha}_t(Q_\lambda)f(t)\,dt\Big)$$

which converges weakly to zero since β_s is continuous on each bounded set of $M^{\bar{\alpha}}(K)$. Hence

$$\int \bar{\alpha}_t(Q)f(t)dt=0 .$$

This implies that $Q=0$. Thus β extends to an automorphism group, say $\bar{\beta}$, of M by Kaplansky's density theorem. Then each $\bar{\beta}_s$ must be continuous in the σ-weak topology. The continuity of $s \mapsto \beta_s$ follows easily (see the proof of Lemma 3.4 in [1]).

The generator of $\bar{\beta}$ is an extension of Δ_1. Since both the generator and Δ_1 have a dense common set of analytic vectors, the generator equals the closure of Δ_1 in the σ-weak topology.

Corollary 6. *δ is well-behaved.*

Proof. Since Δ is a bounded perturbation of Δ_1, Δ is also σ-weakly closable and its closure is a generator. Since $\Delta\circ\pi(x)=\pi\circ\delta(x)$, $x\in A_F$, it follows that

$$\|\pi(x\pm\delta(x))\|\geqq\|\pi(x)\|\ ,\quad x\in A_F\ .$$

Since there is a faithful family of such π by the assumption, it follows that $\|x\pm\delta(x)\|\geqq\|x\|$, $x\in A_F$.

Lemma 7. *Suppose that there exists a compact neighbourhood Ω of $0\in\hat{G}$ such that $\delta(A^\alpha(K))\subset A^\alpha(K+\Omega)$ for any compact subset K of $\hat{G}$. Let $\gamma=\max(\gamma_K,\gamma_\Omega)+2\gamma_0$. Then for any $Q\in M^{\bar{\alpha}}(K)$ and $n=1,2,\cdots$,*

$$\|\Delta^n(Q)\|\leqq(2\gamma)^n n!\ \|Q\|\ .$$

Proof. Note that $\Delta=\Delta_1+\Delta_{ik}$ where $\Delta_{ik}=[ik,\cdot]$, $k\in M^\alpha(\Omega)$, $\|k\|\leqq\gamma_0$ and that $\Delta_1(M^{\bar{\alpha}}(K))\subset M^{\bar{\alpha}}(K)$.

We first expand $(\Delta_1+\Delta_{ik})^n$ into 2^n terms, and apply the formula

$$\begin{aligned}\Delta_1\Delta_{ih}&=[\Delta_1,\Delta_{ih}]+\Delta_{ih}\Delta_1\\&=\Delta_{i\Delta_1(h)}+\Delta_{ih}\Delta_1\end{aligned}$$

to a factor $\Delta_1\Delta_{ih}$ with $h=k$, if any, in each term, by so doing, making each term into two. We repeat this process to a factor $\Delta_1\Delta_{ih}$ with $h=k$, $\Delta_1(k)$, $\Delta_1^2(k)$, $\cdots$, if any, in each resulting term. Eventually every resulting term is a product of inner derivations of the type $\Delta_{i\Delta_1^j(k)}$ $j=0,1,2,\cdots$, multiplied by a power of Δ_2 from the right, i.e., after being multiplied by Q from the right, it is of the form

$$(*)\qquad \Delta_{i\Delta_1^{j_1}(k)}\cdots\Delta_{i\Delta_1^{j_l}(k)}\Delta_1^m(Q)$$

where $l, j_1,\cdots,j_l$, m are non-negative integers and $j_1+\cdots+j_l+l+m=n$.

We estimate the norm of (*). Since $\Delta_1^m(Q)=\Delta(\Delta_1^{m-1}(Q))+\Delta_{ik}(\Delta_1^{m-1}(Q))$ and $\Delta_1^{m-1}(Q)\in M^{\bar{\alpha}}(K)$, one obtains

$$\begin{aligned}\|\Delta_1^m(Q)\|&\leqq\gamma_K\|\Delta_1^{m-1}(Q)\|+2\|k\|\cdot\|\Delta_1^{m-1}(Q)\|\\&\leqq\gamma\|\Delta_1^{m-1}(Q)\|\ .\end{aligned}$$

Hence

$$\|\Delta_1^m(Q)\|\leqq\gamma^m\|Q\|\ .$$

Similarly $\|\Delta_1^j(k)\|\leqq\gamma^j\|k\|\leqq\gamma^{j+1}/2$. Thus the norm of (*) is at most

$$\gamma^{j_1+1}\cdots\gamma^{j_l+1}\gamma^m\|Q\|=\gamma^n\|Q\|\ .$$

We now estimate the number of those resulting terms. Note that the resulting terms are independent of the processes applied. (This fact may be shown by the induction on the number of Δ_1 appearing in a term and by the induction on the positions of Δ_1 when the numbers of Δ_1 are the same. Since using this fact can easily be avoided by making a convention on the order of applying the process, we will not prove it here.) Suppose that one obtains m_{n-1} terms by applying the above procedure to $(\Delta_1+\Delta_{ik})^{n-1}$. Now multiply those terms by $\Delta_1+\Delta_{ik}$ from the left and expand them. The terms multiplied by Δ_{ik} from the left are already of the final form, and so there are m_{n-1} of them. By applying the above procedure to Δ_1 in each of the other terms one obtains at most n terms, because there are at most $n-1$ inner derivations in each term. Hence

$$m_n \leqq m_{n-1} + n m_{n-1} \leqq 2n m_{n-1} .$$

If $m_{n-1} \leqq 2^{n-1}(n-1)!$, then $m_n \leqq 2^n n!$. Since $m_1=2$, this completes the proof.

Lemma 8. *Let f be a real function in $L^1(G)$ such that the support of $\hat{f}$ is compact, and define a *-derivation δ_f on A_F by*

$$\delta_f(x) = \int \alpha_t \circ \delta \circ \alpha_{-t}(x) f(t) dt .$$

Then any element of A_F is analytic for δ_f.

Proof. Let $\Omega = \operatorname{supp} \hat{f}$. Then δ_f satisfies the assumption $\delta_f(A^\alpha(K)) \subset A^\alpha(K+\Omega)$ and also the other properties of δ necessary to derive Lemma 7. (In particular $\delta_f | A^\alpha(K)$ is bounded, e.g., $\|\delta_f | A^\alpha(K)\| \leqq \gamma_K \|f\|_1$.) By Lemma 7,

$$\|\pi \circ \delta_f^n(x)\| \leqq (2\gamma)^n n! \, \|x\|$$

for $x \in A^\alpha(K)$ where $\gamma = \|f\|_1(\max(\gamma_K, \gamma_\Omega) + 2\gamma_0)$. Since γ does not depend on π and there is a faithful family of those π, it follows that

$$\|\delta_f^n(x)\| \leqq (2\gamma)^n n! \|x\| , \quad x \in A^\alpha(K) .$$

This completes the proof.

Corollary 9. *Let f be a real function in $L^1(G)$ such that the support of $\hat{f}$ is compact. Then δ_f is a closable *-derivation and its closure $\bar{\delta}_f$ generates a one-parameter automorphism group of A.*

Proof. By Corollary 6 applied to δ_f, δ_f is well-behaved and so in particular δ_f is closable. By Lemma 8 δ_f has a dense set of analytic vectors. Thus the conclusion follows from a general theory ([4], 3.2.50).

Lemma 10. *Let f be a real function in $L^1(G)$ such that $\hat{f}(0)=0$. Then δ_f is* *bounded and* $\|\delta_f\| \leqq 4\|f\|_1\gamma_0$.

Proof (Proof of Theorem 3.1 in [1]). Let f be a real function in $L^1(G)$ such that $\operatorname{supp}\hat{f}$ is compact and $\hat{f}$ vanishes around 0. By Lemma 4 applied to δ_f instead of δ, one has $k=k^* \in M$ such that $\|k\| \leqq \|f\|_1\gamma_0$ and $\Delta_1 \equiv \Delta-[ik, \cdot]$ commutes with $\bar{\alpha}$. Since the $\bar{\alpha}$-spectrum of Δ (with respect to the action $\Delta \mapsto \bar{\alpha}_t \circ \Delta \circ \bar{\alpha}_{-t}$) does not contain 0, this implies that $\|\Delta_1\| \leqq \|[ik, \cdot]\| \leqq 2\|k\| \leqq 2\|f\|_1\gamma_0$. Hence $\|\Delta\| \leqq \|\Delta_1\| + \|[ik, \cdot]\| \leqq 4\|f\|_1\gamma_0$. Since the bound does not depend on π, it follows that $\|\delta_f\| \leqq 4\|f\|_1\gamma_0$. Now the conclusion follows from the density of those f in the maximal ideal of the real $L^1(G)$ consisting of f with $\hat{f}(0)=0$.

We now complete the proof of the theorem. Let f be a real function in $L^1(G)$ such that $\operatorname{supp}\hat{f}$ is compact and $\hat{f}(0)=1$. Then, as in the proof of Theorem 3.1 in [1], Lemma 10 implies that $\|\delta-\delta_f\| \leqq 8\|f\|_1\gamma_0$. Thus δ is a bounded perturbation of δ_f. Since $\bar{\delta}_f$ is a generator by Corollary 9, $\bar{\delta}$ is also a generator.

Acknowledgements

Part of this work was done while the author was visiting University of Edinburgh. He would like to thank C. J. K. Batty for numerous discussions on this topic.

References

[1] C. J. K. Batty and A. Kishimoto, *Derivations and one-parameter subgroups of C*-dynamical systems*, to appear in J. London Math. Soc.

[2] O. Bratteli, G. A. Elliott and P. E. T. Jorgensen, *Decompositions of unbounded derivations into invariant and approximately inner parts*, J. Reine Angew. Math. **346** (1984), 166–193.

[3] O. Bratteli, F. M. Goodman and P. E. T. Jorgense, *Unbounded derivations tangential to compact groups of automorphisms* II, preprint.

[4] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics* I, Springer, 1979.

[5] E. B. Davies, *A generation theorem for operators commuting with group actions*, preprint.

[6] A. Kishimoto, *Automorphisms and covariant irreducible representaions*, Yokohama Math. J., **31** (1983), 159–168.

[7] A. Kishimoto and D. W. Robinson, *Derivations, Dynamical systems, and spectral restrictions*, to appear in Math. Scand.

[8] G. K. Pedersen, *C*-algebras and their automorphism groups*, Academic Press, 1979.

[9] R. Powers and S. Sakai, *Existence of ground states and KMS states for approximately inner dynamics*, Commun. Math. Phys., **39** (1975), 273–288.

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