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ON MANIFOLDS OF NONNEGATIVE RICCI CURVATURE

By

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§0. Introduction

On a Riemannian manifold M we can define a function $\rho_M: M\to R^+\cup\{+\infty\}$ which gives us interesting geometric properties of M where R^+ is the set of all positive real numbers. The definition of ρ_M will be given in §2. The purpose of this paper is to investigate Riemannian manifolds of nonnegative Ricci curvature with $\rho(M)<+\infty$ where $\rho(M)=\sup\rho_M$.

In the following let M be an n-dimensional ($n\geqq 2$) connected, complete Riemannian manifold of nonnegative Ricci curvature. If $\rho(M)$ is finite then M is compact and the fundamental group of M is finite. M is homeomorphic to a standard sphere of dimension n if M is compact and $d(M)\geqq 2\rho(M)$ where $d(M)$ denotes the diameter of M. We now suppose that M is compact and is not simply connected and that $\rho_M(p)\leqq d_M(p,C(p))$ holds for some point p of M where d_M is the distance function on M and $C(p)$ stands for the cut locus of p in M. Then there exists a homeomorphic involution $\varphi: S^n(1)\to S^n(1)$ of fixed point free and M is homeomorphic to the quotient manifold $S^n(1)/\varphi$ of $S^n(1)$ obtained by identifying each $x\in S^n(1)$ with $\varphi(x)$ where $S^n(1)$ is the n-dimensional Euclidean sphere of radius 1.

In §1 we prepare some lemmas. Lemmas 1.2 and 1.3 are basic lemmas of this paper. In §2 we give the definition of ρ_M. We will show in this section that if M is a connected, complete Riemannian manifold satisfying $K_M\leqq 1$, $\mathrm{Ric}_M\geqq(n-1)\lambda^2$, $1/2<\lambda\leqq 1$, then $\pi\leqq 2\rho_M(p)\leqq\pi/\lambda$ for all $p\in M$ where K_M (resp. Ric_M) denotes the sectional curvature (resp. Ricci curvature) of M, respectively. In the last section of this paper we investigate Riemannian manifolds of nonnegative Ricci curvature with $\rho(M)<+\infty$.

§1. Notations and Lemmas

Throughout this paper we always assume that manifolds and apparatus on them are of class C^∞, unless otherwise stated.

Let M be an n-dimensional ($n\geqq 2$) connected, complete Riemannian manifold with Riemannian metric $\langle\, ,\rangle$. We denote by d_M the distance function on M which is induced from the Riemannian metric of M. We denote by $d(M)$ the diameter of M. For a $p \in M$ and an $r>0$ we put $B(p, r)=\{q \in M;\ d_M(p, q)<r\}$, $\bar{B}(p, r)=\{q \in M;\ d_M(p, q)\leqq r\}$ and $\partial B(p, r)=\{q \in M;\ d_M(p, q)=r\}$. Let $\exp$: $TM \rightarrow M$ be the exponential map from the tangent bundle TM of M to M. For each $p \in M \exp_p$: $T_pM \rightarrow M$ is the restriction of exp to the tangent space T_pM to M at p. If X and Y are orthogonal unit tangent vectors at a point of M then the quantity $K_M(P)=\langle R(X, Y)Y, X\rangle$ is called the sectional curvature of the plane section P determined by X and Y where R denotes the Riemannian curvature tensor of M. Let $e_1, \cdots, e_n$ be an orthonormal basis of the tangent space T_pM at $p \in M$ and let X be a unit tangent vector at p. Then the quantity $\mathrm{Ric}_M(X)=\sum_{i=1}^{n}\langle R(e_i, X)X, e_i\rangle$ is called the Ricci curvature with respect to X. In this paper, we denote by $K_M \leqq \lambda$ if $K_M(P)\leqq \lambda$ holds for all plane sections P to M, and we denote by $\mathrm{Ric}_M \geqq (n-1)\lambda$ if $\mathrm{Ric}_M(X)\geqq(n-1)\lambda$ holds for all unit tangent vectors X to M.

Let N be a Riemannian manifold of dimension n ($n \geqq 2$) and let f: $S \rightarrow N$ be an isometric immersion of an $(n-1)$-dimensional Riemannian manifold S into N. (S, f) is called a minimal hypersurface in N if the trace of the second fundamental form of S is zero everywhere. (S, f) is called totally geodesic if the second fundamental form of S vanishes identically.

In the following we shall prepare some lemmas which will be used in the next sections. Let D be an open metric ball in the n-dimensional ($n \geqq 1$) Euclidean space R^n. Let $(x_1, \cdots, x_n)$ be the standard coordinate system in R^n. Let us consider a Riemannian manifold $N=(D\times(-\tau, \tau), ds^2)$, $\tau>0$, whose line element is given by $ds^2=\sum_{i,j=1}^{n} g_{ij}(x, t)dx_i dx_j+dt^2$. Let ∇ be the Riemannian connection of N induced from the Riemannian metric of N. For each t, $|t|<\tau$, we denote by H_t the mean curvature of the level hypersurface $S_t=\{(x, t);\ x \in D\}$ with respect to $\partial/\partial t$. In case $n=1$, by the mean curvature we mean the geodesic curvature. H_t is given by $H_t=(1/n)\sum_{i,j=1}^{n} g^{ij}\langle\nabla_{\partial/\partial x_i}\partial/\partial x_j, \partial/\partial t\rangle$ where $g^{ij}(x, t)$ is the (i, j)-component of the inverse matrix of $(g_{ij}(x, t))$. We can easily show

$$n\partial H_t/\partial t=\mathrm{Ric}_N(\partial/\partial t)+\|A_t\|^2$$

where $\|A_t\|$ stands for the length of the second fundamental form A_t of S_t. From this formula we have the following.

Lemma 1.1. *Under the situation stated above, suppose* $\mathrm{Ric}_N(\partial/\partial t)\geqq 0$. *Then* $H_t \leqq H_{t'}$ *for any* t, t' *such that* $t<t'$. *If* $H_t=H_{t'}$ *for* t, t' *such that* $t<t'$, *then*

S_r *is totally geodesic for any* $r\in[t, t']$.

Now for a real valued function $u\in C^2(D)$, $|u|<\tau$, let us consider a hypersurface $S=\{(x, u(x));\ x\in D\}$ in N. We put $X_i=\partial/\partial x_i+u_i\partial/\partial t$ and $\tilde g_{ij}(x)=g_{ij}(x, u(x))+u_i(x)u_j(x)$ where $u_i=\partial u/\partial x_i$, $1\leqq i,\ j\leqq n$. Let $\xi=\sum_{i=1}^n \xi^i\partial/\partial x_i+\xi^{n+1}\partial/\partial t$ be the unit normal vector field on S defined by

$$\xi^i=-u^i/(1+\|\nabla u\|^2)^{1/2}\ , \qquad \xi^{n+1}=1/(1+\|\nabla u\|^2)^{1/2}$$

where $\|\nabla u\|^2=\sum_{i,j=1}^n g^{ij}(x, u(x))u_iu_j$ and $u^i=\sum_{j=1}^n g^{ij}(x, u(x))u_j$. Let Λ be the mean curvature of S with respect to ξ. Λ is given by $\Lambda=(1/n)\sum_{i,j=1}^n \tilde g^{ij}\langle\nabla_{X_i}X_j, \xi\rangle$ where $\tilde g^{ij}(x)=g^{ij}(x, u(x))-u^i(x)u^j(x)/(1+\|\nabla u\|^2)$. We have

$$\begin{aligned}(1.1)\qquad &\sum_{i,j=1}^n \{(1+\|\nabla u\|^2)g^{ij}(x, u(x))-u^iu^j\}u_{ij}\\ &=n\Lambda(x)(1+\|\nabla u\|^2)^{3/2}-nH(x, u(x))(1+\|\nabla u\|^2)\\ &\quad+\frac{1}{2}\sum_{i,j=1}^n(\partial g_{ij}/\partial t)(x, u(x))u^iu^j\\ &\quad+\sum_{i,j,k=1}^n\{(1+\|\nabla u\|^2)g^{ij}(x, u(x))-u^iu^j\}\Gamma^k_{ij}(x, u(x))u_k\end{aligned}$$

where $u_{ij}=\partial^2u/\partial x_i\partial x_j$, $nH(x, u(x))=-(1/2)\sum_{i,j=1}^n g^{ij}(x, u(x))(\partial g_{ij}/\partial t)(x, u(x))$ and Γ^k_{ij} denotes the Christoffel's symbol. In (1.1) if we regard Λ as a given continuous function on D, then we can regard (1.1) as a nonlinear differential equation of second order. We put

$$\begin{aligned}(1.2)\qquad &A_{ij}(x, t, p)=(1+\|p\|^2)g^{ij}(x, t)-p^ip^j\\ &B(x, t, p)=n\Lambda(x)(1+\|p\|^2)^{3/2}-nH(x, t)(1+\|p\|^2)+\frac{1}{2}\sum_{i,j=1}^n(\partial g_{ij}/\partial t)(x, t)p^ip^j\\ &\qquad+\sum_{i,j,k=1}^n\{(1+\|p\|^2)g^{ij}(x, t)-p^ip^j\}\Gamma^k_{ij}(x, t)p_k\end{aligned}$$

where $|t|<\tau, p=(p_1, \cdots, p_n)\in R^n$, $\|p\|^2=\sum_{i,j=1}^n g^{ij}(x, t)p_ip_j$, $p^i=\sum_{j=1}^n g^{ij}(x, t)p_j$, $nH(x, t)=-(1/2)\sum_{i,j=1}^n g^{ij}(x, t)(\partial g_{ij}/\partial t)(x, t)$.

Lemma 1.2. *Under the above situation, suppose that* $\mathrm{Ric}_N(\partial/\partial t)\geqq 0$ *and* $\Lambda\leqq H_0$ *in* D. *Let* u *be a solution of the equation* (1.1) *such that* $0\leqq u<\tau$. *If* u *attains the minimum in* D, *then* u *is constant.*

Proof. Put $E=\{x\in D;\ u(x)=m\}$ where m is the minimum of u in D. Suppose $D\neq E$. Then E is not open in D. Therefore we can choose a $x_0\in D\backslash E$ and an open metric ball D_0 in R^n of radius r_0 centered at x_0 so that $D_0\cap E=\emptyset$, $\bar D_0\cap$

$E=\{y_0\}$ and $\bar{D}_0 \subset D$ where $\bar{D}_0=\{x \in R^n;\ \|x-x_0\| \leqq r_0\}$, $\|\ \|$ denotes the standard norm of R^n. Let D_1 be the open metric ball in R^n of radius r_1 centered at y_0 such that $0<r_1<r_0$ and $\bar{D}_1 \subset D$. Then for each $x \in \bar{D}_1$ we have

$$r_2 \leqq \|x-x_0\| \leqq r_3 \tag{1.3}$$

where $r_2=r_0-r_1$ and $r_3=r_0+r_1$. There exists a constant δ $(0<\delta<1)$ satisfying

$$u>m+\delta \quad \text{on} \quad \bar{D}_0 \cap \partial \bar{D}_1 \tag{1.4}$$

where $\partial \bar{D}_1=\{x \in R^n:\ \|x-y_0\|=r_1\}$. Since the matrix $(A_{ij}(x, t, p))$ is positive definite, there are positive constants λ_1 and λ_2 such that

$$\lambda_1 \|X\|^2 \leqq \sum_{i,j=1}^{n} A_{ij}(x, u(x), p(x)) X_i X_j \leqq \lambda_2 \|X\|^2 \tag{1.5}$$

where $x \in \bar{D}_1$, $X=(X_1, \cdots, X_n) \in R^n$, $\|X\|^2=\sum_{i=1}^{n} X_i^2$ and $p(x)=(u_1(x), \cdots, u_n(x))$. On $\bar{D}_1$ we have

$$|B(x, u(x), p(x))-B(x, u(x), 0)| \leqq c\left(\sum_{i=1}^{n} (u_i(x))^2\right)^{1/2}$$

where

$$c=\sup_{\bar{D}_1} \sum_{i=1}^{n} \int_0^1 |\partial B/\partial p_i(x, u(x), tp(x))| dt<+\infty . \tag{1.6}$$

Since $\mathrm{Ric}_N(\partial/\partial t) \geqq 0$, $\Lambda \leqq H_0$ and $0 \leqq u<\tau$, by Lemma 1.1 for any $x \in \bar{D}_1$

$$B(x, u(x), 0)=n(\Lambda(x)-H(x, u(x))) \leqq n(H_0(x)-H_{u(x)}(x)) \leqq 0 .$$

Thus we get

$$B(x, u(x), p(x)) \leqq c\left(\sum_{i=1}^{n} (u_i(x))^2\right)^{1/2}, \quad x \in \bar{D}_1 . \tag{1.7}$$

Define a real valued function h: $D \to R$ by

$$h(x)=\exp(-\alpha \|x-x_0\|^2)-\exp(-\alpha r_0^2) \tag{1.8}$$

where α is a positive constant such that

$$\alpha>\max\{(-\log \delta)/r_2^2, (n\lambda_2+cr_3)/2\lambda_1 r_2^2\} . \tag{1.9}$$

Put $w=u-h$. Since $h<0$ on $\partial \bar{D}_1 \backslash \bar{D}_0$, we have

$$w>m \quad \text{on} \quad \partial \bar{D}_1 \backslash \bar{D}_0 . \tag{1.10}$$

On the other hand, from (1.3), (1.4) and (1.9) we obtain

$$w>m+\delta-\exp(-\alpha r_2^2)>m \quad \text{on} \quad \partial \bar{D}_1 \cap \bar{D}_0 . \tag{1.11}$$

Since $w(y_0)=u(y_0)=m$, by (1.10) and (1.11) $w \mid \bar{D}_1$ attains the minimum in D_1. Let y be a point of D_1 at which $w \mid \bar{D}_1$ attains the minimum. Using (1.7) we have

$$\sum_{i,j=1}^{n} A_{ij}(y, u(y), p(y))(w_{ij}(y)+h_{ij}(y)) \leqq c\Big(\sum_{i=1}^{n} (u_i(y))^2\Big)^{1/2} . \tag{1.12}$$

From (1.8)

$$h_i(y)=-2\alpha z_i \eta \quad (1\leqq i\leqq n) , \qquad h_{ij}(y)=-2\alpha(\delta_{ij}-2\alpha z_i z_j)\eta \quad (1\leqq i, j\leqq n) \tag{1.13}$$

where $z=(z_1, \cdots, z_n)=y-x_0$ and $\eta=\exp(-\alpha \|y-x_0\|^2)$. Since $w \mid \bar{D}_1$ attains the minimum at y, we have

$$u_i(y)=h_i(y) \quad (1\leqq i\leqq n) \tag{1.14}$$

and

$$\sum_{i,j=1}^{n} A_{ij}(y, u(y), p(y))w_{ij}(y) \geqq 0 . \tag{1.15}$$

From (1.3), (1.5), (1.13) and (1.15) we obtain

$$\begin{aligned} &\text{the left hand side of (1.12)} \\ &\geqq 2\alpha\eta(2\alpha\lambda_1 \|z\|^2 - n\lambda_2) \geqq 2\alpha\eta(2\alpha\lambda_1 r_2^2 - n\lambda_2) . \end{aligned} \tag{1.16}$$

By (1.13) and (1.14), $\Big(\sum_{i=1}^{n} (u_i(y))^2\Big)^{1/2} = 2\alpha\eta \|z\| \neq 0$. It follows from (1.3), (1.12) and (1.16) that $2\alpha\lambda_1 r_2^2 - n\lambda_2 \leqq c\|z\| \leqq cr_3$. This contradicts (1.9). Hence we have proved $D=E$. We complete the proof.

In the rest of this section, let M be an n-dimensional ($n\geqq 2$) connected, complete Riemannian manifold and let (W_1, ι_1) and (W_2, ι_2) be connected hypersurfaces embedded in M with unit normal vector fields ξ_1 and ξ_2 respectively, where ι_k denotes the inclusion map, $k=1, 2$. We denote by H_k the mean curvature of W_k with respect to ξ_k, $k=1, 2$. For a subset U of W_1 and a positive τ we put $\perp_\tau(U)=\{t\xi_1(q) \in TM;\ |t|<\tau, q \in U\}$ and $\perp_\tau^+(U)=\{t\xi_1(q) \in TM;\ 0\leqq t<\tau, q \in U\}$.

Lemma 1.3. *Let M, W_1 and W_2 be as above. Suppose that M is of nonnegative Ricci curvature, that is,* $\mathrm{Ric}_M(X)\geqq 0$ *for all unit tangent vectors X to M at every point of M, and suppose that $H_1\geqq 0$ on W_1 and $H_2\leqq 0$ on W_2. Furthermore assume that there is a point p of $W_1 \cap W_2$ satisfying the following conditions:* (1) $\xi_1(p)=\xi_2(p)$, (2) *For an open neighborhood U_1 of p in W_1 and a positive τ such that* $\exp\colon \perp_\tau(U_1)\to M$ *is an embedding there is an open neighborhood of p in W_2 which is contained in* $\exp(\perp_\tau^+(U_1))$. *Then there exists a minimal hypersurface W embedded in M such that $p \in W \subset W_1 \cap W_2$.*

Proof. Let p be a point of $W_1 \cap W_2$ satisfying the conditions (1) and (2) stated above. Choose a local coordinate neighborhood U_1 about p in W_1 and a positive τ so that exp: $\perp_\tau(U_1) \to M$ is an embedding. By Gauss lemma, the line element of $\perp_\tau(U_1)$ induced from M by exp can be expressed by $ds^2 = \sum_{i,j=1}^{n-1} g_{ij}(x, t)dx_i dx_j + dt^2$ where $(x_1, \cdots, x_{n-1})$ is a local coordinate system on U_1 and $|t| < \tau$. By the condition (2) and the implicit function theorem, there exists an open neighborhood V_1 of p in W_1, $V_1 \subset U_1$, which is diffeomorphic to an open metric ball in R^{n-1}, and there exists a real valued function $u \in C^\infty(V_1)$ satisfying the following conditions: $u(p)=0$, $u \geqq 0$ in V_1 and $p \in V_2 := \{\exp_q u(q)\xi_1(q);\ q \in V_1\} \subset W_2$. Now in Lemma 1.2 we replace H_0, Λ and n by H_1, H_2 and $n-1$, respectively. Then we can apply Lemma 1.2 to the present situation. By Lemma 1.2, $u \equiv 0$ in V_1. Then $V_1 = V_2$ and V_1 is a minimal hypersurface in M which is contained in $W_1 \cap W_2$. This completes the proof.

§2. Definition of ρ_M

In this section let M be an n-dimensional $(n \geqq 2)$ connected, complete Riemannian manifold with Riemannian metric $\langle\ ,\ \rangle$. First we shall give the definition of ρ_M: $M \to R^+ \cup \{+\infty\}$. Suppose that for a $p \in M$ and an $r > 0$ $\exp_p: \bar{B}(0_p, r) \to M$ is of maximal rank where $\bar{B}(0_p, r) = \{Y \in T_pM;\ \|Y\| \leqq r\}$, $\|Y\|$ stands for the length of Y. Let X be a unit tangent vector at p and $c_{p,X}$: $[0, \infty) \to M$ the geodesic parametrized by arc length emanating from p with intial direction X. Then the velocity vector $\dot{c}_{p,X}(r)$ is a unit normal vector to the geodesic sphere $S(p, r) = \exp_p(\partial B(0_p, r))$ where $\partial B(0_p, r) = \{Y \in T_pM;\ \|Y\| = r\}$. We denote by $H_X(p, r)$ the mean curvature of $S(p, r)$ with respect to $\dot{c}_{p,X}(r)$. Let Ω_M be the subset of M which consists of all points p of M satisfying the condition: there exists an $r > 0$ such that $\exp_p$: $\bar{B}(0_p, r) \to M$ is of maximal rank and $H_X(p, r) \geqq 0$ for all unit tangent vectors X at p. We now define ρ_M: $M \to R^+ \cup \{+\infty\}$ by

$$\rho_M(p) = \inf\{r > 0;\ H_X(p, r) \geqq 0 \text{ for all } X \in T_pM \quad (\|X\| = 1)\} \text{ if } p \in \Omega_M\,,$$
$$\rho_M(p) = +\infty \quad \text{if} \quad p \in M \backslash \Omega_M\,.$$

We note that $\rho_M(p) > 0$ if $p \in \Omega_M$. We put $\rho(M) = \sup \rho_M$.

Let $\tilde{M}$ be the universal Riemannian covering manifold of M and Π: $\tilde{M} \to M$ the Riemannian covering map. Then $\rho_{\tilde{M}} = \rho_M \circ \Pi$.

Remark 2.1. If M is a connected, complete Riemannian manifold of nonpositive sectional curvature then Ω_M is empty. A typical example of a Riemannian manifold with $\Omega_M \neq \emptyset$ is the Euclidean sphere $S^n(r)$ of radius r. In this case,

$\Omega_M = S^n(r)$ and $\rho_M(p) = \pi r/2$ for any $p \in S^n(r)$.

Now let p be a point of Ω_M and let r be a positive such that $\exp_p: \bar{B}(0_p, r) \to M$ is of maximal rank. Let X be an arbitrary unit tangent vector at p. Choose an orthonormal basis $e_1, \cdots, e_{n-1}$ in the tangent space to $S(p, r)$ at $c_{p,X}(r)$. There exist Jacobi fields $Y_1(t), \cdots, Y_{n-1}(t)$ along $c_{p,X} | [0, r]$ satisfying $Y_i(0) = 0_p$, $Y_i(r) = e_i$ $(1 \leqq i \leqq n-1)$. Using the second variation formula, $H_X(p, r)$ can be expressed by

$$(2.1) \qquad (n-1) H_X(p, r) = -\sum_{i=1}^{n-1} I(Y_i)$$

$$= -\sum_{i=1}^{n-1} \int_0^r \{\|Y_i'(t)\|^2 - \langle R(Y_i(t), \dot{c}_{p,X}(t)) \dot{c}_{p,X}(t), Y_i(t) \rangle\} dt$$

where $Y_i'(t)$ is the covariant derivative of $Y_i(t)$ along $c_{p,X}$. If M is the n-dimensional Euclidean sphere $S^n(1/\lambda)$, $\lambda > 0$, of radius $1/\lambda$, then we have

$$(2.2) \qquad H_X(p, r) = -\lambda \cot \lambda r \,, \qquad 0 < r < \pi/\lambda \,.$$

Lemma 2.1. *Let M be as above. Suppose $K_M \leqq 1$ and $\mathrm{Ric}_M \geqq (n-1)\lambda^2$ $(0 < \lambda \leqq 1)$. Then we have*

$$-\lambda \cot \lambda r \leqq H_X(p, r) \leqq -\cot r \quad (0 < r < \pi)$$

for all unit tangent vectors X at every point p of M. If $H_X(p, r) = -\lambda \cot \lambda r$ (resp. $-\cot r$) for some $X \in T_p M$ $(\|X\| = 1)$, then $K_M(P(t)) = \lambda^2$ (resp. 1) for all plane sections $P(t)$ containing $\dot{c}_{p,X}(t)$ $(0 \leqq t \leqq r)$.

Proof. Since $K_M \leqq 1$, for each $p \in M$ and an r $(0 < r < \pi)$ $\exp_p: \bar{B}(0_p, r) \to M$ is of maximal rank. Fix an r, $0 < r < \pi$. Let p be a point of M. Let X be an arbitrary unit tangent vector at p. Choose an orthonormal basis $e_1, \cdots, e_{n-1}$ in the tangent space to $S(p, r)$ at $c_{p,X}(r)$. There are Jacobi fields $Y_1(t), \cdots, Y_{n-1}(t)$ along $c_{p,X} | [0, r]$ satisfying $Y_i(0) = 0_p$, $Y_i(r) = e_i$ $(1 \leqq i \leqq n-1)$. We extend $e_1, \cdots, e_{n-1}$ to parallel vector fields $e_1(t), \cdots, e_{n-1}(t)$ along $c_{p,X} | [0, r]$, respectively. Put $Z_i(t) = (\sin \lambda t / \sin \lambda r) e_i(t)$, $0 \leqq t \leqq r$, $1 \leqq i \leqq n-1$. Since $I(Y_i) \leqq I(Z_i)$ $(1 \leqq i \leqq n-1)$ and $\mathrm{Ric}_M \geqq (n-1)\lambda^2$, we have $H_X(p, r) \geqq -\lambda \cot \lambda r$. If $H_X(p, r) = -\lambda \cot \lambda r$, then $Y_i(t) = Z_i(t)$ $(0 \leqq t \leqq r)$, $1 \leqq i \leqq n-1$. From this we obtain $K_M(e_i(t) \wedge \dot{c}_{p,X}(t)) = \lambda^2$, $0 \leqq t \leqq r$, $1 \leqq i \leqq n-1$. Moreover we can show that $K_M(P(t)) = \lambda^2$ for all plane sections $P(t)$ containing $\dot{c}_{p,X}(t)$ $(0 \leqq t \leqq r)$. Similary, $K_M \leqq 1$ implies $H_X(p, r) \leqq -\cot r$ for all unit tangent vectors X at every point p of M. If $H_X(p, r) = -\cot r$ for some $X \in T_p M$ $(\|X\| = 1)$, then $K_M(P(t)) = 1$ for all plane sections $P(t)$ containing $\dot{c}_{p,X}(t)$ $(0 \leqq t \leqq r)$.

Lemma 2.1 implies the following.

Proposition 2.1. *Let M be an n-dimensional $(n \geqq 2)$ connected, complete*

Riemannian manifold. If $K_M \leqq 1$ *and* $\mathrm{Ric}_M \geqq (n-1)\lambda^2$ $(1/2<\lambda\leqq 1)$, *then* $\Omega_M = M$. *Moreover,* ρ_M *is continuous and* $\pi \leqq 2\rho_M(p) \leqq \pi/\lambda$ *for all* $p \in M$.

Remark 2.2. We note that there are Riemannian manifolds satisfying $\mathrm{Ric}_M \geqq (n-1)\lambda^2$ $(0<2\lambda\leqq 1)$ and $\Omega_M = M$. For example, the Riemannian product manifold $M=S^n(1)\times S^2(1)$ $(n=3, 4)$ satisfies such conditions. In this case, $\rho(M)<\pi$.

Proposition 2.2. *Let* M *be an* n*-dimensional* $(n\geqq 2)$ *connected, complete Riemannian manifold of positive Ricci curvature. If* Ω_M *is not empty, then* M *is compact and the fundamental group* $\pi_1(M)$ *is finite.*

Proof. Let p be a point of Ω_M. Suppose that M is not compact. Then there is a geodesic $c\colon [0, \infty)\to M$ parametrized by arc length emanating from p with $d_M(p, c(t))=t$ for all $t>0$. By the definition of Ω_M there exists an $r'>0$ such that $\exp_p\colon \bar{B}(0_p, r')\to M$ is of maximal rank and $H_X(p, r')\geqq 0$ for all $X\in T_pM$ $(\|X\|=1)$. Since M is of positive Ricci curvature, using Lemma 1.1 we can choose an $r>r'$ so that $\exp_p\colon \bar{B}(0_p, r)\to M$ is of maximal rank and $H_X(p, r)>0$ for all $X\in T_pM$ $(\|X\|=1)$. For each t $(t>r)$ let $V(t)$ be a connected open neighborhood of $-(t-r)\dot{c}(t)$ in $\partial B(0_{c(t)}, t-r)$ such that $\exp_{c(t)}\colon CV(t)\to M$ is an embedding where $CV(t)=\{sY;\ 0\leqq s\leqq 1,\ Y\in V(t)\}$. We put $W(t)=\exp_{c(t)} V(t)$ $(t>r)$. Denote by $H(t)$ the mean curvature (with respect to $\dot{c}(r)$) of $W(t)$ at $c(r)$. Then, $H(t)\geqq H_{\dot{c}(o)}(p, r)$. Since $\mathrm{Ric}_M>0$, $H(t)<1/(t-r)$ for all $t>r$. Thus we get $H_{\dot{c}(o)}(p, r)<1/(t-r)$ for all $t>r$. We obtain $H_{\dot{c}(o)}(p, r)=0$ as $t\to\infty$. This contradicts $H_{\dot{c}(o)}(p, r)>0$. Therefore, M is compact. Let $\tilde{M}$ be the universal Riemannian covering manifold of M. It is easy to see $\Omega_{\tilde{M}}\neq\emptyset$. By the same argument as above, we see that $\tilde{M}$ is compact. Hence $\pi_1(M)$ is finite.

§3. Manifolds with $\rho(M)<+\infty$

Theorem 3.1. *Let* M *be an* n*-dimensional* $(n\geqq 2)$ *connected, complete Riemannian manifold of nonnegative Ricci curvature. If there exist distinct points* p *and* q *of* M *such that* $d_M(p, q)\geqq \rho_M(p)+\rho_M(q)$, *then* M *is homeomorphic to a standard sphere of dimension* n.

Proof. Let p and q be distinct points of M such that $d_M(p, q)\geqq\rho_M(p)+\rho_M(q)$. We put $A_p=\{X\in T_pM;\ \|X\|=1,\ \exp_p dX=q\}$ where $d:=d_M(p, q)=\rho_M(p)+\rho_M(q)+2r$, $r\geqq 0$. By completeness of M, A_p is a nonempty closed subset in the unit sphere $\partial B(0_p, 1)$ in T_pM. We shall show A_p is open in $\partial B(0_p, 1)$. Let X be an arbitrary unit tangent vector contained in A_p and let $c\colon [0, d]\to M$ be the minimal geodesic from p to q with initial direction X. Since c is minimal, each $c(t)$ $(0<t<d)$ is

not a conjugate point of p along c. Hence we can choose a connected open neighborhood U_X of X in $\partial B(0_p, 1)$ so that $\exp_p$: $\tilde{U}_X \to M$ is an embedding where $\tilde{U}_X = \{tZ \in T_pM;\ 0 \leqq t \leqq \rho_M(p)+r,\ Z \in U_X\}$. By the same reason we can choose a connected open neighborhood U_Y of $Y=-\dot{c}(d)$ in $\partial B(0_q, 1)$ so that $\exp_q$: $\tilde{U}_Y \to M$ is an embedding where $\tilde{U}_Y = \{tZ' \in T_qM;\ 0 \leqq t \leqq \rho_M(q)+r,\ Z' \in U_Y\}$. Then $W_1 = \exp_p((\rho_M(p)+r)U_X)$ and $W_2 = \exp_q((\rho_M(q)+r)U_Y)$ are connected hypersurfaces embedded in M such that $c(\rho_M(p)+r) \in W_1 \cap W_2$, where $(\rho_M(p)+r)U_X = \{(\rho_M(p)+r)Z \in T_pM;\ Z \in U_X\}$ and $(\rho_M(q)+r)U_Y = \{(\rho_M(q)+r)Z' \in T_qM;\ Z' \in U_Y\}$. Let ξ_1 and ξ_2 be unit normal vector fields on W_1 and W_2 respectively which are defined by $\xi_1(c_{p,Z}(\rho_M(p)+r)) = \dot{c}_{p,Z}(\rho_M(p)+r)$ $(Z \in U_X)$ and $\xi_2(c_{q,Z'}(\rho_M(q)+r)) = -\dot{c}_{q,Z'}(\rho_M(q)+r)$ $(Z' \in U_Y)$. We denote by H_i the mean curvature of W_i with respect to ξ_i, $i=1, 2$. Using Lemma 1.1, $H_1 \geqq 0$ on W_1 and $H_2 \leqq 0$ on W_2. Moreover, W_1 and W_2 satisfy the other hypotheses in Lemma 1.3. Hence, by Lemma 1.3 there exists a connected minimal hypersurface W embedded in M such that $c(\rho_M(p)+r) \in W \subset W_1 \cap W_2$. We can choose open neighborhoods V_X of X in $\partial B(0_p, 1)$ and V_Y of Y in $\partial B(0_q, 1)$ such that $\exp_p((\rho_M(p)+r)V_X) = \exp_q((\rho_M(q)+r)V_Y) \subset W$. This implies $V_X \subset A_p$. Hence A_p is open in $\partial B(0_p, 1)$. Therefore, $A_p = \partial B(0_p, 1)$. Then we see that $\exp_{p|B(0_p, d)}$: $B(0_p, d) \to B(p, d)$ is a diffeomorphism and $M = B(p, d) \cup \{q\}$. It is now clear that M is homeomorphic to a standard sphere of dimension n.

Remark 3.1. Let M be as in Theorem 3.1. Suppose that there exist distinct points p and q of M such that $\rho_M(p)+\rho_M(q) \leqq d_M(p, q)$. From the proof of the above theorem we see that for each t, $0<t<d := d_M(p, q)$, $M = B(p, t) \cup B(q, d-t) \cup S(p, t)$, $S(p, t) = \partial B(p, t) = \partial B(q, d-t)$ and $S(p, t)$ is a hypersurface embedded in M. Suppose now $d-\rho_M(p)-\rho_M(q)=2r>0$. Using Lemma 1.1, the mean curvature of $\partial B(p, t)$, $\rho_M(p) \leqq t < d$, with respect to the outer unit normal vector is nonnegative and the mean curvature of $\partial B(q, t)$, $\rho_M(q) \leqq t < d$, with respect to the outer unit normal vector is nonnegative. By Lemma 1.1, $S(p, t)$ is totally geodesic for each t, $\rho_M(p) \leqq t \leqq \rho_M(p)+2r = d-\rho_M(q)$. Therefore there is an isometric imbedding from the Riemannian product manifold $S(p, \rho_M(p)) \times [0, 2r]$ into M. If $d_M(p, q) = \rho_M(p)+\rho_M(q)$, then $S(p, \rho_M(p))$ is a minimal hypersurface in M. We see that if M is of positive Ricci curvature then $d_M(p, q) = \rho_M(p)+\rho_M(q)$.

Remark 3.2. Using a similar method as the proof of Theorem 3.1 we can show Cheng's theorem ([3]) which is a generalization of Toponogov Sphere Theorem.

As a consequence of Theorem 3.1 we have the following.

Corollary 3.1. *Let M be an n-dimensional* $(n \geqq 2)$ *connected, compact Riemannian manifold of nonnegative Ricci curvature. If* $d(M) \geqq 2\rho(M)$, *then M*

is homeomorphic to a standard sphere of dimension n.

Corollary 3.2. *Let M be an n-dimensional $(n\geqq 2)$ connected, complete Riemannian manifold of positive Ricci curvature. If $\rho(M)$ is finite, then M is compact and $d(M)\leqq 2\rho(M)\leqq\pi/\lambda$ where λ is a positive constant such that $\lambda^2=\inf\{\mathrm{Ric}_M(X)/(n-1);\ X\in T_pM\ (\|X\|=1), p\in M\}$.*

Proof. Since M is of positive Ricci curvature and $\rho(M)$ is finite, by Proposition 2.2 M is compact. From Remark 3.1 we see $d(M)\leqq 2\rho(M)$. Let p be an arbitrary point of M. Choose an $r>0$ so that $\exp_p\colon \bar{B}(0_p, r)\to M$ is of maximal rank. By a similar method as in the proof of Lemma 2.1, $H_X(p, r)\geqq -\lambda\cot\lambda r$ for all $X\in T_pM$ $(\|X\|=1)$ where λ is a positive constant such that $\lambda^2=\inf\{\mathrm{Ric}_M(X)/(n-1);\ X\in T_qM\ (\|X\|=1), q\in M\}$. Suppose $2\rho_M(p)>\pi/\lambda$. Then $H_X(p, \rho_M(p))>0$ for all $X\in T_pM$ $(\|X\|=1)$. There exists an r' such that $0<r'<\rho_M(p)$ and $H_X(p, r')>0$ for all $X\in T_pM$ $(\|X\|=1)$. This contradicts the definition of $\rho_M(p)$. Hence $2\rho_M(p)\leqq\pi/\lambda$. This completes the proof.

From Proposition 2.2 and Corollaries 3.1, 3.2 we have the following.

Theorem 3.2. *Let M be an n-dimensional $(n\geqq 2)$ connected, complete Riemannian manifold. Suppose that $K_M\leqq 1$ and $\mathrm{Ric}_M\geqq(n-1)\lambda^2$, $1/2<\lambda\leqq 1$. Then $\pi\leqq 2\rho(M)\leqq\pi/\lambda$ and $d(M)\leqq 2\rho(M)$. If $d(M)=2\rho(M)$, then M is homeomorphic to a standard sphere of dimension n.*

Theorem 3.3. *Let M be an n-dimensional $(n\geqq 2)$ connected, complete Riemannian manifold satisfying the condition $K_M\leqq 1$ and $\mathrm{Ric}_M\geqq(n-1)\lambda^2$ $(1/2<\lambda\leqq 1)$.*

(1) *If $d(M)=2\rho(M)=\pi$, then M is isometric to the n-dimensional Euclidean sphere $S^n(1)$ of radius 1.*

(2) *If $d(M)=2\rho(M)=\pi/\lambda$, then M is isometric to the n-dimensional Euclidean sphere $S^n(1/\lambda)$ of radius $1/\lambda$.*

Proof. We shall prove (1). Since M is compact, we can choose points p and q of M such that $d_M(p, q)=d(M)=\pi$. Since M is of positive Ricci curvature, $\rho_M(p)+\rho_M(q)=d_M(p, q)=2\rho(M)$ (see Remark 3.1). This implies $\rho_M(p)=\rho_M(q)=\rho(M)=\pi/2$. From the proof of Theorem 3.1 we see that $M=B(p, \pi/2)\cup B(q, \pi/2)\cup S(p, \pi/2)$, $\partial B(p, \pi/2)=\partial B(q, \pi/2)=S(p, \pi/2)$ and $S(p, \pi/2)$ is a hypersurface embedded in M. Since $H_X(p, \pi/2)\geqq 0$ for all $X\in T_pM$ $(\|X\|=1)$, by Lemma 2.1 $H_X(p, \pi/2)=-\cot(\pi/2)=0$ for all $X\in T_pM$ $(\|X\|=1)$. Then for each $X\in T_pM$ $(\|X\|=1)$ $K_M(P(t))=1$ $(0\leqq t\leqq\pi/2)$ where $P(t)$ is an arbitrary plane section containing $\dot{c}_{p,X}(t)$. This implies that $\bar{B}(p, \pi/2)$ is isometric to a closed metric ball of radius $\pi/2$ in the n-dimensional Euclidean sphere $S^n(1)$ of radius 1. Similary, $\bar{B}(q, \pi/2)$ is isometric to a closed metric ball of radius $\pi/2$ in $S^n(1)$. Then we see that M is isometric to $S^n(1)$.

By the same method as above we can prove (2).

Remark 3.3. We note that (2) of the above theorem also follows from Cheng's theorem ([3]).

Theorem 3.4. *Let M be an n-dimensional ($n\geqq 2$) connected, complete Riemannian manifold of nonnegative Ricci curvature. If $\rho(M)$ is finite, then M is compact and the fundamental group $\pi_1(M)$ of M is finite.*

Proof. Let $\tilde{M}$ be the universal Riemannian covering manifold of M with covering map Π. Since $\rho_{\tilde{M}}=\rho_M\circ\Pi$, $\rho(\tilde{M})$ is finite. We shall show that $\tilde{M}$ is compact. If $\tilde{M}$ is not compact, then we can choose distinct points p and q of $\tilde{M}$ such that $d_{\tilde{M}}(p, q)>\rho_{\tilde{M}}(p)+\rho_{\tilde{M}}(q)$. By Theorem 3.1 $\tilde{M}$ is homeomorphic to a standard sphere. This is a contradiction. Hence $\tilde{M}$ is compact. This completes the proof.

Theorem 3.5. *Let M be an n-dimensional ($n\geqq 2$) connected, compact Riemannian manifold of nonnegative Ricci curvature. Suppose that M is not simply connected and that $\rho_M(p)\leqq d_M(p, C(p))$ holds for some $p\in M$, where $C(p)$ stands for the cut locus of p in M. Then there exists a homeomorphic involution φ: $S^n(1)\to S^n(1)$ of fixed point free and M is homeomorphic to the quotient manifold $S^n(1)/\varphi$ of $S^n(1)$ obtained by identifying each $x\in S^n(1)$ with $\varphi(x)$.*

Proof. Let p be a point of M such that $\rho_M(p)\leqq d_M(p, C(p))$. Let $\tilde{M}$ be the universal Riemannian covering manifold of M and Π: $\tilde{M}\to M$ the Riemannian covering map. Let Γ be the deck transformation group of $\tilde{M}$ corresponding to the fundamental group $\pi_1(M, p)$. Each element of $\Gamma_1=\Gamma\backslash\{\text{identity}\}$ acts on $\tilde{M}$ as an isometry of fixed point free. Let p_1 be a point of $\Pi^{-1}(p)$. There exists a $\sigma\in\Gamma_1$ such that $d_{\tilde{M}}(p_1, \sigma(p_1))\leqq d_{\tilde{M}}(p_1, \gamma(p_1))$ for any $\gamma\in\Gamma_1$. We put $p_2=\sigma(p_1)$ and $d=d_{\tilde{M}}(p_1, p_2)$. Since $\rho_M(p)\leqq d_M(p, C(p))$ and $\rho_{\tilde{M}}=\rho_M\circ\Pi$, $d\geqq 2\rho_M(p)=\rho_{\tilde{M}}(p_1)+\rho_{\tilde{M}}(p_2)$. By the same method as the proof of Theorem 3.1, $\exp_{p_1}|B(0_{p_1}, d)$ is diffeomorphic and $\tilde{M}=B(p_1, d)\cup\{p_2\}$. Then we see $\Gamma=\{\text{identity}, \sigma\}$. Let s be a point of $S^n(1)$ and let Φ: $T_{p_1}\tilde{M}\to T_sS^n(1)$ be a linear isometry. We now define a map $\tilde{f}:\tilde{M}\to S^n(1)$ by $\tilde{f}(x)=\exp_s(\Phi((\pi/d)\Psi(x)))$ for $x\in\tilde{M}\backslash\{p_2\}$ and $\tilde{f}(p_2)=-s$ where $\Psi=(\exp_{p_1}|\mathrm{B}(0_{p_1}, d))^{-1}$ and $-s$ denotes the antipodal point of s in $S^n(1)$. Then $\tilde{f}$ is homeomorphic. Let φ: $S^n(1)\to S^n(1)$ be a map defined by $\varphi=\tilde{f}\circ\sigma\circ\tilde{f}^{-1}$. We see that φ is a homeomorphic involution of fixed point free. Let $S^n(1)/\varphi$ be the quotient manifold of $S^n(1)$ obtained by identifying each $x\in S^n(1)$ with $\varphi(x)$. Define a map f: $M\to S^n(1)/\varphi$

by $f(q)=[\tilde{f}(\tilde{q})]$ where $\tilde{q}\in \Pi^{-1}(q)$ and $[\tilde{f}(\tilde{q})]$ stands for the equivalence class containing $\tilde{f}(\tilde{q})$. It is easy to see that f is homeomorphic. We complete the proof.

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