YOKOHAMA MATHEMATICAL
JOURNAL VOL. 32, 1984

ISOMETRIC MINIMAL IMMERSIONS OF SPHERES INTO SPHERES ISOTROPIC UP TO SOME ORDER

By

YOSIO MUTŌ

(Received April 10, 1984)

1. Introduction

An immersion f of an m-dimensional shere S^m into an M-dimensional sphere $S^M(r)$ is called an isometric minimal immersion $f: S^m(1)\rightarrow S^M(r)$ if $f: S^m\rightarrow S^M(r)$ is a minimal immersion and, at the same time, $f: S^m(1)\rightarrow S^M(r)$ is an isometric immersion. In the present paper we study some properties of isometric minimal immersions f_s which are said to be of order s and which send a standard m-sphere $S^m(1)$ into a sphere $S^{n-1}(r)$ such that $m\geqq 3$, $s\geqq 4$ and

$$n=(2s+m-1)(s+m-2)!/(s!(m-1)!)\ ,$$
$$r^2=m/(s(s+m-1))\ .$$

We consider $S^m(1)$ as the unit hypersphere of R^{m+1} and $S^{n-1}(r)$ as a hypersphere of R^n with radius r and with center at the origin. We fix an orthonormal basis $\{e_1, \cdots, e_{m+1}\}$ in R^{m+1} and an orthonormal basis $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ in R^n. Then the image $f_s(x)$, $x\in S^m(1)$, is expressed by n coordinates $f^A(u(x))$, $A=1, \cdots, n$, where $u(x)=u^1(x)e_1+\cdots+u^{m+1}(x)e_{m+1}$, $\|u(x)\|=1$, and $f^A(u)$ are homogeneous harmonic polynomials of degree s in $u^1, \cdots, u^{m+1}$ satisfying

$$\sum_A (f^A)^2=r^2\ .$$

Besides, $f^A(u(x))$ must satisfy the isometry condition

$$\sum_A (\partial f^A/\partial x^\mu)(\partial f^A/\partial x^\lambda)=g_{\mu\lambda}$$

where x^λ ($\lambda=1, \cdots, m$) are local coordinates of the point x of $S^m(1)$ and $g_{\mu\lambda}$ are the local components of the standard Riemannian metric g on $S^m(1)$.

Such minimal immersions were first studied by M. do Carmo and N. Wallach [1]. On the other hand the notion of isotropic immersions was introduced by B. O'Neill [4]. Recently K. Tsukada studied the cases where f_s becomes isotropic [6]. He studied also the relation between helical geodesic immersions and isotropic immersions.

The present author also studied isometric minimal immersions f_s: $S^m(1) \to S^{n-1}(r)$ by his own way [2] and found that there exists a linear space $D^m_{s,s}$ of some bi-symmetric tensors of bi-degree (s, s). This space is essentially the space W_2 of do Carmo and Wallach and to each equivalence class of immersions f_s there corresponds an element of $D^m_{s,s}$. This space is a linear subspace of $B^m_{s,s}$ which is also a space of some bi-symmetric tensors of bi-degree (s, s).

In the present paper j-isotropic elements of $D^m_{s,s}$ are defined and higher fundamental forms of an immersion f_s are investigated when the element C of $D^m_{s,s}$ associated with f_s is j-isotropic. For that purpose higher fundamental forms of a standard minimal immersion h_s are first obtained in explicit form. It is proved that higher fundamental forms $B_2, \cdots, B_j$ of an immersion f_s are isotropic if and only if C associated with f_s is j-isotropic.

In § 2 we reproduce some of the results obtained in [2]. There we explain definitions and notations most frequently used in the present paper. In § 3 isotropic elements of $D^m_{s,s}$ are defined and their properties are studied. In studying properties of higher fundamental forms it became clear that some properties of spherical harmonics play an important role. So these together with some properties of symmetric tensors were studied in [3] and the results are cited in § 4. These are used in § 5 in order to construct some symmetric tensor fields of isometric minimal immersions which are closely related to higher fundamental forms. § 6 is devoted to some study on the unit element U of $B^m_{s,s}$, which we use in §§ 7, 8 in order to get higher fundamental forms of a standard minimal immersion. Thus an explicit formula is obtained for each fundamental form. In § 9 we get some of the higher fundamental forms of an isometric minimal immersion. § 10 is devoted to isotropic property of higher fundamental forms.

The author wishes to express his hearty thanks to Professor K. Ogiue for his kind encouragement and to Dr. K. Tsukada whose study invited the autor to the present study.

2. Preliminaries

We consider $S^m(1)$ as the unit hypersphere of R^{m+1} where we have fixed an orthonormal basis $\{e_1, \cdots, e_{m+1}\}$. On the other hand we take in R^n an orthonormal basis $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ and a hypersphere $S^{n-1}(r)$ where the center is the origin and the radius is r. n and r are those given in § 1. Indices used most often in the present paper are

$$A, B, \cdots = 1, \cdots, n\,,$$
$$P, Q, \cdots = 1, \cdots, n-1\,,$$

$$a, b, \cdots, h, i, \cdots = 1, \cdots, m+1 \,,$$
$$\alpha, \beta, \cdots, \kappa, \lambda, \cdots = 1, \cdots, m$$

for which the usual summation convention is used if possible. But this does not mean that these letters cannot be used otherwise.

We use the position vector $X = X^A \tilde{e}_A$ for a point of R^n and, at the same time, local coordinates $(Y^1, \cdots, Y^{n-1})$ for a point of $S^{n-1}(r)$. As we consider $S^{n-1}(r)$ as a hypersphere of R^n as stated above, we have $X^A = X^A(Y^1, \cdots, Y^{n-1})$ or shortly $X^A = X^A(Y)$ for a point in a coordinate neighborhood of $S^{n-1}(r)$. We use local coordinates $(x^1, \cdots, x^m)$ for a point of $S^m(1)$ assuming that any immersion f we consider in the present study sends each coordinate neighborhood of $S^m(1)$ we take into a coordinate neighborhood of $S^{n-1}(r)$ so that f can be expressed locally by $Y^P = Y^P(x^1, \cdots, x^m)$ or $Y^P = Y^P(x)$ for short where $P = 1, \cdots, n-1$.

We use the position vector $u = u^i e_i$ for a point of $S^m(1)$ hence $u(x) = u^i(x) e_i$ for a point x of $S^m(1)$.

Let $f \colon S^m(1) \to S^{n-1}(r)$ be an immersion such that

$$f(x) = f^A(u(x)) \tilde{e}_A \,, \qquad \sum_A (f^A(u(x)))^2 = r^2$$

where $f^A(u)$ are homogeneous harmonic polynomials of degree s in u, namely, in $u^1, \cdots, u^{m+1}$. Then we have

$$\Delta_m f^A = \lambda_s f^A \,, \quad \lambda_s = s(s+m-1) \tag{2.1}$$

where Δ_m is the Laplacian on the standard sphere $S^m(1)$. f is called an immersion of order s and we denote it by f_s. As a result of a well-known theorem of Takahashi [5] we see that, if f is an isometric minimal immersion $S^m(1) \to S^M(\rho)$ where $S^M(\rho)$ is a hypersphere of some radius ρ in R^{M+1}, then f is necessarily an immersion of some order s.

The functions $u^h(x)$ are eigenfunctions of Δ_m satisfying $\Delta_m u^h = m u^h$. It is well-known that, for each eigenfunction ϕ satisfying $\Delta_m \phi = \lambda_s \phi$ there exists just one homogeneous harmonic polynomial

$$F = F_{i_1 \cdots i_s} u^{i_1} \cdots u^{i_s}$$

of degree s such that

$$\phi(x) = F_{i_1 \cdots i_s} u^{i_1}(x) \cdots u^{i_s}(x) \,.$$

The number $n = (2s+m-1)(s+m-2)!/(s!(m-1)!)$ is the dimension of the space H_s^{m+1} of homogeneous harmonic polynomials of degree s. It is also clear that a homogeneous harmonic polynomial F of degree s determines a symmetric tensor

t such that $t(v, \cdots, v)=F(v, \cdots, v)$. This fact allows us to use the letter F for a symmetric tensor too when it satisfies

$$\sum_i F(e_i, e_i, v_3, \cdots, v_s)=0$$

for arbitrary vectors $v_3, \cdots, v_s$ of R^{m+1}.

Definition of tensors F^A. Suppose we have an immersion f_s of order s. From what we have stated above we can define n symmetric tensors F^A of degree s such that $f^A(u(x))=F^A(u(x), \cdots, u(x))$ and

$$\sum_i F^A(e_i, e_i, v_3, \cdots, v_s)=0 . \tag{2.2}$$

F^A are called the tensors of degree s associated with the immersion f_s of order s.

Definition of $B^m_{s,s}$. The linear space $B^m_{s,s}$ is defined as the space such that $C\in B^m_{s,s}$ if and only if the tensor C of degree $2s$ satisfies the following conditions (i), (ii), (iii) where $v_1, \cdots, v_{2s}$ are arbitrary vectors of R^{m+1}:

(i) $C(v_1, \cdots, v_s; v_{s+1}, \cdots, v_{2s})$ is symmetric both in $v_1, \cdots, v_s$ and in $v_{s+1}, \cdots, v_{2s}$,

(ii) $C(v_1, \cdots, v_s; v_{s+1}, \cdots, v_{2s})=C(v_{s+1}, \cdots, v_{2s}; v_1, \cdots, v_s)$,

(iii) $\sum_i C(e_i, e_i, v_3, \cdots, v_s; v_{s+1}, \cdots, v_{2s})=0$.

$B^m_{s,s}$ is called the space of bi-symmetric harmonic tensors of bi-degree (s, s).

Definition of $D^m_{s,s}$. We define a linear subspace $D^m_{s,s}$ of $B^m_{s,s}$ as follows: $C\in D^m_{s,s}$ if and only if $C\in B^m_{s,s}$ and satisfies

(iv) $C(w, w, v, \cdots, v; v, \cdots, v)=0$ for arbitrary vectors v and w of R^{m+1}.

Definition of $f_{s,s}$. When f_s is an immersion of order s, we define a tensor $f_{s,s}$ of degree $2s$ by

$$f_{s,s}=\sum_A F^A\otimes F^A . \tag{2.3}$$

$f_{s,s}$ belongs to $B^m_{s,s}$ and is called the tensor of degree $2s$ associated with the immersion f_s.

It is to be minded that, hitherto, isometry condition is not taken into account.

The set of isometric minimal immersions f_s of order s was classified by do Carmo and Wallach [1] into three classes, namely, standard minimal immersions, non standard full isometric minimal immersions and non-full isometric minimal immersions.

We take an isometric minimal immersion f_s and a standard minimal immersion h_s. $f_{s,s}$ associated with the immersion f_s is common to all isometric minimal immersions belonging to the same equivalence class $[f_s]$ as f_s (see [2] Theorem 3.

3). The tensor of degree $2s$ associated with standard minimal immersions h_s is denoted by $h_{s,s}$. Then we have $f_{s,s}-h_{s,s}\in D^m_{s,s}$ ([2] § 6). Conversely, if $\hat{d}_{s,s}\in D^m_{s,s}$ and if $t_1<t<t_2$ where (t_1, t_2) is a certain interval depending on $\hat{d}_{s,s}$, then $h_{s,s}+t\hat{d}_{s,s}$ is the tensor $f_{s,s}$ of degree $2s$ associated with some isometric minimal immersion $f_s: S^m(1)\to S^{n-1}(r)$. $C=f_{s,s}-h_{s,s}$ is then called the element of $D^m_{s,s}$ associated with the isometric minimal immersion f_s.

Now, in terms of the position vector $X=X^A\tilde{e}_A$, the local coordinates Y^P ($P=1, \cdots, n-1$) and the local coordinates x^κ ($\kappa=1, \cdots, m$), we can write for an isometric minimal immersion

$$X^A(x)=X^A(Y(x))=F^A(u(x), \cdots, u(x)) . \tag{2.4}$$

Thus, using the notations $X_P{}^A=\partial X^A/\partial Y^P$, $X_\lambda{}^A=\partial X^A/\partial x^\lambda$, $Y_\lambda{}^P=\partial Y^P/\partial x^\lambda$, we have

$$\begin{aligned} X_\lambda{}^A&=X_P{}^A Y_\lambda{}^P \\ &=sF^A(\partial_\lambda u(x), u(x), \cdots, u(x)) \end{aligned} \tag{2.5}$$

where $\partial_\lambda=\partial/\partial x^\lambda$. Let us use ∇ as the symbol of covariant differentiation in the sense of Van der Waerden-Bortolotti with respect to $S^m(1)=(S^m, g)$ and $S^{n-1}(r)=(S^{n-1}, \tilde{g})$. Hence ∇ may be the Riemannian connection of $S^m(1)$ or of $S^{n-1}(r)$ according to cases and $\nabla_\lambda u=\partial_\lambda u$. Then as f_s is an isometric immersion, we have

$$\nabla_\mu X_\lambda{}^A=Y_\mu{}^Q Y_\lambda{}^P\nabla_Q X_P{}^A+X_P{}^A\nabla_\mu Y_\lambda{}^P$$

where

$$\sum_A X_Q{}^A X_P{}^A=\tilde{g}_{QP} , \qquad \nabla_Q X_P{}^A=-r^{-2}\tilde{g}_{QP}X^A ,$$

hence

$$\nabla_\mu X_\lambda{}^A=X_P{}^A\nabla_\mu Y_\lambda{}^P-r^{-2}g_{\mu\lambda}X^A \tag{2.6}$$

because of $g_{\mu\lambda}=\tilde{g}_{QP}Y_\mu{}^Q Y_\lambda{}^P$.

u^h are eigenfunctions of the Laplacian on $S^m(1)$ and satisfy

$$\sum_i (u^i)^2=u_i u^i=1 , \tag{2.7}$$

$$\nabla_\mu u^i\nabla^\mu u^h=\delta^{ih}-u^i u^h , \tag{2.8}$$

$$\nabla_\mu u_i\nabla_\lambda u^i=g_{\mu\lambda} , \tag{2.9}$$

$$\nabla_\mu \Delta_\lambda u^h=-g_{\mu\lambda}u^h \tag{2.10}$$

where $u_i=u^i$ and $\nabla^\mu=g^{\mu\lambda}\nabla_\lambda$.

Thus, in view of (2.10), we get from (2.5)

$$\nabla_\mu X_\lambda{}^A=s(s-1)F^A(\nabla_\mu u, \nabla_\lambda u, u, \cdots, u)-sg_{\mu\lambda}X^A . \tag{2.11}$$

Let us define $X^A_{\mu\lambda}$ by

$$X^A_{\mu\lambda}=\nabla_\mu X_\lambda{}^A+r^{-2}g_{\mu\lambda}X^A\ , \tag{2.12}$$

namely, by

$$X^A_{\mu\lambda}=s(s-1)F^A(\nabla_\mu u, \nabla_\lambda u, u, \cdots, u)+(r^{-2}-s)g_{\mu\lambda}X^A\ . \tag{2.12$'$}$$

Then we have, in view of (2.6),

$$X^A_{\mu\lambda}=X_P{}^A\nabla_\mu Y_\lambda{}^P\ . \tag{2.13}$$

The tensors of degree s associated with a standard minimal immersion h_s are denoted by H^A. Hence we have

$$f_{s,s}=\sum_A F^A\otimes F^A\ ,\qquad h_{s,s}=\sum_A H^A\otimes H^A$$

and

$$\sum_A F^A\otimes F^A=\sum_A H^A\otimes H^A+C\ ,\quad C\in D^m_{s,s}\ . \tag{2.14}$$

The unit element U of $B^m_{s,s}$ is defined in § 4 of [2]. Its formula is

$$\begin{aligned}U(w_1, \cdots, w_s; v_1, \cdots, v_s)&\\ =a_0\mathscr{S}_w\langle w_1, v_1\rangle\cdots\langle w_s, v_s\rangle&+\sum_{q=1}^{\sigma}a_q\mathscr{S}_w\mathscr{S}_v\langle w_1, w_2\rangle\cdots\langle w_{2q-1}, w_{2q}\rangle\\ &\langle v_1, v_2\rangle\cdots\langle v_{2q-1}, v_{2q}\rangle\\ &\langle w_{2q+1}, v_{2q+1}\rangle\cdots\langle w_s, v_s\rangle\end{aligned} \tag{2.15}$$

where $a_0, a_1, \cdots, a_\sigma$ are certain numbers, $\langle\ ,\ \rangle$ denotes the ordinary inner product in R^{m+1}, $\mathscr{S}_w$ (resp. $\mathscr{S}_v$) is the symmetrizer with respect to $w_1, \cdots, w_s$ (resp. $v_1, \cdots, v_s$) and σ $(=[s/2])$ is the largest integer satisfying $2\sigma\leqq s$. U and H^A satisfy (see [2] § 6)

$$U=(1/c')\sum_A H^A\otimes H^A \tag{2.16}$$

where $c'=r^2/(a_0+a_1+\cdots+a_\sigma)$.

Notations. We prefer to use notations such as $g(\mu\lambda)$, $\varphi(\omega\cdots\kappa)$ for $g_{\mu\lambda}$, $\varphi_{\omega\cdots\kappa}$ and so on sometimes. We also use abbreviated notations defined as follows:

$$\begin{aligned}F^A(\lambda)&=F^A(\nabla_\lambda u, u, \cdots, u)\ ,\\ F^A(\omega\cdots\lambda)&=F^A(\nabla_\omega u, \cdots, \nabla_\lambda u, u, \cdots, u)\ ,\\ U(\omega\cdots\lambda)&=U(\nabla_\omega u, \cdots, \nabla_\lambda u, u, \cdots, u; u, \cdots, u)\ ,\\ U(\omega\cdots\lambda; \delta\cdots\alpha)&=U(\nabla_\omega u, \cdots, \nabla_\lambda u, u, \cdots, u; \nabla_\delta u, \cdots, \nabla_\alpha u, u, \cdots, u)\ .\end{aligned}$$

If $S_i\in T_i{}^0(S^m(1))$ is a symmetric tensor field and t is a tangent vector field of $S^m(1)$, $S_i(t, \cdots, t)$ is denoted by $S_i(t)$. Similarly we define $F_i{}^A(t)$, $X_i{}^A(t)$, $(DX_i{}^A)(t)$ by

$$F_i^A(t)=F_i^A(t,\cdots,t,u,\cdots,u) \quad (i\text{-linear in } t)\,,$$
$$F_0^A(t)=F^A(u,\cdots,u)=f^A \quad (\text{do not depend on } t)\,,$$
$$X_i^A(t)=X_i^A(t,\cdots,t) \quad (i\text{-linear in } t)\,,$$
$$(DX_i^A)(t)=\nabla_t(X_i^A)(t))-iX_i^A(t,\cdots,t,\nabla_t t) \quad ((i+1)\text{-linear in } t)$$

where $X_i^A \in T_i^0(S^m(1))$. $\langle\ ,\ \rangle^{\sim}$ denotes the inner product in the tangent space of $S^{n-1}(r)$ with the Riemannian metric $\tilde{g}$ and $\langle\ ,\ \rangle^E$ denotes the ordinary inner product in R^n.

3. Isotropic elements of $D^m_{s,s}$

Definition 3.1. We define the function $\Gamma_{p,q}$: $R^{m+1}\times\cdots\times R^{m+1}\times R^{m+1}\to R$ by $\Gamma_{p,q}(w_1,\cdots,w_{p+q};v)=C(w_1,\cdots,w_p,v,\cdots,v;w_{p+1},\cdots,w_{p+q},v,\cdots,v)$.

Definition 3.2. We define the function $C_{p,q}$: $R^{m+1}\times R^{m+1}\to R$ by restricting $\Gamma_{p,q}$ to the domain $w_1=w_2=\cdots=w_{p+q}$ so that

$$C_{p,q}(w,v)=C(w,\cdots,w,v,\cdots,v;w,\cdots,w,v,\cdots,v)\,.$$

Thus $C_{p,q}(w,v)$ is of bi-degree (p,q) in w, $C_{p,q}(w,v)=C_{s-p,s-q}(v,w)$ and $C_{p,q}=C_{q,p}$.

Definition 3.3. We say that $C\in D^m_{s,s}$ is isotropic of order p when $C_{p,p}(w,v)$ does not depend on the choice of the orthonormal set $\{v,w\}$.

$C_{p,q}$ is identically zero for any element C of $D^m_{s,s}$ if $p+q\leqq 3$ (see [2] § 6, § 7). Thus we have

$$C_{p,q}=0 \quad \text{if} \quad p+q\leqq 3 \quad \text{or} \quad p+q\geqq 2s-3 \tag{3.1}$$

and every element of $D^m_{s,s}$ is isotropic of order 1.

Let $C^*_{p,q}$ be the function obtained from $C_{p,q}$ when the domain is restricted to orthonormal sets $\{v,w\}$. Considering the rotation

$$v'=v\cos\theta+w\sin\theta\,, \qquad w'=-v\sin\theta+w\cos\theta\,, \tag{3.2}$$

we can deduce from $C^*_{3,0}=C^*_{2,1}=0$ the identities

$$(s-3)C^*_{4,0}+sC^*_{3,1}=3C^*_{2,0}\,,$$
$$(s-2)C^*_{3,1}+(s-1)C^*_{2,2}=2C^*_{1,1}+C^*_{2,0}\,.$$

As we have $C^*_{2,0}=C^*_{1,1}=0$, we get

$$C^*_{3,1}=-((s-1)/(s-2))C^*_{2,2}\,,$$
$$C^*_{4,0}=(s(s-1)/((s-2)(s-3)))C^*_{2,2}$$

and this proves that, if C is isotropic of order 2, then $C^*_{3,1}$ and $C^*_{4,0}$ are also constants.

Thus we can put, if $\{v, w\}$ is orthonormal,

$$C(w, w, w, w, v, \cdots, v; v, \cdots, v)=k$$

where k is a constant. Let the set $\{v, w_1, w_2\}$ be orthonormal and put $w=w_1 \cos\theta+w_2 \sin\theta$. Then we get

$$C(w_1, w_1, w_2, w_2, v, \cdots, v; v, \cdots, v)=k/3\ .$$

Thus we have

$$\begin{aligned}
&C(e_1, e_1, e_2, e_2, e_1, \cdots, e_1; e_1, \cdots, e_1)=0\ ,\\
&C(e_2, e_2, e_2, e_2, e_1, \cdots, e_1; e_1, \cdots, e_1)=k\ ,\\
&C(e_a, e_a, e_2, e_2, e_1, \cdots, e_1; e_1, \cdots, e_1)=k/3
\end{aligned}$$

for $a=3, \cdots, m+1$. As C satisfies (iii) we get $k+(m-1)k/3=0$ which proves $k=0$ and the following lemma.

Lemma 3.1. *If an element C of $D^m_{s,s}$ is isotropic of order 2, then $C_{4,0}=C_{3,1}=C_{2,2}=0$ identically.*

That this lemma is valid for any set $\{v, w\}$ is proved as follows. Any vector w is a linear combination of v and a vector w' normal to v. In view of the identities (3.1) we easily get the lemma.

Returning to the orthonormal set $\{v, w\}$ and the rotation (3.2), we get from

$$C_{4,0}=C_{3,1}=C_{2,2}=0 \tag{3.3}$$

the identities

$$\begin{gathered}
(s-4)C^*_{5,0}+sC^*_{4,1}=0\ ,\\
(s-3)C^*_{4,1}+(s-1)C^*_{3,2}=0\ ,\\
C^*_{3,0}=0
\end{gathered}$$

if $s\geqq 5$. Thus we get, in view of (3.1) and (3.3),

$$C_{5,0}=C_{4,1}=C_{3,2}=0\ . \tag{3.4}$$

The results obtained above prove the following corollary.

Corollary 3.2. *Suppose $s\geqq 5$. If an element C of $D^m_{s,s}$ is isotropic of order 2, then $C_{p,q}=0$ identically for any integers $p, q\geqq 0$ such that $p+q\leqq 5$.*

If $s=4$, $C_{5,0}$ does not exist, but (3.4) is valid in the sense, $C_{4,1}=C_{3,2}=0$, which is equivalent to $C_{3,0}=C_{2,1}=0$.

Extending the method used in deducing Lemma 3.1 and Corollary 3.2 to more general cases, we easily obtain the following theorem where j-isotropic elements are defined as follows:

Definition 3.4. Let C be an element of $D^m_{s,s}$, $m\geqq 3$, $s\geqq 4$. C is said to be j-isotropic if C is isotropic of order $2, 3, \cdots, j$ simultaneously where $j\leqq\sigma=[s/2]$.

Theorem 3.3. *Let C be a j-isotropic element of $D^m_{s,s}$. Then we have $C_{p,q}=0$ if $p+q\leqq 2j+1$. If C is σ-isotropic, then C vanishes.*

It is easy to deduce from (3.1) that every element C of $D^m_{s,s}$ satisfies

$$C(a, b, c, v, \cdots, v; v, \cdots, v)=0 ,$$
$$C(a, b, v, \cdots, v; c, v, \cdots, v)=0$$

for arbitrary vectors a, b, c, v of R^{m+1} (see [2] § 7). Replacing v by $v+td$ where v and d are arbitrary vectors of R^{m+1} and t is a variable number, we get

$$(s-3)C(a, b, c, d, v, \cdots, v; v, \cdots, v)+sC(a, b, c, v, \cdots, v; d, v, \cdots, v)=0 ,$$
$$(s-2)C(a, b, d, v, \cdots, v; c, v, \cdots, v)+(s-1)C(a, b, v, \cdots, v; c, d, v, \cdots, v)=0 ,$$

hence

$$C(a, b, c, v, \cdots, v; d, v, \cdots, v)=-((s-3)/s)C(a, b, c, d, v, \cdots, v; v, \cdots, v) ,$$
$$C(a, b, v, \cdots, v; c, d, v, \cdots, v)=-((s-2)/(s-1))C(a, b, c, v, \cdots, v; d, v, \cdots, v) .$$

Suppose C is isotropic of order 2. Then $C_{4,0}=0$ and we have $C(a, b, c, d, v, \cdots, v; v, \cdots, v)=0$ as C satisfies (i). Thus we have the following lemma.

Lemma 3.4. *Let C be isotropic of order 2. Then*

$$C(a, b, c, d, v, \cdots, v; v, \cdots, v)=0 ,$$
$$C(a, b, c, v, \cdots, v; d, v, \cdots, v)=0 ,$$
$$C(a, b, v, \cdots, v; c, d, v, \cdots, v)=0$$

hold for any vectors a, b, c, d, v of R^{m+1}.

This lemma states that, if C is isotropic of order 2, then $\Gamma_{p,q}$ is identically zero for $p+q\leqq 4$. In order to get the following corollary from Corollary 3.2 and Lemma 3.4, we can use the process used in deducing Lemma 3.4.

Corollary 3.5. *Let C be isotropic of order 2. Then $\Gamma_{p,q}$ vanishes identically if $p+q\leqq 5$.*

Extending further the method used above we can deduce the following theorem.

Theorem 3.6. *Let C be a j-isotropic element of $D^m_{s,s}$ where $j\leqq\sigma$. Then $\Gamma_{p,q}$ vanishes identically if $p+q\leqq 2j+1$.*

We also get the following theorem.

Theorem 3.7. *Let C be j-isotropic where $2j+2\leqq s$ and let p, q be a pair of* *numbers such that* $p+q=2j+2$. *If* $C^*_{p,q}(w, v)$ *does not depend on* w, *then* C *is* $(j+1)$*-isotropic.*

Proof. As C is j-isotropic, we have $C_{p,q}(w, v)=0$ if $p+q\leqq 2j+1$. Hence we have

$$(s-p)C_{p+1,q}+(s-q)C_{p,q+1}=0$$

for any p, q such that $p+q=2j+1$. As no one of $s-p$ and $s-q$ vanishes, we see that we have to prove only the following assertion. If $C^*_{2j+2,0}(w,v)$ does not depend on w, then C is $(j+1)$-isotrpic. Thus we assume

$$C(w, \cdots, w, v, \cdots, v; v, \cdots, v)=c(v) .$$

Then, taking an orthonormal set $\{v, w_1, w_2\}$ and putting $w=w_1 \cos\theta+w_2 \sin\theta$, we get

$$C(w_2, w_2\, w_1, \cdots, w_1, v, \cdots, v; v, \cdots, v)=c(v)/(2j+1) .$$

Thus we have

$$C(e_1, e_1, e_2, \cdots, e_2, e_1, \cdots, e_1; e_1, \cdots, e_1)=0 ,$$
$$C(e_2, e_2, e_2, \cdots, e_2, e_1, \cdots, e_1; e_1, \cdots, e_1)=c(v) ,$$
$$C(e_a, e_n, e_2, \cdots, e_2, e_1, \cdots, e_1; e_1, \cdots, e_1)=c(v)/(2j+1)$$

where the third formula is obtained by putting $w_1=e_2$, $v=e_1$, $w_2=e_a$ $(a=3, \cdots, m+1)$ in the formula obtained above. From this result we get, in view of (iii), $c(v)=0$ which proves Theorem 3.7.

Definition 3.5. From the tensors F^A defined in § 2 we define functions $F_r{}^A(w, v)$ which are of degree r in w and of degree $s-r$ in v by

$$F_r{}^A(w, v)=F^A(w, \cdots, w, v, \cdots, v) .$$

Similarly we define $H_r{}^A$ from H^A. $F_r{}^A(w, u)$ (resp. $H_r{}^A(w, u)$) where u is the position vector of $x\in S^m(1)$ are denoted by $F_r{}^A(w)$ (resp. $H_r{}^A(w)$) for short.

Definition 3.6. We define $\Upsilon_{p,q}$: $R^{m+1}\times\cdots\times R^{m+1}\times R^{m+1}\to R$ by $\Upsilon_{p,q}(w_1, \cdots, w_{p+q}, v)=U(w_1, \cdots, w_p, v, \cdots, v; w_{p+1}, \cdots, w_{p+q}, v, \cdots, v)$ and $U_{p\,q}$: $R^{m+1}\times R^{m+1}\to R$ by restricting $\Upsilon_{p,q}$ to the domain $w_1=\cdots=w_{p+q}$, thus $U_{p,q}(w, v)=U(w, \cdots, w, v, \cdots, v; w, \cdots, w, v, \cdots, v)$. Hence, in view of (2.15), $U_{p,q}(w, v)$ has constant value depending only on p and q if the set $\{w, v\}$ is orthonormal. Besides $U_{p,q}=0$ if $p+q$ is odd.

From Definition 3.5 and (2.14) we get

$$\sum_A F_q{}^A(w, v)F_r{}^A(w, v)=\sum_A H_q{}^A(w, v)H_r{}^A(w, v)+C_{q,r}(w, v) . \tag{3.5}$$

On the other hand we have, in view of (2.16),

$$\sum_A H_q{}^A(w, v) H_r{}^A(w, v) = c' U_{q,r}(w, v) . \tag{3.6}$$

Thus we have the following theorem.

Theorem 3.8. *Let C be the element of $D^m_{s,s}$ associated with an isometric minimal immersion $f_s: S^m(1) \to S^{n-1}(r)$. The following condition (α) is equivalent to (β).*

(α) *C is j-isotropic*

(β) *$\sum_A F_q{}^A(w, v) F_r{}^A(w, v)$ is independent of the choice of the orthonormal set $\{w, v\}$ for all q, r such that $q + r \leqq 2j + 1$.*

4. Some properties of spherical harmonics and symmetric tensors

In this paragraph we give some results obtained in another paper [3]. Thus proof is omitted here.

A spherical harmonic φ of order s on the m-sphere is considered as a homogeneous harmonic polynomial $\tilde{\varphi}$ of degree s in R^{m+1} restricted to the unit hypersphere $S^m(1)$. A homogeneous harmonic polynomial $\tilde{\varphi}$ of degree s determines a symmetric tensor t of degree s in R^{m+1} satsfying $t(u, \cdots, u) = \tilde{\varphi}(u, \cdots, u)$ and

$$\sum_i t(e_i, e_i, v, \cdots, v) = 0 \tag{4.1}$$

where $\{e_1, \cdots, e_{m+1}\}$ is an orthonormal basis of R^{m+1}. We use the same letter φ for this tensor too.

From a spherical harmonic φ on $S^m(1)$ we get

$$\begin{aligned} \partial\varphi/\partial x^\lambda &= s\varphi(\partial u/\partial x^\lambda, u, \cdots, u) \\ &= s\varphi(\nabla_\lambda u, u, \cdots, u) \end{aligned}$$

where u is the position vector of a point of $S^m(1)$ as in §2. Let $t_1, \cdots, t_i$ be arbitrary tangent vector fields of $S^m(1)$. Then we can define a symmetric tensor field φ_i on $S^m(1)$ by

$$\varphi_i(t_1, \cdots, t_i) = \varphi(t_1, \cdots, t_i, u, \cdots, u)$$

and get

$$\varphi_{\lambda_1 \cdots \lambda_i} = \varphi(\nabla_{\lambda_1} u, \cdots, \nabla_{\lambda_i} u, u, \cdots, u)$$

as the local components since a tangent vector t can be written locally as $t = t^\lambda \nabla_\lambda u$. We may write $\varphi(\lambda_1 \cdots \lambda_i)$ for the local components and adopt the abbreviation

$$\varphi_i(t, \cdots, t) = \varphi_i(t) .$$

From the tensor fields φ_0 $(=\varphi)$, $\varphi_1, \cdots, \varphi_s$ we define for each integer $i = 0, 1, \cdots, s$ a symmetric tensor field Φ_i as follows.

Definition 4.1. Let $x_{i,q}$ $(q=0, 1, \cdots, k)$ be given by

$$((i-2q+2)(i-2q+1)/2)x_{i,q-1}=q(m+2i-2q-2)x_{i,q}\,, \quad x_{i,0}=1 \tag{4.2}$$

where $k=[i/2]$. Then Φ_i satisfies

$$\Phi_i(t)=s(s-1)\cdots(s-i+1)\sum_{q=1}^{k} x_{i,q}\varphi_{i-2q}(t) \tag{4.3}$$

for any local unit tangent vector field t. Especially $\Phi_0(t)=\varphi_0(t)=\varphi$.

Definition 4.2. $D\varphi_i$ is a symmetric tensor field of degree $i+1$ satisfying

$$(D\varphi_i)(t)=\nabla_t(\varphi_i(t))-i\varphi_i(t, \cdots, t, \nabla_t t)$$

for an arbitrary tangent vector field t. Thus the local components of $D\varphi_i$ are $\mathscr{S}_{\lambda,i+1}\nabla_{\lambda_1}\varphi(\lambda_2\cdots\lambda_{i+1})$ where $\mathscr{S}_{\lambda,i+1}$ is the symmetrizer with respect to $\lambda_1, \cdots, \lambda_{i+1}$. $D\Phi_i$ is defined similarly.

Lemma 4.1. *Let $y_{i,q}$ $(q=1, \cdots, h\ (=[(i+1)/2]))$ be determined by*

$$\begin{aligned} &y_{i,q}=(s-i+2q)x_{i,q}-(i-2q+2)x_{i,q-1}-(s-i)x_{i+1,q} \quad \textit{if} \quad q=1, \cdots, k\ (=[i/2])\,, \\ &y_{i,h}=-x_{i,k}-(s-i)x_{i+1,h} \quad \textit{which appear only when } i \textit{ is odd}\,. \end{aligned} \tag{4.4}$$

Then the following identity holds at every point of $S^m(1)$ and for any unit tangent vector t,

$$(D\Phi_i)(t)-\Phi_{i+1}(t)=s(s-1)\cdots(s-i+1)\left[\sum_{q=1}^{k} y_{i,q}\varphi_{i+1-2q}(t)+y_{i,h}\varphi_0\right]. \tag{4.5}$$

Lemma 4.2. *The local components $\Phi(\lambda_1\cdots\lambda_i)$ of the symmetric tensor field Φ_i satisfy*

$$g^{\nu\mu}\Phi(\nu\mu\lambda_3\cdots\lambda_i)=0\,. \tag{4.6}$$

Theorem 4.3. *Let i be any integer $1\leqq i\leqq s-1$. Then we have*

$$\Phi_{i+1}(t)=(D\Phi_i)(t)-(s-i+1)y_{i,1}\Phi_{i-1}(t) \tag{4.7}$$

at every point of $S^m(1)$ and for any unit tangent vector t.

Remark. The key of the proof of Lemma 4.1 is

$$\nabla_\mu\varphi(\lambda_1\cdots\lambda_i)=(s-i)\varphi(\mu\lambda_1\cdots\lambda_i)-i\mathscr{S}_{\lambda,i}g(\mu\lambda_1)\varphi(\lambda_2\cdots\lambda_i)\,. \tag{4.8}$$

We consider for a while the Euclidean m-space R^m with a basis $\{f_1, \cdots, f_m\}$ and put $h_{\mu\lambda}=\langle f_\mu, f_\lambda\rangle$. Let P_i $(=P_{i,0})$ be a symmetric tensor of degree i with components $P_i(\lambda_1\cdots\lambda_i)=P_i(f_{\lambda_1}, \cdots, f_{\lambda_i})$. We define from P_i a series of symmetric tensors $P_{i,q}$ ($q=1, \cdots, k$ $(=[i/2])$) of degree $i-2q$ inductively by

$$-h^{\nu\mu}P_{i,q}(\nu\mu\lambda_{2q+3}\cdots\lambda_i)=P_{i,q+1}(\lambda_{2q+3}\cdots\lambda_i)$$

where $h^{\nu\mu}$ are determined by $h^{\nu\mu}=h^{\mu\nu}$, $h^{\nu\mu}h_{\mu\lambda}=\delta_\lambda{}^\nu$ and $q=0, 1, \cdots, k-1$.

Lemma 4.4. *Let Q_i be a symmetric tensor of degree i defined from $P_{i-2p,q}$ ($q=0, 1, \cdots, k-p$) by*

$$Q_i(t)=\sum_{q=0}^{k-p}\alpha_{i,p+q}P_{i-2p,q}(t) \tag{4.9}$$

where $h_{\nu\mu}t^\nu t^\mu=1$. Then, putting $\alpha_{i,p-1}=0$, we have

$$\begin{aligned}&-(i(i-1)/2)Q_{i,1}(t)\\&=\sum_{q=0}^{k-p}\{r(m+2i-2r-2)\alpha_{i,r}-((i-2r+2)(i-2r+1)/2)\alpha_{i,r-1}\}P_{i-2p,q}(t)\end{aligned} \tag{4.10}$$

where $r=p+q$.

This lemma is used when we prove Lemma 4.2.

We use the next lemma to get a property of the unit element U of $B^m_{i,s}$.

Lemma 4.5. *Let S_i be a symmetric tensor of degree i and T_{i-2} be a symmetric tensor of degree $i-2$ such that $S_i(t)=T_{i-2}(t)$ when t satisfies $h_{\mu\lambda}t^\mu t^\lambda=1$. Then we have*

$$S_i(t)+\sum_{q=1}^{k}x_{i,q}S_{i,q}(t)=0\ , \quad k=[i/2]\ .$$

5. Symmetric tensor fields $X_i{}^A$ and $\dot{X}_i{}^A$

As $F^A(u, \cdots, u)$ as well as $H^A(u, \cdots, u)$ are spherical harmonics of order s when u is the position vector of the moving point of $S^m(1)$, we can apply the results of § 4 to F^A and H^A as follows.

We have defined in § 3 $F_i{}^A$ and $H_i{}^A$ (see Definition 3.5). Taking a local unit tangent vector field t we get $F_i{}^A(t)$ and $H_i{}^A(t)$.

Definition 5.1. We define symmetric tensor fields $X_i{}^A$ and $\dot{X}_i{}^A$ by

$$X_i{}^A(t)=s(s-1)\cdots(s-i+1)\sum_{q=0}^{k}x_{i,q}F^A_{i-2q}(t)\ , \tag{5.1}$$

$$\dot{X}_i{}^A(t)=s(s-1)\cdots(s-i+1)\sum_{q=0}^{k}x_{i,q}H^A_{i-2q}(t)\ . \tag{5.1°}$$

The analogy between Definition 4.1 and Definition 5.1 is obvious. Thus we get, as an analogy of Lemma 4.1,

$$(DX_i{}^A)(t)-X^A_{i+1}(t)=s(s-1)\cdots(s-i+1)\left[\sum_{q=1}^{k} y_{i,q}F^A_{i+1-2q}(t)+y_{i,h}X^A\right] \tag{5.2}$$

where $X^A=F^A(u,\cdots,u)$, and a similar formula for a standard minimal immersion.

Denoting the local components of the tensor fields $X_i{}^A$ by $X^A(\lambda_1\cdots\lambda_i)$ we get

$$g^{\nu\mu}X^A(\nu\mu\lambda_3\cdots\lambda_i)=0 \tag{5.3}$$

from Lemma 4.2. A similar formula is obtained for a standard minimal immersion.

From Theorem 4.3 we get the following theorem.

Theorem 5.1. *The tensor fields* $X_i{}^A$ *and* $\mathring{X}_i{}^A$ *satisfy*

$$X^A_{i+1}(t)=(DX_i{}^A)(t)-(s-i+1)y_{i,1}X^A_{i-1}(t)\ , \tag{5.4}$$

$$\mathring{X}^A_{i+1}(t)=(D\mathring{X}_i{}^A)(t)-(s-i+1)y_{i,1}\mathring{X}^A_{i-1}(t) \tag{5.4°}$$

where t is an arbitrary unit tangent vector and $i=1,\cdots,s-1$.

6. Some property of the unit element U

Let us consider at a fixed point x of $S^m(1)$ and let v be a unit tangent vector of $S^m(1)$ at x.

According to the definition of $U(\lambda_1\cdots\lambda_i;\kappa_1\cdots\kappa_h)$ in § 2 and the equation (2.16) we have

$$U(\lambda_1\cdots\lambda_i;\kappa_1\cdots\kappa_h)v^{\kappa_1}\cdots v^{\kappa_h}=(1/c')\sum_A H^A(\lambda_1\cdots\lambda_i)H_h{}^A(v)\ .$$

Supposing h is fixed we denote the left hand side by $U(\lambda_1\cdots\lambda_i\colon v)$. Then we get

$$g^{\nu\mu}U(\nu\mu\lambda_3\cdots\lambda_i\colon v)=-U(\lambda_3\cdots\lambda_i\colon v)$$

as H^A, being spherical harmonics, satisfy

$$g^{\nu\mu}H^A(\nu\mu\lambda_3\cdots\lambda_i)=-H^A(\lambda_3\cdots\lambda_i)$$

because of (2.8) and (4.1).

On the other hand, as we have (2.15) for U and as u^i satisfy $u_i\nabla_\lambda u^i=0$ and (2.9), there exists a tensor V_{i-2} of degree $i+h-2$ satisfying

$$U(\lambda_1\cdots\lambda_i;\kappa_1\cdots\kappa_h)=\mathscr{S}_{\lambda,i}g(\lambda_1\lambda_2)V_{i-2}(\lambda_3\cdots\lambda_i;\kappa_1\cdots\kappa_h)$$

if $i\geqq h+2$. Thus we get

$$U(\lambda_1\cdots\lambda_i\colon v)=\mathscr{S}_{\lambda,i}g(\lambda_1\lambda_2)V_{i-2}(\lambda_3\cdots\lambda_i\colon v)$$

where

$$V_{i-2}(\lambda_3\cdots\lambda_i\colon v)=V_{i-2}(\lambda_3\cdots\lambda_i;\kappa_1\cdots\kappa_h)v^{\kappa_1}\cdots v^{\kappa_h}\ .$$

This shows that the tensor $U_i{}^*$ of degree i with components $U(\lambda_1 \cdots \lambda_i\colon v)$ and the tensor V^*_{i-2} of degree $i-2$ with components $V_{i-2}(\mu_1 \cdots \mu_{i-2}\colon v)$ satisfy $U_i{}^*(t) = V^*_{i-2}(t)$ for any unit tangent vector t.

Thus, as an application of Lemma 4.5, we get

$$U_i{}^*(t) + \sum_{q=1}^{k} x_{i,2} U^*_{i-2q}(t) = 0\,, \quad k = [i/2]\,.$$

Hence, using the notation defined by

$$U_{i;h}(t, v) = U(\lambda_1 \cdots \lambda_i;\, \kappa_1 \cdots \kappa_h) t^{\lambda_1} \cdots t^{\lambda_i} v^{\kappa_1} \cdots v^{\kappa_h}\,,$$

we get the following lemma.

Lemma 6.1. *Let t be any unit tangent vector and v be any tangent vector. If $i \geqq h+2$, the unit element U of $B^m_{i,s}$ satisfies*

$$U_{i;h}(t, v) + \sum_{q=1}^{k} x_{i,q} U_{i-2q;h}(t, v) = 0\,.$$

7. The property of symmetric tensors $\dot{X}_i{}^A$

We prove following five lemmas where t, t_1, t_2 are arbitrary local unit tangent vector fields.

Lemma 7.1. *The tensor fields $\dot{X}_i{}^A$ satisfy*

$$\sum_A \dot{X}_i{}^A(t_1) H_j{}^A(t_2) = 0 \quad \text{if} \quad i > j\,. \tag{7.1}$$

Proof. From Lemma 6.1 we get, if $i \geqq h+2$,

$$\sum_A H_i{}^A(t) H_h{}^A(v) + \sum_{q=1}^{k} x_{i,q} \sum_A H^A_{i-2q}(t) H_h{}^A(v) = 0\,,$$

hence

$$\sum_A \dot{X}_i{}^A(t) H_h{}^A(v) = 0\,.$$

On the other hand, if $i = h+1$, $\sum_A H^A_{i-2q}(t) H_h{}^A(v)$ vanishes identically as $i-2q+h$ is odd. This proves the lemma.

Corollary 7.2. *The tensor fields $\dot{X}_i{}^A$ and $\dot{X}_j{}^A$ satisfy*

$$\sum_A \dot{X}_i{}^A(t_1) \dot{X}_j{}^A(t_2) = 0 \quad \textit{if} \quad i \neq j\,. \tag{7.2}$$

Proof. This is almost immediate.

Lemma 7.3. *If $j \leqq i-2$, we have*

$$\sum_A H^A(\mu_1 \cdots \mu_j) \nabla_\lambda \dot{X}^A(\lambda_1 \cdots \lambda_i) = 0\,, \tag{7.3}$$

$$\sum_A \mathring{X}^A(\mu_1 \cdots \mu_j)\nabla_\lambda \mathring{X}^A(\lambda_1 \cdots \lambda_i)=0 . \tag{7.4}$$

Proof. In view of Lemma 7.1 we have

$$\sum_A H^A(\mu_1 \cdots \mu_j)\nabla_\lambda \mathring{X}^A(\lambda_1 \cdots \lambda_i)=-\sum_A \mathring{X}^A(\lambda_1 \cdots \lambda_i)\nabla_\lambda H^A(\mu_1 \cdots \mu_j) .$$

As we have, in view of (4.8),

$$\nabla_\lambda H^A(\mu_1 \cdots \mu_j)=(s-j)H^A(\lambda\mu_1 \cdots \mu_j)-j\mathscr{S}_{\mu,j}g(\lambda\mu_1)H^A(\mu_2 \cdots \mu_j) ,$$

we get the lemma by virtue of Lemma 7.1 and Corollary 7.2.

Lemma 7.4. *No one of the symmetric tensor fields $\mathring{X}_i^A$ vanishes identically.*

Proof. As a result of Lemma 9 of [3], if $\mathring{X}_i^A$ vanishes identically for some A and i, then $H^A(u, \cdots, u)$ vanishes identically for this number A. As h_s is a standard minimal immersion, this cannot occur.

Lemma 7.5. *If $i=1, \cdots, s-1$, we have*

$$\mathring{X}^A_{i+1}(t)=(D\mathring{X}_i{}^A)(t)-(s-i+1)y_{i,1}\mathring{X}^A_{i-1}(t) . \tag{7.5}$$

Proof. This is a direct result of Theorem 4.3.

8. Higher fundamental forms of a standard minimal immersion

The i-th fundamental form B_i $(i=2, \cdots, s)$ of an isometric minimal immersion f_s: $S^m(1)\to S^{n-1}(r)$ is a symmetric form, $B_i \in T_0{}^1(S^{n-1}(r))\otimes T_i{}^0(S^m(1))$, and satisfies the following conditions. These conditions completely determime B_i (see [1], [6]).

For convenience sake we define B_1 by $B_\lambda{}^P=Y_\lambda{}^P$. B_2 has local components $B_{\mu\lambda}{}^P=\nabla_\mu Y_\lambda{}^P$ which may be written $B^P(\mu\lambda)$.

Local components of B_i are also written $B^P(\lambda_1 \cdots \lambda_i)$. The condition to be satisfied by B_i is given as follows when lower fundamental forms B_j, $j<i$, are already given.

We define $B_1{}^*$ and $B_2{}^*$ by their local components $Y_\lambda{}^P$ and $\nabla_\mu Y_\lambda{}^P$, hence $B_1{}^*=B_1$, $B_2{}^*=B_2$. $B_j{}^*$ where $j\leqq i$ is defined by its local components

$$B^{*P}(\lambda_1 \cdots \lambda_j)=\mathscr{S}_{\lambda,j}\nabla_{\lambda_1}B^P(\lambda_2 \cdots \lambda_j) .$$

When tangent vectors $t_1, \cdots, t_i$ of $S^m(1)$ at x are given, $B_j{}^*(t, \cdots, t_j)$ is a tangent vector of $S^{n-1}(r)$ at $Y=f_s(x)$. When $t_1, \cdots, t_i$ range over $T_x(S^m(1))$ arbitrarily, $B_j{}^*(t_1, \cdots, t_j)$ spans a linear subspace of $T_Y(S^{n-1}(r))$ which we denote by $L_j{}^*$. The linear subspace of $T_Y(S^{n-1}(r))$ spanned by $L_1{}^*, \cdots, L_j{}^*$ is denoted by L_j. The projection to the linear subspace of $T_Y(S^{n-1}(r))$ complementary and normal to L_j is denoted by N_j. Then $B_i=N_{i-1}B_i{}^*$.

Theorem 8.1. *Let h_s: $S^m(1)\to S^{n-1}(r)$ be a standard minimal immersion and H^A be the tensors associated with h_s. Then the symmetric tensor fields $\mathring{X}_i^A$ and the i-th fundamental form $\mathring{B}_i$ of the immersion h_s are related by*

$$\mathring{X}_i^A(t)=X_P{}^A\mathring{B}_i{}^P(t) \tag{8.1}$$

where $\mathring{B}_i{}^P(t)=\mathring{B}_i{}^P(t,\cdots,t)$ and $X^A(Y(x))=H^A(u(x),\cdots,u(x))$.

Proof. (8.1) is valid if $i=1,2$. Assume (8.1) is valied for $2\leqq i\leqq h$ where h is a natural number less than s. Then we have, in view of $\nabla_Q X_P{}^A=-r^{-2}\tilde{g}_{QP}X^A$ and $B_1{}^Q(t)\mathring{B}_i{}^P(t)\tilde{g}_{QP}=0$, where $X^A=\mathring{X}^A$, $B_1{}^Q=\mathring{B}_1{}^Q$,

$$\begin{aligned}(D\mathring{X}_i{}^A)(t)&=(t^\mu Y_\mu{}^Q\nabla_Q X_P{}^A)\mathring{B}_i{}^P(t)+X_P{}^A(D\mathring{B}_i{}^P)(t)\\&=X_P{}^A(D\mathring{B}_i{}^P)(t)\,.\end{aligned}$$

From this and Lemma 7.5 we get

$$\mathring{X}^A_{i+1}(t)=X_P{}^A(D\mathring{B}_i{}^P)(t)-(s-i+1)y_{i,1}X_P{}^A\mathring{B}^P_{i-1}(t)\ ,$$

hence $\mathring{X}^A_{h+1}(t)=X_P{}^A\hat{B}^P_{h+1}(t)$ where $\hat{B}^P_{h+1}(t)$ is defined by

$$\hat{B}^P_{h+1}(t)=(D\mathring{B}_h{}^P)(t)-(s-h+1)y_{h,1}\mathring{B}^P_{h-1}(t)\ .$$

On the other hand, in view of Corollary 7.2, we have $\sum_A \mathring{X}^A_{h+1}(t_1)\mathring{X}_j{}^A(t_2)=0$ for $j=1,\cdots,h$, hence

$$\tilde{g}_{QP}\hat{B}^Q_{h+1}(t_1)\mathring{B}_j{}^P(t_2)=0\ .$$

This proves that $\hat{B}_{h+1}$ is the $(h+1)$-th fundamental form $\mathring{B}_{h+1}$. Repeating this process we can prove Theorem 8.1.

9. Higher fundamental forms of an isometric minimal immersion

We consider an isometric minimal immersion f_s: $S^m(1)\to S^{n-1}(r)$ where the element C of $D^m_{s,s}$ associated with f_s is j-isotropic. Then, from (2.14) and Theorem 3.6, we get, if $p+q\leqq 2j+1$,

$$\sum_A F_q{}^A(t_1)F_p{}^A(t_2)=\sum_A H_q{}^A(t_1)H_p{}^A(t_2)\ ,$$

hence

$$\sum_A X_q{}^A(t_1)X_p{}^A(t_2)=\sum_A \mathring{X}_q{}^A(t_1)\mathring{X}_p{}^A(t_2)$$

and also, in view of (5.4),

$$\sum_A X_q{}^A(t_1)(DX^A_{p-1})(t_2)=\sum_A \mathring{X}_q{}^A(t_1)(D\mathring{X}^A_{p-1})(t_2)\ .$$

Using the above result, Lemma 7.1, Corollary 7.2 and the method used in proving 7.3, we get the following lemma.

Lemma 9.1. *Let C be j-isotropic. Then we have*

$$\sum_A X_i{}^A(t_1)F_h{}^A(t_2)=0\ , \qquad \sum_A X_i{}^A(t_1)X_h{}^A(t_2)=0 \quad \textit{if} \quad h<i\leqq j+1\ ,$$

$$\sum_A F^A(\mu_1\cdots\mu_h)\nabla_\lambda X^A(\lambda_1\cdots\lambda_i)=0 \qquad \textit{and}$$

$$\sum_A X^A(\mu_1\cdots\mu_h)\nabla_\lambda X^A(\lambda_1\cdots\lambda_i)=0 \quad \textit{if} \quad h+1<i\leqq j+1\ .$$

Now we can prove the following theorem.

Theorem 9.2. *Let f_s be an isometric minimal immersion: $S^m(1)\to S^{n-1}(r)$ such that the element C of $D^m_{s,s}$ associated with f_s is j-isotropic. If $i\leqq j+1$, the symmetric tensor fields $X_i{}^A$ and the i-th fundamental form B_i of the immersion f_s are related by*

$$X_i{}^A(t)=X_P{}^AB_i{}^P(t)\ . \tag{9.1}$$

Proof. As B_1 and B_2 are given by $B_\lambda{}^P=Y_\lambda{}^P$ and $B^P(\mu\lambda)=\nabla_\mu Y_\lambda{}^P$ respectively, (9.1) is valid if $i\leqq 2$. Assume (9.1) is valid for $i=3,\cdots,h$ where h is a natural number $\leqq j$. Then we get, in view of $\nabla_Q X_P{}^A=-r^{-2}\tilde{g}_{QP}X^A$,

$$\begin{aligned}(DX_i{}^A)(t)&=(t^\mu Y_\mu{}^Q\nabla_Q X_P{}^A)B_i{}^P(t)+X_P{}^A(DB_i{}^P)(t)\\ &=X_P{}^A(DB_i{}^P)(t)\ ,\end{aligned}$$

hence, as in the case of a standard minimal immersion h_s,

$$X^A_{h+1}(t)=X_P{}^A\{(DB_h{}^P)(t)-(s-h+1)y_{h,1}B^P_{h-1}(t)\}\ .$$

On the other hand, in view of Lemma 9.1, we have

$$\sum_A X^A_{h+1}(t_1)X_i{}^A(t_2)=0 \quad \text{for} \quad i=1,\cdots,h$$

as $h\leqq j$. This proves that (9.1) is valid for $i=h+1$ as in the case of h_s. In such a way Theorem 9.2 is proved.

$i=j+1$ is the best value for i in (9.1), for we can prove the following theorem.

Theorem 9.3. *Let f_s be an isometric minimal immersion $S^m(1)\to S^{n-1}(r)$. If the tensors $X_i{}^A$ and the i-th fundamental form B_i satisfy* (9.1) *for $i=2,\cdots,j+1$, then the element C of $D^m_{s,s}$ associated with f_s is j-isotropic.*

Proof. (9.1) is always valid for $i=2$ and C is always 1-isotropic. Hence we need not consider the case $j=1$. If (9.1) is valid for $i=3$, we get, for any unit tangent vector t, $\sum_A X_3{}^A(t)X_1{}^A(t)=0$ from $\tilde{g}_{QP}B_3{}^QB_1{}^P=0$. Hence, in view of $s^2\sum_A F_1{}^A(t)\cdot F_1{}^A(t)=1$, we find that $\sum_A F_3{}^A(t)F_1{}^A(t)$ is a constant which depends neither on t nor on u. On the other hand, as $\sum_A H_3{}^A(t)H_1{}^A(t)=c'U_{3,1}(t,u)$ is also a constant of the same property, so is also $C_{3,1}(t,u)$. Hence we get $C_{3,1}(w,v)=$ const for orthonormal vectors w, v of R^{m+1} and C is 2-isotropic by virture of Theorem 3.7.

Let h be less than j and assume that the following assertion has been proved to be true. If (9.1) is valid for $i=3,\cdots,h+1$, then C is h-isotropic. Let us assume furthermore that (9.1) holds for $i=3,\cdots,h+2$. Then

$$\sum_A X^A_{h+2}(t)X_p{}^A(t)=0 \quad \text{for} \quad p=0,1,\cdots,h\ ,$$

hence

$$\sum_A X^A_{h+2}(t)F_h{}^A(t)=0\ .$$

On the other hand, as $\sum_A H_q{}^A(t)H_p{}^A(t)$ does not depend on u and the unit tangent vector t, and C is h-isotropic, the same is true for $\sum_A F_q{}^A(t)F_h{}^A(t)$ if $q\leqq h+1$. Thus we get, in view of (5.1),

$$\sum_A F^A_{h+2}(t)F_h{}^A(t)=\text{const}$$

which depends neither on t nor on u. Hence $C_{h+2,h}(t,u)$ has the same property. C being h-isotropic, this proves that C is $(h+1)$-isotropic by virture of Theorem 3.7. This process can be repeated until we get Theorem 9.3.

10. Isotropic fundamental forms

The i-th fundamental form B_i is said to be isotropic if $\langle B_i(t), B_i(t)\rangle^\sim$, namely, $\tilde{g}_{QP}B_i{}^Q(t)B_i{}^P(t)$ does not depend on the unit tangent vector t at each point x of $S^m(1)$. If $\langle B_i(t), B_i(t)\rangle^\sim$ depends neither on t nor on x, B_i is said to be constant isotropic. Similarly the R^n valued i-th symmetric form $X_i=X_i{}^A\tilde{e}_A$ is said to be isotropic or constant isotropic according as $\langle X_i(t), X_i(t)\rangle^E$, namely, $\sum_A X_i{}^A(t)X_i{}^A(t)$ does not depend on the unit tangent vector t at each point x of $S^m(1)$ or depends neither on t nor on x.

A proposition which treats a little more general cases than the following one was proved by Tsukada [6].

Proposition 10.1. *The j-th fundamental form of an isometric minimal immersion f_s (resp. standard minimal immersion h_s): $S^m(1)\to S^{n-1}(r)$ is denoted by B_j (resp. $\mathring{B}_j$). Let i be an integer such that $2\leqq i\leqq$minimum (degree of f_s, degree of h_s). If B_p is isotropic for $p=2,\cdots,i$, then*

$$\langle B_p(t_1,\cdots,t_p), B_p(t_{p+1},\cdots,t_{2p})\rangle^\sim=\langle \mathring{B}_p(t_1,\cdots,t_p), \mathring{B}_p(t_{p+1},\cdots,t_{2p})\rangle^\sim\ .$$

A standard minimal immersion h_s is an immersion such that the element C of $D^m_{s,s}$ associated with h_s vanishes, hence $[s/2]$-isotropic. Hence, for a standard minimal immersion, we have, in view of (7.1),

$$\begin{aligned}\langle \mathring{B}_i(t), \mathring{B}_i(t)\rangle^{\sim} &= \langle \mathring{X}_i(t), \mathring{X}_i(t)\rangle^{E}\\ &= s(s-1)\cdots(s-i+1)\sum_A \mathring{X}_i{}^A(t)H_i{}^A(t)\\ &= (s(s-1)\cdots(s-i+1))^2\sum_A H_i{}^A(t)\sum_{q=0}^{k} x_{i,q}H^A_{i-2q}(t)\\ &= (s(s-1)\cdots(s-i+1))^2 c' \sum_{q=0}^{k} x_{i,q}U_{i,i-2q}(t,u)\,.\end{aligned}$$

As $U_{p,q}(t,u)$ does not depend on the choice of the orthonormal set $\{t,u\}$, $\mathring{B}_i$ is constant isotropic [6].

Definition 10.1. The set B of fundamental forms $B_2, B_3, \cdots$ is said to be j-isotropic when $B_2, \cdots, B_j$ are isotropic.

We study the relation between the case C is j-isotropic and the case B is j-isotropic.

Theorem 10.2. *Let f_s: $S^m(1)\to S^{n-1}(r)$ be an isometric minimal immersion such that the associated element C of $D^m_{s,s}$ is j-isotropic. Then the set B is j-constant isotropic.*

Proof. As C is j-isotropic, (9.1) is valid for $i\leqq j+1$. Thus we get for $i\leqq j$

$$\begin{aligned}\langle B_i(t), B_i(t)\rangle^{\sim} &= \sum_A X_i{}^A(t)X_i{}^A(t)\\ &= (s(s-1)\cdots(s-i+1))^2\sum_A F_i{}^A(t)\sum_{q=0}^{k} x_{i,q}F^A_{i-2q}(t)\\ &= (s(s-1)\cdots(s-i+1))^2\sum_A H_i{}^A(t)\sum_{q=0}^{k} x_{i,q}H^A_{i-2q}(t)\\ &= \langle \mathring{B}_i(t), \mathring{B}_i(t)\rangle^{\sim}\end{aligned}$$

since $C_{p,q}(t,u)=0$ when $p+q\leqq 2j+1$. As $\mathring{B}_i$ is constant isotropic, B_i is also constant isotropic.

In order to prove the converse of Theorem 10.2 without leaning on Proposition 10.1, we first prove the following lemma.

Lemma 10.3. *If B_2 is isotropic, then C is 2-isotropic.*

Proof. As we have $\sum_A F_2{}^A(t)X^A=-(s(s+m-1))^{-1}$ and $\sum_A (X^A)^2=r^2$, we get

$$\sum_A (X_2{}^A(t))^2-\sum_A (\mathring{X}_2{}^A(t))^2=(s(s-1))^2\sum_A [(F_2{}^A(t)+(1/m)X^A)^2-(H_2{}^A(t)+(1/m)\mathring{X}^A)^2]$$

$$=(s(s-1))^2[\sum_A (F_2{}^A(t))^2-\sum_A (H_2{}^A(t))^2]$$
$$=(s(s-1))^2C_{2,2}(t, u) .$$

As $X_2{}^A=X_P{}^A B_2{}^P$, $\mathring{X}_2{}^A=\mathring{X}_P{}^A\mathring{B}_2{}^P$ where B_2 and $\mathring{B}_2$ are isotropic, $C_{2,2}(t, u)$ does not depend on the unit tangent vector t. Thus C is 2-isotropic by Theorem 3.7.

Remark. If we use Proposition 10.1, we get $C_{2,2}(t, u)=0$ immediately and need not use Theorem 3.7.

Next was assume that the converse of Theorem 10.2 is valid for $j\leqq a$ where a is some natural number. Thus we consider the case C is a-isotropic and B is $(a+1)$-isotropic. Then we have, in view of Theorem 9.2, $X_{a+1}^A=X_P{}^A B_{a+1}^P$ and $\mathring{X}_{a+1}^A=\mathring{X}_P{}^A\mathring{B}_{a+1}^P$. On the other hand we have

$$\sum_A (X_{a+1}^A(t))^2-\sum_A (\mathring{X}_{a+1}^A(t))^2$$
$$=(s(s-1)\cdots(s-a))^2\left[\sum_A\left[F_{a+1}^A(t)+\sum_{q=1}^b x_{a+1,q}F_{a+1-2q}^A(t)\right]^2\right.$$
$$\left.-\sum_A\left[H_{a+1}^A(t)+\sum_{q=1}^b x_{a+1,q}H_{a+1-2q}^A(t)\right]^2\right]$$

where $b=[(a+1)/2]$, and this does not depend on the unit tangent vector t by virtue of the assumption. Besides,

$$\sum_A F_p{}^A(t)F_q{}^A(t)-\sum_A H_p{}^A(t)H_q{}^A(t)=C_{p,q}(t, u)$$

vanishes if $p+q\leqq 2a+1$ by Theorem 3.3. Hence

$$C_{a+1,a+1}(t, u)=\sum_A (F_{a+1}^A(t))^2-\sum_A (H_{a+1}^A(t))^2$$

does not depend on t and C is $(a+1)$-isotropic.

The following theorem is easily deduced from the above result.

Theorem 10.4. *Let f_s: $S^m(1)\to S^{n-1}(r)$ be an isometric minimal immersion such that the set B of fundamental forms is j-isotropic. Then the element C of $D_{t,s}^m$ associated with f_s is j-isotropic.*

The following corollary is immediately obtained.

Corollary 10.5. *Let the set B be j-isotropic. Then B is j-constant isotropic. The R^n valued i-th form X_i satisfies $X_i{}^A(t)=X_P{}^A B_i{}^P(t)$ if $i\leqq j+1$.*

From Theorem 10.4 we see that, if B is $[s/2]$-isotropic, then f_s is a standard minimal immersion [6].

References

[1] M. do Carmo and N. Wallach, *Minimal immersions of spheres into spheres*, Ann. of Math., **93** (1971), 43-62.
[2] Y. Mutō, *Some properties of isometric minimal immersions of spheres into spheres*, Kodai Math. J., **6** (1983), 308-332.
[3] Y. Mutō, *Some properties of spherical harmonics*, to appear in Yokohama Math. J.
[4] B. O'Neill, *Isotropic and Kaehler immersions*, Canad. J. Math., **17** (1965), 907-915.
[5] T. Takahashi, *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan, **18** (1966), 380-385.
[6] K. Tsukada, *Isotropic minimal immersions of spheres into spheres*, J. Math. Soc. Japan, **35** (1983), 355-379.

34-6-904, 5-chome, Koishikawa,
Bunkyo-ku, Tokyo 112
Japan