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ANNIHILATING MEASURES FOR DOUGLAS ALGEBRAS

By

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1. Introduction

Let D be the open unit disc in the complex plane $\mathbf{C}$ and ∂D be its boundary. We denote by m the normalized Lebesgue measure on ∂D. Let L^∞ be the space of bounded measurable functions on ∂D with respect to m. For a function f in L^∞, $\|f\|$ denotes the essential supremum norm. H^∞ denotes the space of bounded analytic functions in D. Now we regard H^∞ as a (essentially) uniformly closed subalgebra of L^∞ by considering its boundary functions. We denote by $M(B)$ the maximal ideal space of a commutative Banach algebra B. We put $X=M(L^\infty)$. We identify L^∞ with $C(X)$ the algebra of continuous functions on X. We denote by $\hat{m}$ the lifting measure of m from ∂D to X, that is, $\hat{m}$ is the probability measure on X such that $\int_X f\, d\hat{m}=\int_{\partial D} f\, dm$ for every f in L^∞. A uniformly closed subalgebra between H^∞ and L^∞ is called a *Douglas algebra*. $H^\infty+C$ is the smallest Douglas algebra containing H^∞ properly, where C is the space of continuous functions on ∂D. [8], [10] and [19] are convenient references for H^∞ and Douglas algebras.

Throughout this paper, B will represent a Douglas algebra. A measure μ on X is called an *annihilating measure* for B, which we write $\mu\perp B$, if $\int_X f\, d\mu=0$ for every f in B. We denote by $B^\perp$ the set of annihilating measures for B. Also $\operatorname{supp}\mu$ and $\|\mu\|$ denote the support set and the total variation of the measure μ respectively. To study the properties of Douglas algebras, we need to know the properties of annihilating measures on X for Douglas algebras. Recall that the theorems of general uniform algebras deeply depend on annihilating measures (see [3] and [6]). For example, an interpolation set, a peak set and the essential set for B can be described by means of annihilating measures for B. Let E be a closed subset of X. Here E is called an *interpolation set* for B if the restriction algebra of B onto E, $B_{|E}$, coincides with $C(E)$, the space of continuous functions on E, a *peak set* for B if there is a function f in B such that $f=1$ on E and $|f|<1$ on $X\backslash E$, and a *weak peak set* for B if E is the intersection of some peak sets for B. Also E is called the *essential set* for B if E is the smallest closed subset of X for which if $f\in L^\infty$ vanishes on E, then $f\in B$. We denote by Γ the essential set for B. For measures μ and λ on X, $\lambda\ll\mu$ means that λ is absolutely continuous with respect to μ. While, $\lambda\perp\mu$ means that λ and μ are mutually singular.

The following is the key theorem of this paper.

Theorem 2.1. *Let $B\supset H^\infty+C$ and $\{\mu_n\}_{n=1}^\infty$ be a sequence of measures on X such that $\mu_n\in B^\perp$ for every n. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of measures on X such that $\lambda_n\ll\mu_n$ for every n. Then there is a Blaschke product b such that $b\lambda_n\in B^\perp$ for every n.*

This is a dual version of Axler-Sundberg's factorization theorem ([2], [20]). We will obtain the other results of this paper as applications of the above theorem.

In Section 2, we will prove our key theorem. In Section 3, we will give some results of interpolation sets and weak peak sets for B. Theorem 3.1 gives a characterization of interpolation sets by means of representing measures. Here, a measure μ_x on X is called a representing measure for a point x in $M(B)$ if $\int_X f\,d\mu_x=f(x)$ for every f in B. As a corollary, we will get that a union set of two interpolation sets for B is also an interpolation set. We will give in Theorem 3.2 that every peak interpolation set for B is contained in an open-closed interpolation set. As a corollary, we will get that there are no peak interpolation sets for H^∞. In Section 4, we will give some remarks on the essential set for B. It is important to know the essential set for a given Douglas algebra (see [12], [13]). We will give the essential sets for some concrete Douglas algebras.

In Section 5, we will study M-ideals of L^∞/H^∞. Let Y be a Banach space and let Z be its closed subspace. Z is called an *M-ideal* of Y if there is a projection P from the dual space of Y, Y^*, onto the annihilating subspace of Z in Y^*, $Z^\perp$, such that

$$\|x\|=\|Px\|+\|x-Px\| \quad \text{for every } x \text{ in } Y^*\,.$$

The projection P satisfying the above conditions is called an L-projection ([1]). We already know that if B/H^∞ is an M-ideal of L^∞/H^∞, then B has the best approximation property and some other properties (see [14], [22], [23]). So it is important to determine a Douglas algebra B such that B/H^∞ is an M-ideal of L^∞/H^∞. Theorem 5.1 is a characterization of B so that B/H^∞ is an M-ideal of L^∞/H^∞, which is given in [7] (it is obtained by the authors independently). As corollaries of Theorem 5.1, we will get some known theorems in [14], [21] and [22]. And we will give a Douglas algebra B such that B/H^∞ is not an M-ideal of L^∞/H^∞ (Theorem 5.2). In [15], Luecking and Younis gave the following conjecture: If B/H^∞ is an M-ideal of L^∞/H^∞, then $B=(H^\infty+C)_E$ for some weak peak set E of X for $H^\infty+C$. Here, for a weak peak set E for B, we put $B_E=\{f\in L^\infty; f_{|E}\in B_{|E}\}$, then B_E becomes a Douglas algebra. Corollary 5.9 will shed light on this conjecture, and Corollary 5.10 will give an interesting result: If a measure μ on X annihilates $H^\infty+C$, then $\hat{m}(\operatorname{supp}\mu)=0$.

In Section 6, we will give some results of interpolation sets and weak peak sets. For h in H^∞, we put $Z(h)=\{x\in M(H^\infty+C); h(x)=0\}$. If q is an interpolating Blaschke product, then $Z(q)$ is an interpolation set for H^∞. We will prove that for every interpolating Blaschke product q, there exists an interpolating Blaschke product b such that $Z(q)\cup Z(b)$ is not an interpolation set for H^∞(Proposition 6.1). Also we will give a measure μ on X with $\mu\in B^\perp$ so that $\operatorname{supp}\mu$ is not a weak peak set for B. Then we will give a closed G_δ-set E of X with $\hat{m}(E)=0$ so that E is not a peak set for H^∞.

2. Proof of the key theorem

For a measure μ on X in $(H^\infty)^\perp$, we put $\mu=\mu_a+\mu_s$, where $\mu_a\ll\hat{m}$ and $\mu_s\perp\hat{m}$. By Hoffman and Singer's theorem ([10, p. 186]), we have $\mu_a\in(H^\infty)^\perp$ and $\mu_s\in(H^\infty+C)^\perp$. We identify $H_0{}^1=\{f\in L^1(m); f\perp H^\infty\}$, the usual Hardy space, with $\{f\in L^1(\hat{m}); f\perp H^\infty\}$. Then we have $(H^\infty)^\perp=H_0^1\oplus(H^\infty+C)^\perp$.

To prove Theorem 2.1, we need some lemmas. In [20], Sundberg gave a refinement of Axler's factorization theorem [2].

Lemma 2.1. *For a sequence $\{f_n\}_{n=1}^\infty$ in L^∞, there is a Blaschke product b such that $bf_n\in H^\infty+C$ for every n.*

For a subset J of L^∞, $[J]$ denotes the uniformly closed subalgebra of L^∞ generated by J.

Lemma 2.2. *If $B\supset H^\infty+C$ and $\{f_n\}_{n=1}^\infty\subset L^\infty$, then there is a Blaschke product b such that $b[B,f_n;\ n=1,2,\cdots]\subset B$.*

Proof. We put

$$J=\{f_{n_1}^{k_1}f_{n_2}^{k_2}\cdots f_{n_i}^{k_i};\quad k_i,n_i\ \text{ are positive integers and }\ i=1,2,\cdots\}\,.$$

By Lemma 2.1, there is a Blaschke product b such that $bJ\subset H^\infty+C$. Then we have

$$b[B,f_n;\ n=1,2,\cdots]=b[B,J]\subset[B,H^\infty+C]\subset B\,.$$

Corollary 2.1. *Let $B\supset H^\infty+C$ and $\{f_n\}_{n=1}^\infty\subset L^\infty$. Then there is a Blaschke product b such that*

$$b\mu\in[B,f_n;\ n=1,2,\cdots]^\perp \quad \textit{for every} \quad \mu\in B^\perp .$$

Theorem 2.1. *Let $B\supset H^\infty+C$ and $\{\mu_n\}_{n=1}^\infty$ be a sequence of measures on X such that $\mu_n\in B^\perp$ for every n. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of measures on X such that $\lambda_n\ll\mu_n$ for every n. Then there exists a Blaschke product b such that $b\lambda_n\in B^\perp$ for every n.*

Proof. For each n, we put

$$d\lambda_n=f_n d|\mu_n|\,, \quad \text{where} \quad f_n\in L^1(|\mu_n|)\,, \quad \text{and}$$
$$d\mu_n=\phi_n d|\mu_n|\,, \quad \text{where} \quad |\phi_n|=1 \quad \text{a.e.} \quad d|\mu_n|\,.$$

Since $\phi_n C(X)=\phi_n L^\infty$ is dense in $L^1(|\mu_n|)$, there is a sequence $\{f_{n,k};\ k=1,2,\cdots\}$ in L^∞ such that

$$f_{n,k}\phi_n\to f_n \quad \text{in} \quad L^1(|\mu_n|)\text{-norm as} \quad k\to\infty\,. \tag{1}$$

By Lemma 2.1, there exists a Blaschke product b such that

$$bf_{n,k}\in H^\infty+C \quad \text{for every} \quad n,k=1,2,\cdots\,.$$

Then for each n, we get $\mu_n\perp bf_{n,k}B$ for every $k=1,2,\cdots$. This means that $bf_{n,k}\mu_n=bf_{n,k}\phi_n|\mu_n|\in B^\perp$. Then by (1), we get

$$b\lambda_n=bf_n|\mu_n|\in B^\perp \quad \text{for every} \quad n\,.$$

3. Interpolation sets

For a closed subset E of X and a measure μ on X, we denote by $\mu_{|E}$ the restriction of the measure μ onto E. The following lemma is known as Glicksberg's peak set theorem for general uniform algebras [9].

Lemma 3.1. *Let E be a closed subset of X.*

(a) *E is a weak peak set for B if and only if $\mu_{|E}\in B^\perp$ for every $\mu\in B^\perp$.*

(b) *E is a weak peak interpolation set for B if and only if $\mu_{|E}=0$ for every $\mu\in B^\perp$.*

Lemma 3.2. *Let E be a closed subset of X.*

(a) *If E is an interpolation set for $H^\infty+C$, then $\hat{m}(E)=0$.*

(b) *If $\hat{m}(E)=0$ and E is a weak peak set for $H^\infty+C$, then E is a weak peak set for H^∞.*

(c) *If E is a weak peak interpolation set $H^\infty+C$, then so is for H^∞.*

Proof. (a) Suppose that $\hat{m}(E)\neq 0$. By [6, p. 18], E has an open-closed interior $\mathring{E}$ such that $\hat{m}(\mathring{E})=\hat{m}(E)$. Then $H^\infty+C$ has an open-closed interpolation set. By Axler's theorem [2], we have $|H^\infty+C|=|L^\infty|$ on X, and so there is a function h

in $H^\infty+C$ with $|h|=\chi_{\dot{E}}$, where $\chi_{\dot{E}}$ is the characteristic function of $\dot{E}$. Then we have

$$H^\infty+C\supset h(H^\infty+C)=\{f\in L^\infty; f=0 \text{ on } \dot{E}^c\} .$$

This implies that the essential set for $H^\infty+C$ is contained in $\dot{E}^c$. But this is a contradiction, because X is the essential set for $H^\infty+C$ (see in Section 4).

(b) For $\mu\in(H^\infty)^\perp$, we put $\mu=\mu_a+\mu_s$, where $\mu_a\ll\hat{m}$ and $\mu_s\perp\hat{m}$. Since $\mu_s\in(H^\infty+C)^\perp$, we have $\mu_{s|E}\in(H^\infty+C)^\perp$ by Lemma 3.1 (a). Since $\mu_{a|E}=0$, we get

$$\mu_{|E}=\mu_{s|E}\in(H^\infty+C)^\perp\subset(H^\infty)^\perp .$$

Again by Lemma 3.1 (a), E is a weak peak set for H^∞.

(c) By (a), we get $\hat{m}(E)=0$. Using Lemma 3.1 (b), we can lead the conclusion by the same way as (b).

The following are corollaries of Theorem 2.1.

Corollary 3.1. *If E is a closed subset of X such that $\overline{B_{|E}}=C(E)$, where $\overline{B_{|E}}$ is the uniform closure of $B_{|E}$, then E is a weak peak interpolation set for B.*

Proof. First, suppose that $B\supset H^\infty+C$. Let $\mu\in B^\perp$. By Theorem 2.1, there exists a Blaschke product b such that $b\mu_{|E}\in B^\perp$. Since $\overline{B_{|E}}=C(E)$, we get $b\mu_{|E}=0$. Hence $\mu_{|E}=0$. This implies that E is a weak peak interpolation set for B by Lemma 3.1 (b). Next, suppose that $B=H^\infty$. Then we have

$$C(E)=\overline{H^\infty{}_{|E}}\subset\overline{(H^\infty+C)_{|E}}\subset C(E) .$$

Consequently we get $\overline{(H^\infty+C)_{|E}}=C(E)$. By the first part, E is a weak peak interpolation set for $H^\infty+C$. Immediately, we get the conclusion by Lemma 3.2 (c).

Corollary 3.2. *Let E be a closed subset of X. If E is an interpolation set for B, then E is a weak peak set for B.*

Corollary 3.3. *For a sequence $\{f_n\}_{n=1}^\infty$ in L^∞, both B and $[B,f_n;\ n=1,2,\cdots]$ have the same interpolation sets of X.*

The following theorem is a characterization of an interpolation set for B.

Theorem 3.1. *Let E be a closed subset of X. Then the following assertions are equivalent.*

(a) *E is not an interpolation set for B.*

(b) *There is a point x in $M(B)\backslash X$ such that* $\operatorname{supp}\mu_x\subset E$.

(c) *There is a point x in $M(B)\backslash X$ such that $\mu_x(E)\neq 0$.*

Proof. Put $B_0=\{f\in L^\infty; f_{|E}\in\overline{B_{|E}}\}$. Then B_0 is a Douglas algebra.

(a)$\Rightarrow$(b) Suppose that E is not an interpolation set for B. By Corollary 3.1, we have $B_0 \neq L^\infty$. By Chang-Marshall's theorem ([4, 16]), there is a point x in $M(B_0)\backslash X$ such that

$$B_{0|\mathrm{supp}\,\mu_x} = H^\infty{}_{|\mathrm{supp}\,\mu_x} \quad \text{and} \quad \mathrm{supp}\,\mu_x \subset E\,.$$

Since $H^\infty \subset B \subset B_0$, we have $B_{|\mathrm{supp}\,\mu_x} = H^\infty{}_{|\mathrm{supp}\,\mu_x}$. This implies $x \in M(B)$. This leads us the assertion (b).

(b)$\Rightarrow$(c) It is trivial.

(c)$\Rightarrow$(a) Let $x \in M(B)\backslash X$ such that $\mu_x(E) \neq 0$. If x is contained in D, then $B = H^\infty$ and $\mu_x \ll \hat{m}$. This means that $\hat{m}(E) \neq 0$. By Lemma 3.2 (a), E is not an interpolation set for $H^\infty + C$ and so is for H^∞. If x is not contained in D, then we have $x \in M(B+C)$. Clearly, $B+C = H^\infty + C$ if $B = H^\infty$, and $B+C = B$ if $B \neq H^\infty$. By [10, p. 179], there is a Blaschke product b such that $b(x)=0$. Then we get $b\mu_x \in (B+C)^\perp$ and $(b\mu_x)_{|E} \neq 0$. By Lemma 3.1 (b), E is not a weak peak interpolation set for $B+C$, and then E is neither for B. Now the assertion is proved by Corollary 3.2.

As an immediate corollary from Theorem 3.1 (c), we get the following. The corresponding result is not true for general uniform algebras.

Corollary 3.4. *Let E_1 and E_2 be closed subsets of X. If both E_1 and E_2 are interpolation set for B, then the union $E_1 \cup E_2$ is an interpolation set for B.*

Remark 3.1. If we take closed subsets E_1 and E_2 of $M(H^\infty + C)$ instead of X, then Corollary 3.4 is not true for H^∞ (see Section 6).

Theorem 3.2. *If E is a peak interpolation set of X for B, then there is an open-closed subset U of X such that*

(a) *$E \subset U$, and*

(b) *U is an interpolation set for B.*

Proof. If E is open-closed, we do not need to prove. Now we assume that E is not open-closed. Suppose that there is not an open-closed subset U satisfying (a) and (b). Since E is a peak set, E is a G_δ-set and hence there is a sequence $\{U_n\}_{n=1}^\infty$ of open-closed subsets of X such that

(1) $U_{n+1} \subsetneqq U_n$ for every n,

(2) $\bigcap_n U_n = E$, and

(3) U_n is not an interpolation set for B for every n.

By (3) and Corollary 3.1, we have $\overline{B_{|U_n}} \neq C(U_n)$. Thus there is a sequence $\{\mu_n\}_{n=1}^\infty$ of measures such that

(4) $\operatorname{supp}\mu_n \subset U_n$,

(5) $\|\mu_n\|=1$ and $\mu_n \in B^{\perp}$ for every n.

By Theorem 2.1, there is a Blaschke product b such that

(6) $b|\mu_n| \in B^{\perp}$ for every n.

Let μ_0 be a weak-* cluster measure of $\{|\mu_n|;\ n=1,2,\cdots\}$. Then $\|\mu_0\|=1$ by (5), and $\operatorname{supp}\mu_0 \subset E$ by (1), (2) and (4). By (6), we have $|\mu_n| \in \bar{b}B^{\perp}$, consequently we get $\mu_0 \in \bar{b}B^{\perp}$, that is, $b\mu_0 \in B^{\perp}$. This means that E is not an interpolation set for B and this contradiction leads us the conclusion.

Corollary 3.5. *There is not a peak interpolation set E for B such that $E \cap \Gamma \neq \emptyset$.*

Corollary 3.6. *If $\Gamma = X$, then B has no peak interpolation sets.*

Corollary 3.7. *For a sequence $\{f_n\}_{n=1}^{\infty}$ in L^{∞}, $[H^{\infty}, f_n;\ n=1,2,\cdots]$ does not have any peak interpolation sets.*

Proof. This follows from the fact that X is the essential set for $[H^{\infty}, f_n;\ n=1,2,\cdots]$ (see Section 4).

4. Essential sets.

Here we will study the essential set Γ for B. To see the properties of Γ, the following lemma is a basic one ([3, p. 146]).

Lemma 4.1. *Γ coincides with the closure of $\cup\{\operatorname{supp}\mu;\ \mu \in B^{\perp}\}$.*

Since $H_0^1 \subset (H^{\infty})^{\perp}$ and $X = \operatorname{supp} f$ for every $f \in H_0^1 (f \neq 0)$, X is the essential set for H^{∞}. Also X is the essential set for $H^{\infty}+C$. Because if the essential set for $H^{\infty}+C$ is a proper subset of X, $H^{\infty}+C$ has a non-trivial idempotent, and so $M(H^{\infty}+C)$ is not connected. But this is a contradiction [10, p. 188].

In [23, Theorem 2], Younis gave that both B and $[B, f_n;\ n=1,2,\cdots]$, $f_n \in L^{\infty}$, have the same essential set. This is an easy consequence of Corollary 2.1 and Lemma 4.1. In [23, Proposition 3], he proved the following to answer a question in [16]: If S is a peak set for $H^{\infty}+C$, then S is the essential set for $(H^{\infty}+C)_S$. More generally we have

Proposition 4.1. *If S is a closed G_{δ}-set of X, then $S \cap \Gamma$ is the essential set for the Douglas algebra $\{f \in L^{\infty};\ f_{|S} \in \overline{B_{|S}}\}$.*

Proof. If we put $B_0 = \{f \in L^{\infty};\ f_{|S} \in \overline{B_{|S}}\}$, then B_0 is a Douglas algebra. Let E be the essential set for B_0. It is easy to see $E \subset S$. Since $B \subset B_0$, we have $E \subset \Gamma$ and $E \subset \Gamma \cap S$. Now suppose that $E \subsetneqq \Gamma \cap S$. Then there exists an open-closed subset U of X such that $E\cap U=\emptyset$ and $\Gamma\cap S\cap U\neq\emptyset$. Since $S\cap U$ is an interpolation set for B_0, we have

$$C(S\cap U)=B_{0|S\cap U}\subset\overline{B_{|S\cap U}}\subset C(S\cap U)\ .$$

By Corollary 3.1, $S\cap U$ is a weak peak interpolation set for B. Since $S\cap U$ is a G_δ-set, $S\cap U$ is a peak set for B. Then $S\cap U$ belongs to $X\backslash\Gamma$ by Corollary 3.5. Thus we get $\Gamma\cap S\cap U=\emptyset$, but this is a contradiction.

Remark 4.1. Proposition 4.1 is not true for a weak peak set S for B. For, let $B=H^\infty$ and $S=X_1\cup\{x\}$, where $x\in X$, $x\notin X_1$ and X_1 is the fiber at $\lambda=1$. Then S is a weak peak set for H^∞. But X_1 is the essential set for H_S^∞.

In [23, Proposition 2], Younis proved that $\Gamma_0=\cup\{\text{supp}\,\mu_x;\ x\in M(B)\backslash X\}$ is dense in Γ. We note that this fact also follows from the results in Section 3: It is easy to see $\Gamma_0\subset\Gamma$. If Γ_0 is not dense in Γ, we take an open-closed subset U of X such that $\Gamma_0\cap U=\emptyset$ and $\Gamma\cap U\neq\emptyset$. Then $\Gamma\cap U$ is not an interpolation set. By Theorem 3.1, we have $\Gamma_0\cap U\supset\Gamma_0\cap\Gamma\cap U\neq\emptyset$. But this is a contradiction.

5. M-ideals of L^∞/H^∞

Let B_1 and B_2 be closed subspaces of continuous functions on a compact Hausdorff space with $B_1\subsetneqq B_2\subsetneqq C(K)$. B_2/B_1 is called an M-ideal of $C(K)/B_1$ if there is an L-projection P: $B_1^\perp\to B_2^\perp$ (onto) with the property

$$\|\mu\|=\|P\mu\|+\|\mu-P\mu\|\quad\text{for every}\quad \mu\in B_1^\perp\ .$$

Gamelin, Marshall, Younis and Zame [7] give the following characterization of M-ideals in $C(K)/B_1$.

Theorem 5.1. *B_2/B_1 is an M-ideal of $C(K)/B_1$ if and only if for each $\mu\in B_1^\perp$ there exists $f_\mu\in L^1(|\mu|)$ such that*

(a) $f_\mu^2=f_\mu$ a.e. $d|\mu|$,

(b) $\mu-f_\mu\mu\perp B_2^\perp$, *and*

(c) $f_\mu\mu\in B_2^\perp$.

If $(f_\mu;f_\mu\in L^1(|\mu|),\mu\in B_1^\perp\}$ satisfies (a), (b) and (c), we call it the *system of idempotents* for B_2/B_1.

Corollary 5.1. *Let B_1, B_2 and B_3 be closed subspaces with $B_1\subsetneqq B_2\subsetneqq B_3\subsetneqq C(K)$. Suppose that B_2/B_1 is an M-ideal of $C(K)/B_1$. Then B_3/B_1 is an M-ideal of $C(K)/B_1$ if and only if B_3/B_2 is an M-ideal of $C(K)/B_2$.*

Proof. Since $B_3^\perp\subset B_2^\perp\subset B_1^\perp$, the necessary part follows from the definition of M-ideals. Suppose that B_3/B_2 is an M-ideal. Let $\{f_\mu; f_\mu \in L^1(|\mu|), \mu \in B_1^\perp\}$ be the system of idempotents for B_2/B_1 and let $\{g_\nu; g_\nu \in L^1(|\nu|), \nu \in B_2^\perp\}$ be the system of idempotents for B_3/B_2. We identify f_μ with the measure $f_\mu \mu$. Then it is easy to see that $\{f_\mu g_{f_\mu}; \mu \in B_1^\perp\}$ is the system of idempotents for B_3/B_1.

On the rest of this section, we study the case when B_i are Douglas algebras. The following three known results are easy consequences of Theorem 5.1.

Corollary 5.2 ([14]). *$H^\infty+C/H^\infty$ is an M-ideal of L^∞/H^∞.*

Proof. We have that $H^{\infty\perp}=H_0^1 \oplus (H^\infty+C)^\perp$, For $\mu \in H^{\infty\perp}$, we put $\mu=\mu_a+\mu_s$, where $\mu_a \ll \hat{m}$ and $\mu_s \perp \hat{m}$. Then there is an idempotent $f_\mu \in L^1(|\mu|)$ such that $\mu_a=f_\mu \mu$. Then it is easy to see that $\{f_\mu; \mu \in H^{\infty\perp}\}$ is the system of idempotents for $H^\infty+C/H^\infty$.

Corollary 5.3 ([21]). *If E is a weak peak non-interpolation set for B, then B_E/B is an M-ideal of L^∞/B.*

Proof. For $\mu \in B^\perp$, we put

$$f_\mu=\begin{cases} 1 & \text{on } E \\ 0 & \text{on } X\backslash E\,. \end{cases}$$

It is trivial that the essential set for B_E is contained in E by the definition of B_E (see Section 1). Also it is easy to see that $\{f_\mu; \mu \in B^\perp\}$ is the system of idempotents for B_E/B. The condition (b) follows from Lemma 4.1. The condition (c) follows from Lemma 3.1 (a).

The following corollary is given by Marshall and Zame in their unpublished note without using Theorem 5.1. But we shall prove it using Theorem 5.1.

Corollary 5.4. *Let B_0 be a Douglas algebra with $B_0 \supset H^\infty+C$* and $\{f_n\}_{n=1}^\infty \subset L^\infty$. *If B satisfies $B_0 \subsetneqq B \subset [B_0, f_n;\, n=1, 2, \cdots]$, then*

(a) *B/B_0 is not an M-ideal of L^∞/B_0, and*

(b) *B/H^∞ is not an M-ideal of L^∞/H^∞.*

Proof. Suppose that B/B_0 is an M-ideal of L^∞/B_0. Let $\{f_\mu; \mu \in B_0^\perp\}$ be the system of idempotents for B/B_0. Then we have $f_\mu=1$ by Corollary 2.1 and Theorem 5.1 (b). This implies $B_0=B$. But this is a contradiction and we get (a). To see (b), suppose that B/H^∞ is an M-ideal. By the definition of M-ideals, B/B_0 becomes an M-ideal of L^∞/B_0. But this contradicts (a).

Corollary 5.4 gives us the following result which has a connection with [13, Section 5].

Corollary 5.5. *If $B \supsetneqq H^\infty+C$ and B/H^∞ is an M-ideal of L^∞/H^∞, then*

(a) *there are no maximal subalgebras between H^∞ and B, and*

(b) *for every Douglas algebra* B_1 *with* $B_1 \subsetneqq B$, B/B_1 *is not separable.*

Proof. (a) Suppose that B_0 is a maximal subalgebra with $H^\infty \subset B_0 \subsetneqq B$. Then $B=[B_0, f]$ for some $f \in B$. By Corollary 5.4, B/H^∞ is not an M-ideal of L^∞/H^∞.

(b) If B/B_1 is separable, then $B=[B_1, f_n; n=1, 2, \cdots]$ for some sequence $\{f_n\}_{n=1}^\infty$ in B. Also by Corollary 5.4, we get a contradiction.

Remark 5.1. In Section 6, we will give a Douglas algebra B which has a maximal subalgebra between H^∞ and B.

Using Theorem 2.1, we shall give examples of Douglas algebras B such that B/H^∞ are not M-ideals, which are not covered by Corollary 5.4.

Theorem 5.2. *Let B be a Douglas algebra. If there exists a closed subset E of X and another Douglas algebra B_1 satisfying the following conditions* (a)-(d), *then B/H^∞ is not an M-ideal of L^∞/H^∞.*

(a) $B_1 \supset H^\infty + C$.

(b) *E is a weak peak set for B.*

(c) *E is not a weak peak set for B_1.*

(d) $B_{|E} = \overline{B_{1|E}}$.

Proof. By (c), we have $\overline{B_{1|E}} \neq B_{1|E}$ ([6, p. 65]). By (b) and (d), we get $B_E \supsetneqq B_1$. We shall see

(1) B_E/B_1 is not an M-ideal of L^∞/B_1.

To see (1), suppose that B_E/B_1 is an M-ideal of L^∞/B_1. By Theorem 5.1, there is the system of idempotents $\{f_\mu; f_\mu \in L^1(|\mu|), \mu \in B_1^\perp\}$. For each $\mu \in B_1^\perp$, by Theorem 2.1 there is a Blaschke product b such that $b\mu_{|E} \in B_1^\perp$. By (d), we have $b\mu_{|E} \in B_E^\perp$. This implies that $f_\mu = \chi_E$, the characteristic function for E. Then $\chi_E \mu \in B_E^\perp$ by (c) of Theorem 5.1, and then $\chi_E \mu \in B_1^\perp$ by (d). By Lemma 3.1 (a), E is a weak peak set for B_1 This contradicts (c). Thus we get (1).

To prove our assertion, suppose that B/H^∞ is an M-ideal of L^∞/H^∞. By (c), (d) and Corollary 3.1, E is not an interpolation set for B. By (b) and Corollary 5.3, B_E/B is an M-ideal of L^∞/B. We note that $H^\infty \subsetneqq B \subset B_E$. If $B=B_E$, then B_E/H^∞ is an M-ideal of L^∞/H^∞ by our assumption. If $B \neq B_E$, then also B_E/H^∞ is an M-ideal of L^∞/H^∞ by Corollary 5.1. Since $H^\infty \subset B_1 \subsetneqq B_E$, B_E/B_1 is an M-ideal of L^∞/B_1 by the definition of M-ideals. But this contradicts (1).

The following is a direct corollary of Theorem 5.2.

Corollary 5.6. *Let E be a closed subset of X which is not a weak peak set for $H^\infty + C$. We put $B = \{f \in L^\infty; f_{|E} \in \overline{H^\infty + C_{|E}}\}$, then B/H^∞ is not an M-ideal of L^∞/H^∞.*

Remark 5.1. If E is a closed subset of X and E is not a weak peak set for $H^\infty+C$, then we have

$$\{f\in L^\infty;\, f_{|E}\in \overline{H^\infty+C_{|E}}\}=\{f\in L^\infty;\, f_{|E}\in \overline{H^\infty{}_{|E}}\}\ .$$

Corollary 5.7. *Let $f\in L^\infty$ be a peaking function for a closed subset E of X, that is, $f=1$ on E and $|f|<1$ on $X\backslash E$. We put $B=[B_0(1-f), H^\infty]$ for any Douglas algebra B_0. If E is not a weak peak set for $H^\infty+C$, then B/H^∞ is not an M-ideal of L^∞/H^∞.*

Proof. Since $f\in B$, E is a peak set for B. Since E is not a weak peak set for $H^\infty+C$, we have $H^\infty+C\subsetneqq B$ and $B_{|E}=\overline{H^\infty{}_{|E}}=\overline{H^\infty+C_{|E}}$. If we put $B_1=H^\infty+C$, then every assumption of Theorem 5.2 is satisfied.

The following is a special case of Corollary 5.7.

Corollary 5.8. *If E is a proper open-closed subset of X and B is a Douglas algebra, then $[B\chi_E, H^\infty]/H^\infty$ is not an M-ideal of L^∞/H^∞.*

Proof. Every proper open-closed subset of X is not a peak set for $H^\infty+C$.

In [15], Luecking and Younis gave the following conjecture of M-ideals: If B/H^∞ in an M-ideal of L^∞/H^∞, then $B=(H^\infty+C)_E$ for some weak peak set E of X for $H^\infty+C$. The following theorem sheds light on this conjecture.

Theorem 5.3. *Let $\{\mu_n\}_{n=1}^\infty$ be a sequence of measures on X such that $\mu_n\in B^\perp$ for every n, and let E be the closure of $\cup\{\operatorname{supp}\mu_n;\ n=1, 2, \cdots\}$ in X. If B/H^∞ is an M-ideal of L^∞/H^∞, then $B_{|E}=H^\infty{}_{|E}$.*

To see this, we need the following lemma proved in [11, Theorem 3].

Lemma 5.1. *Let Z be an M-ideal of a Banach space Y. Then for each y in Y, Z coincides with the linear span (not closed) of $\{x\in Z;\ \operatorname{dist}(y, Z)=\|y-x\|\}$, where $\operatorname{dist}(y, Z)=\inf\{\|y-x\|;\ x\in Z\}$.*

Proof of Theorem 5.3. By Theorem 2.1, there is a Blaschke product b such that

$$(1)\qquad b\,|\mu_n|\in B^\perp \quad \text{for every}\quad n\ .$$

Then it is easy to see that $\bar{b}\notin B$ and $\operatorname{dist}(\bar{b}, B)=1$. We may now assume that $\|\mu_n\|=1$ for every n. Put

$$J=\{f\in B;\ \|\bar{b}-f\|=\operatorname{dist}(\bar{b}, B)\}\ .$$

Since B/H^∞ is an M-ideal of L^∞/H^∞ by our assumption, Lemma 5.1 implies that the linear span of

(2) $$\{g+H^{\infty};\ g\in B \text{ and } \|\bar{b}-g+H^{\infty}\|=\operatorname{dist}(\bar{b}+H^{\infty}, B/H^{\infty})\}$$

coincides with B. Let g be a function in B satisfying (2). Then there is $h\in H^{\infty}$ such that $\|\bar{b}-g+H^{\infty}\|=\|\bar{b}-g-h\|$, because H^{∞} has the best approximation property. Since $\operatorname{dist}(\bar{b}+H^{\infty}, B/H^{\infty})=\operatorname{dist}(\bar{b}, B)$, we get $g+h\in J$. Thus we have

(3) the linear span of $\{J+H^{\infty}\}$ coincides with B.

Let $f\in J$. Then $\int_X fbd|\mu_n|=0$ by (1). Since $1=\|\bar{b}-f\|=\|1-bf\|$, we get

$$1=\int_X (1-bf)d|\mu_n|\leqq\|1-bf\|=1 .$$

This implies that $bf=0$ a.e. $d|\mu_n|$, and then $f=0$ on $\operatorname{supp}\mu_n$ for every n. Consequently, we get $f=0$ on E. This means that $B_{|E}=H^{\infty}{}_{|E}$ by (3).

The following corollary answers partially to the M-ideal conjecture.

Corollary 5.9. *Suppose that B/H^{∞} is an M-ideal of L^{∞}/H^{∞}. If Γ coincides with the closure of $\cup\{\operatorname{supp}\mu_n;\ n=1,2,\cdots\}$ for some sequence of measures $\{\mu_n\}_{n=1}^{\infty}$ in $B^{\perp}$, then Γ is the weak peak set for H^{∞} and $B=H_{\Gamma}{}^{\infty}$.*

Proof. Since Γ is the weak peak set for B ([3, p. 145]), $B_{|\Gamma}$ is closed. By Theorem 5.3, $H^{\infty}{}_{|\Gamma}=B_{|\Gamma}$ is closed. Immediately we get $B=H_{\Gamma}{}^{\infty}$.

As a corollary of Theorem 5.3, we get an interesting result of annihilating measures for $H^{\infty}+C$.

Corollary 5.10. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of measures on X such that $\mu_n\in(H^{\infty}+C)^{\perp}$ for every n. If we put E the closure of $\cup\{\operatorname{supp}\mu_n;\ n=1,2,\cdots\}$ in X, then $\hat{m}(E)=0$.*

Proof. Suppose that $\hat{m}(E)>0$. Then there exists a function f in C such that $f\neq 0$ on E and $f=0$ on some subset of E with positive measure for $\hat{m}$. Since $H^{\infty}+C/H^{\infty}$ is an M-ideal of L^{∞}/H^{∞} by Corollary 5.2, there is a function F in H^{∞} such that $F=f$ on E by Theorem 5.3. Then we get $F=0$, because F vanishes on the $\hat{m}$-positive set. But this is a contradiction and thus we get the conclusion.

6. Some examples

In this section, we will give some examples related the previous sections. In Corollary 3.4, we proved that a union set of two interpolation sets of X for H^{∞} is also an interpolation set. First, we will study a union set for some interpolation sets of $M(H^{\infty})$ for H^{∞}. For two points x and y in $M(H^{\infty})$, we put

$$\rho(x, y)=\sup\{|f(x)|; f\in H^\infty, \|f\|<1, f(y)=0\}\ .$$

If z and w are points in D, then we have $\rho(x, y)=|z-w|/|1-\bar{w}z|$. A sequence $\{z_n\}_{n=1}^\infty$ in D is called an interpolating sequence if $\{\{h(z_n)\}_{n=1}^\infty; h\in H^\infty\}=l^\infty$. If $\{z_n\}_{n=1}^\infty$ is an interpolating sequence, a Blaschke product

$$q(z)=\prod_{n=1}^{\infty}\frac{-\bar{z}_n}{|z_n|}\cdot\frac{z-z_n}{1-\bar{z}_n z}$$

is called an interpolating Blaschke product associated with zeros $\{z_n\}_{n=1}^\infty$. The following lemma is well known as Carleson's theorem (see [8, p. 287]).

Lemma 6.1. *Let $\{z_n\}_{n=1}^\infty$ be a sequence in D. Then the following conditions are equivalent.*

(a) *$\{z_n\}_{n=1}^\infty$ is an interpolating sequence.*

(b) *There is a positive constant δ such that*

$$(\#)\qquad \delta\leqq\inf_n (1-|z_n|^2)\,|q'(z_n)|=\inf_n\prod_{k\neq n}\rho(z_n, z_k)\ ,$$

where q is an interpolating Blaschke product associated with zeros $\{z_n\}_{n=1}^\infty$.

(c) *There is a positive constant σ such that $\rho(z_n, z_k)\geqq\sigma$ for every $n\neq k$ and $\sum_{n=1}^{\infty}(1-|z_n|)\delta_{z_n}$ is a Carleson measure, where a positive measure μ on D is called a Carleson measure if $\int_{\partial D}|f|d\mu\leqq C\|f\|_1$ for every f in H^1, the usual Hardy space.*

By Lemma 6.1 and the following lemma, it is easy to see that for each interpolating sequence, there exists an interpolating sequence such that a union of these two interpolating sequences is not an interpolating sequence.

Lemma 6.2 (see [8, p. 310]). *Let $\{z_n\}_{n=1}^\infty$ be an interpolating sequence such that $0<\delta\leqq\inf_n\prod_{k\neq n}\rho(z_n, z_k)$, and let $\{w_n\}_{n=1}^\infty$ be a sequence in D such that $\rho(w_n, z_n)\leqq\delta/3$ for every n. Then $\{w_n\}_{n=1}^\infty$ is an interpolating sequence.*

A typical interpolating set of $M(H^\infty+C)$ for H^∞ is obtained by using an interpolating sequence as follows.

Lemma 6.3 ([10, p. 205]). *Let q be an interpolating Blaschke product associated with zeros $\{z_n\}_{n=1}^\infty$. Then we have $Z(q)=\mathrm{cl}(\{z_n\}_{n=1}^\infty)\backslash\{z_n\}_{n=1}^\infty$, and $Z(q)$ is an interpolation set for H^∞, where $\mathrm{cl}(\{z_n\}_{n=1}^\infty)$ is the weak-*closure of $\{z_n\}_{n=1}^\infty$ in $M(H^\infty)$.*

The following proposition proves that a union set of two interpolation sets of $M(H^\infty+C)$ for H^∞ is not an interpolation set for H^∞.

Proposition 6.1. *For each interpolating Blaschke product q, there is an inter-*

polating Blaschke product b *such that* $Z(q)\cup Z(b)$ *is not an interpolation set for* H^∞.

To prove Proposition 6.1, we need the following lemmas.

Lemma 6.4. *If* E *is an interpolation set of* $M(H^\infty)$ *for* H^∞, *then* $\inf\{\rho(x, y); x, y\in E, x\neq y\}>0$.

Lemma 6.5 (see [8, pp. 404–405]). *Let* $\{z_n\}_{n=1}^\infty$ *be an interpolating sequence with* $\delta>0$ *in* (#), *and let* q *be an interpolating Blaschke product associated with zeros* $\{z_n\}_{n=1}^\infty$. *Then for each* $\lambda>0$ *satisfying* $0<\lambda\leqq\delta/3$, *there exists* $r=r(\lambda)>0$ *such that* $\{z\in D; |q(z)|<r\}$ *is the union of pairwise disjoint domains* V_n, $z_n\in V_n$ *and* $V_n\subset\{z\in D; \rho(z, z_n)<\lambda\}$.

Proof of Proposition 6.1. Let $\{L_m\}_{m=1}^\infty$ be a sequence of disjoint subsets of positive integers such that

$$L_m \text{ is an infinite subset for } m=1, 2, \cdots . \tag{1}$$

We put $L_m=\{n_{m,k}\}_{k=1}^\infty$. If we put

$$U_n^m=\{z\in D; \rho(z, z_n)<\delta/3m\} ,$$

then for each fixed m, U_n^m $(n=1, 2, \cdots)$ are disjoint subsets. By Lemma 6.5, there is $r_m>0$ such that $\{z\in D; |q(z)|<r_m\}$ is the union of pairwise disjoint domains V_n^m, $z_n\in V_n^m$ and

$$V_n^m\subset U_n^m . \tag{2}$$

We may assume that $\{r_m\}_{m=1}^\infty$ is a decreasing sequence. Since V_n^m is a connected subset, we have $\{|q(z)|; z\in V_n^m\}=[0, r_m)$. Then there is $\zeta_{m,k}\in D$ such that

$$\zeta_{m,k}\in V_{n_{m,k}}^m , \quad \text{and} \tag{3}$$

$$r_{m+1}<|q(\zeta_{m,k})|<r_m . \tag{4}$$

Since $V_{n_{m,k}}^m\subset U_{n_{m,k}}^m\subset U_{n_{m,k}}^1$ by (2), $\{\zeta_{m,k}\}_{m,k=1}^\infty$ is an interpolating sequence by Lemma 6.2. Let b be an interpolating Blaschke product associated with zeros $\{\zeta_{m,k}\}_{m,k=1}^\infty$. To show that $Z(b)\cup Z(q)$ is not an interpolation set for H^∞, let x_m be one of the points in $\mathrm{cl}\,(\{z_{n_{m,k}}\}_{k=1}^\infty)\backslash\{z_{n_{m,k}}\}_{k=1}^\infty$ and let y_m be one of the points in $\mathrm{cl}\,(\{\zeta_{m,k}\}_{k=1}^\infty)\backslash\{\zeta_{m,k}\}_{k=1}^\infty$ for each m. Since $q(x_m)=0$ and $r_{m+1}\leqq|q(y_m)|\leqq r_m$ by (4), we have $x_m\neq y_m$. By (2) and (3), we have $\rho(\zeta_{m,k}, z_{n_{m,k}})<\delta/3m$ for $k=1, 2, \cdots$. Then there exists x_m in $\mathrm{cl}\,(\{z_{n_{m,k}}\}_{k=1}^\infty)\backslash\{z_{n_{m,k}}\}_{k=1}^\infty$ and y_m in $\mathrm{cl}\,(\{\zeta_{m,k}\}_{k=1}^\infty)\backslash\{\zeta_{m,k}\}_{k=1}^\infty$ such that

$$\rho(y_m, x_m)\leqq\varlimsup_{k\to\infty}\rho(\zeta_{m,k}, z_{n_{m,k}})\leqq\delta/3m .$$

This means that $y_m, x_m\in Z(b)\cup Z(q)$ and $\rho(y_m, x_m)\to 0$ $(m\to\infty)$. By Lemma 6.4, $Z(b)\cup Z(q)$ is not an interpolation set for H^∞.

Here we give one more remark on an interpolating Blaschke product.

Proposition 6.2. *For each interpolating Blaschke product q, there is a non-interpolating simple Blaschke product b such that $Z(b)=Z(q)$, where simple means that every zero point in D of b has zero's order* 1.

Proof. For each n, we take an open subset U_n of D such that

(1) $z_n \in D_n$,

(2) $\sup\{|q(z)|;\ z \in U_n\} \to 0\ (n\to\infty)$,

(3) $U_n \subset \{z \in D;\ \rho(z, z_n) < \delta/3n\}$.

Let $\{w_n\}_{n=1}^{\infty}$ be a sequence with $w_n \in U_n$ and $w_n \neq z_n$. By Lemma 6.2, $\{w_n\}_{n=1}^{\infty}$ is an interpolating sequence, we put b_0 the interpolating Blaschke product with zeros $\{w_n\}_{n=1}^{\infty}$. By (2), we have $q(w_n)\to 0$ $(n\to\infty)$, and so that $q=0$ on $\mathrm{cl}\,(\{w_n\}_{n=1}^{\infty})\backslash\{w_n\}_{n=1}^{\infty}$. Thus we get $Z(q)\supset Z(b_0)$ by Lemma 6.3. We put $b=qb_0$, then b is a Blaschke product with simple zeros $\{w_n, z_n\}_{n=1}^{\infty}$ and

$$Z(b)=Z(q)\cup Z(b_0)=Z(q)\ .$$

By (3), we have $\rho(w_n, z_n)\to 0$ $(n\to\infty)$. By Lemma 6.1, b is not an interpolating Blaschke product.

It is known that there are no Douglas algebras which are maximal among the proper Douglas algebras (see [10, p. 194]). While, it is proved in [5] that for every Douglas algebra B such that $H^\infty+C\subsetneqq B$, there exists another Douglas algebra B' such that $H^\infty+C\subsetneqq B'\subsetneqq B$. The following proposition proves that for some Douglas algebra B with $B\supsetneqq H^\infty+C$, there exists a Douglas algebra which is maximal among proper Douglas algebras contained in B. The idea of the proof can be found in [17].

Proposition 6.3. *There exist two Douglas algebras B_1 and B_2 with $H^\infty+C\subsetneqq B_1\subsetneqq B_2\subsetneqq L^\infty$ such that there are no proper Douglas algebras between B_1 and B_2.*

Proof. Let B_0 be a Douglas algebra with $B_0\subsetneqq L^\infty$. By Chang-Marshall's theorem, there is an inner function ϕ such that $\bar{\phi}\notin B_0$. We put $B_2=[B_0, \bar{\phi}]$. By [20], we have $B_2\subsetneqq L^\infty$. Let Λ be the family of Douglas algebras B_λ such that

$$B_0\subset B_\lambda\subsetneqq B_2 \quad \text{and} \quad \bar{\phi}\notin B_\lambda\ .$$

Let $\{B_\alpha\}_\alpha$ be a totally ordered subset in Λ, where the order in Λ is defined by inclusion. We denote by B' the closed subalgebra generated by $\{B_\alpha\}_\alpha$. Since $\|\bar{\phi}-h\|\geqq 1$ for every $h\in B_\alpha$, we have $\|\bar{\phi}-g\|\geqq 1$ for every $g\in B'$, because $\{B_\alpha\}_\alpha$ is the increasing family. Thus we get $\bar{\phi}\notin B'$. By Zorn's lemma, there exists a maximal Douglas algebra B_1 in Λ. Suppose that B is a Douglas algebra with

$B_1 \subsetneq B \subset B_2$. By our construction, we get $\bar{\phi} \in B$. Since $B_0 \subset B_1 \subset B$, we have $B_2 \supset B \supset [B_0, \bar{\phi}] = B_2$. This leads us the assertion.

In the last part of this section, we give two results of weak peak sets. In [22], Younis gave that if B is a Douglas algebra and μ is an extreme point of the unit ball of $B^{\perp}$, then $\operatorname{supp} \mu$ is a weak peak set for B.

Proposition 6.4. *Let B be a Douglas algebra such that $B \subsetneq L^{\infty}$. Then there exists a measure μ on X such that*

(a) *$\mu \in B^{\perp}$ and*

(b) *$\operatorname{supp} \mu$ is not a weak peak set for B.*

Proof. Since $B \subsetneq L^{\infty}$, there is a representing measure μ_x for $x \in M(B) \backslash X$. Then there is a Blaschke product b such that $b(x)=0$ ([10, p. 179]). It is easy to see that $b\mu_x \in B^{\perp}$. We put $W = \{y \in \operatorname{supp} \mu_x ; \operatorname{Re} b(y) \geqq 0\}$. Then clearly $b\mu_{x|W} \notin B^{\perp}$, because $\int_W b d\mu_x \neq 0$. If we put $S = \operatorname{supp}(b\mu_{x|W})$, then $b\mu_{x|S} = b\mu_{x|W}$ and $b\mu_{x|S} \notin B^{\perp}$. Hence by Lemma 3.1 (a), S is not a weak peak set for B. By Theorem 2.1, we get a Blaschke product b_0 such that $b_0 b \mu_{x|S} \in B^{\perp}$. Now we put $\mu = b_0 b \mu_{x|S}$. Then μ satisfies (a) and (b).

In [18, Proposition 6.3], Pełczyński stated that there is a closed non-empty G_δ-set F of X with $\hat{m}(F)=0$ which is not a peak set for H^{∞}. But he did not give a concrete example.

Proposition 6.5. *Let U be an open-closed subset of X. If $X_\lambda \supsetneq U \cap X_\lambda \neq \emptyset$, where X_λ is the fiber at $\lambda \in \partial D$, then $U \cap X_\lambda$ is a closed G_δ-set with $\hat{m}(U \cap X_\lambda)=0$, and $U \cap X_\lambda$ is not a peak set for H^{∞}.*

Proof. Suppose that $U \cap X_\lambda$ is a peak set for H^{∞}. Let f be a function in H^{∞} such that $f=1$ on $U \cap X_\lambda$ and $|f|<1$ on $X \backslash (U \cap X_\lambda)$. Since $U \cap X_\lambda$ is an open-closed subset of X_λ, we have

$$\sup \{|f(x)| ; x \in X_\lambda \backslash (U \cap X_\lambda)\} < 1 .$$

Hence a sequence of functions $\{(f_{|X_\lambda})^n\}_{n=1}^{\infty}$ converges uniformly to a characteristic function $\chi_{U \cap X_\lambda}$ on X_λ. Since $f_{|X_\lambda} \in H^{\infty}{}_{|X_\lambda}$, this implies that $\chi_{U \cap X_\lambda} \in H^{\infty}{}_{|X_\lambda}$. But this is a contradiction, because $H^{\infty}{}_{|X_\lambda}$ does not have non-trivial idempotents ([10, p. 188]).

The part of this work was done while the both authors were Visiting Scholars at the University of California, Berkeley.

References

[1] E. Alfsen and E. Effros, *Structure in real Banach spaces* I, II, Ann. of Math., **96** (1972), 98–173.

[2] S. Axler, *Factorization of L^∞ functions*, Ann. of Math., **106** (1977), 567–572.
[3] A. Browder, *Introduction to Function Algebras*, Benjamin, New York, 1969.
[4] S.-Y. Chang, *A characterization of Douglas algebras*, Acta Math., **137** (1976), 81–89.
[5] K. Clancey and W. Cutrer, *Subalgebras of Douglas algebras*, Proc. Amer. Math. Soc., **40** (1973), 102–106.
[6] T. W. Gamelin, *Uniform Algebras*, Prentice Hall, Englewood Cliffs, New Jersey, 1969.
[7] T. W. Gamelin, D. Marshall, R. Younis and W. Zame, Function theory and M-ideals, preprint.
[8] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York and London, 1981.
[9] I. Glicksberg, *Measures orthogonal to algebras and sets of antisymmetry*, Trans. Amer. Math. Soc., **105** (1962), 415–435.
[10] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice Hall, Englewood Cliffs, New Jersey, 1962.
[11] R. Holmes, B. Scrantan, and J. Ward, *Approximation from the space of compact operators and other M-ideals*, Duke Math. J., **42** (1975), 259–269.
[12] K. Izuchi, *Extreme points of unit balls of quotients of L^∞ by Douglas algebras*, to appear in Illinois J. Math..
[13] K. Izuchi and R. Younis, *On the quotient space of two Douglas algebras*, preprint.
[14] D. Luecking, *The compact Hankel operators from an M-ideal in the space of Hankel operators*, Proc. Amer. Math. Soc., **79** (1980), 222–224.
[15] D. Luecking and R. Younis, *Quotients of L^∞ by Douglas algebras and best approximation*, Trans. Amer. Math. Soc., **276** (1983), 699–706.
[16] D. Marshall, *Subalgebras of L^∞ containing H^∞*, Acta Math., **137** (1976), 91–98.
[17] T. Nakazi, *A note on maximality theorems of some subalgebras of H^∞*, preprint.
[18] A. Pełczyński, *Banach spaces of Analytic Functions and Absolutely Summing Operators*, Amer. Math. Soc., Providence, Rhode Island, 1977.
[19] D. Sarason, *Function Theory on the Unit Circle*, Virginia Poly. Inst. and State Univ., Blacksburg, Virginia, 1978.
[20] C. Sundberg, *A note on algebras between L^∞ and H^∞*, Rocky Mountain J., **11** (1981), 333–336.
[21] R. Younis, *Best approximation in certain Douglas algebras*, Proc. Amer. Math. Soc., **80** (1980), 639–642.
[22] R. Younis, *Properties of certain algebras between L^∞ and H^∞*, J. Functional Analysis, **44** (1981), 381–387.
[23] R. Younis, *M-ideals of L^∞/H^∞ and support sets*, to appear in Illinois J. Math..

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