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A GENERALIZED BLACKWELL RENEWAL THEOREM

BY

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1. Introduction and results

Let F be a nonlattice distribution function with finite and positive mean $\mu=\int_{-\infty}^{\infty} xdF(x)$. For sequences $\{a(n)\}_{n\in N}$ of positive numbers we define the generalized renewal measure

$$V(I)=\sum_{n=1}^{\infty} a(n)F^{*n}(I) ,$$

where I is a bounded interval on $\boldsymbol{R}$ and $F^{*n}(I)=F^{*n}(b)-F^{*n}(a)$ for $I=(a, b]$, $F^{*n}(x)$ being the n-th convolution of F. In this paper, we intend to give some asymptotic results of Blackwell type, i.e., we shall examine the asymptotic behaviour of

$$V((t, t+h])=\sum_{n=1}^{\infty} a(n)F^{*n}((t,t+h]) , \quad h>0 , \tag{1.1}$$

as $t\to\infty$.

Let us recall the following renewal theorem by Kalma [5].

Theorem A. ([5] Theorems 5.17 and 5.18).

(i) *Let* $a(n)=n^{\theta}$, $\theta>0$, *in* (1.1), *and suppose that when* θ *is an integer*

$$\int_{-\infty}^{0} |x|^{\theta+1}dF(x)<\infty$$

and when θ *is not an integer*

$$\int_{-\infty}^{0} |x|^{[\theta]+2}dF(x)<\infty . \tag{1.2}$$

Then

$$V((t, t+h])\sim\frac{h}{\mu^{\theta+1}}t^{\theta} \quad as \quad t\to\infty . \tag{1.3}$$

(ii) *Let* $a(n)=n^{\theta}$, $\theta<0$, *and suppose that*

(1.4) $$\int_0^\infty x^{|\theta|} dF(x) < \infty \ .$$

Then (1.3) *holds.*

In this paper, we shall generalize Theorem A (i) to the case where $\{a(n)\}$ is regularly varying at infinity, imposing similar conditions. We also improve and extend Theorem A (ii).

Theorem 1. *Suppose* $a(x)=x^\theta L(x)$, $\theta \in \boldsymbol{R}$, *where* $L(x)$ *is a slowly varying function.*

(i) *Case* $\theta > -1$. *If*

(1.2) $$\int_{-\infty}^0 |x|^{[\theta]+2} dF(x) < \infty \ ,$$

then

(1.5) $$V((t, t+h]) \sim \frac{h}{\mu^{\theta+1}} a(t) \quad as \quad t \to \infty \ .$$

(ii) *Case* $\theta=-1$. *Suppose that*

(1.6) $$\int_{-\infty}^0 |x|^2 dF(x) < \infty$$

and further

(a) L *is monotone decreasing and* $\int_0^\infty \frac{x}{L(x)} dF(x) < \infty$

or

(b) $x^{1+\delta}(1-F(x)) \to 0$ *as* $x \to \infty$ *for some* $\delta > 0$.

Then we have (1.5).

(iii) *Case* $\theta < -1$, *integer. Suppose* L *is constant or* (1.6) *holds, and further suppose*

(1.7) $$1-F(x) = o(a(x)) \quad as \quad x \to \infty \ .$$

Then have (1.5).

(iv) *Case* $\theta < -1$, *non-integer. If* (1.7) *holds, then* (1.5) *holds.*

Remarks. (1) In (i), if $-1 < \theta < 0$, then condition (1.2) is automatically satisfied because of the finite mean. Statement (i) means that if θ is non-integer, we can generalize $a(x)=x^\theta$ to general slowly varying functions under the same moment condition (1.2)

(2) Condition (1.7) is weaker than

$$\int_0^\infty \frac{1}{a(x)} dF(x) < \infty \ ,$$

which is corresponding to condition (1.4). Hence if $a(x)=x^\theta$, $\theta<-1$, our statements (iii) and (iv) also relax Kalma's moment condition (1.4) to get his result. If $\theta<-1$ and θ is non-integer, as seen in (iv), we can generalize $a(x)=x^\theta$ to general slowly varying functions under weaker condition (1.7).

(3) For lattice distribution, a similar theorem can be formulated and proved.

Generalized renewal measures $V(I)$ have been studied by many authors (e.g., Embrechts-Omey [2], Greenwood-Omey-Teugels [3], Heyde [4], Kalma [5], Kawata [6] and Smith [7]). Recently in [1] we have proved similar results for the distribution function F concentrated on $[0, \infty)$. Our present Theorem 1 generalizes some parts of the results in [1] to the general case where $F((-\infty, 0))\geqslant 0$. The main idea of the proof is quite different.

2. Preliminaries.

Lemma 1. *Let* $a(x)=x^\theta L(x)$, $\theta\leqslant 0$. *If*

$$1-F(x)=o(a(x)) \quad as \quad x\to\infty \ ,$$

then for each n,

$$1-F^{*n}(x)=o(a(x)) \quad as \quad x\to\infty \ .$$

Proof. Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables having common distribution function F. Then for $x>0$, since

$$\bigcap_{i=1}^{n} \left\{X_i\leqslant\frac{x}{n}\right\}\subset\{S_n\leqslant x\} \ ,$$

where $S_n=X_1+X_2+\cdots+X_n$, we have

$$0\leqslant 1-F^{*n}(x)=P\{S_n>x\}\leqslant n\left(1-F\left(\frac{x}{n}\right)\right) .$$

Hence $1-F(x)=o(a(x))$ implies $1-F^{*n}(x)=o(a(x))$.

Lemma 2. *Let* $b(x)$ *be a regularly varying function at infinity with index* θ, $\theta\in\boldsymbol{R}$ *and suppose that a stochastic process* $\eta(t)$ *converges to* $c(0<c<\infty)$ *in probability as* $t\to\infty$. *Then*

$$\frac{b(\eta(t)t)}{b(t)}\to c^\theta \quad in\ probability\ at \quad t\to\infty \ .$$

Proof. We have for any $\varepsilon>0$ and $\delta>0$

$$P\left\{\left|\frac{b(\eta(t)t)}{b(t)}-c^{\theta}\right|>\varepsilon\right\}=P\left\{\left|\frac{b(\eta(t)t)}{b(t)}-c^{\theta}\right|>\varepsilon,\ |\eta(t)-c|\leqslant\delta\right\}$$
$$+P\left\{\left|\frac{b(\eta(t)t)}{b(t)}-c^{\theta}\right|>\varepsilon,\ |\eta(t)-c|>\delta\right\}\equiv I_1+I_2\,,$$

say. Since $\eta(t)\to c$ in probability, $I_2\to 0$ as $t\to\infty$ for any $\delta>0$.

Next consider I_1. For fixed $0<\alpha<\beta<\infty$, we know that

$$\frac{b(yt)}{b(t)}\to y^{\theta}$$

uniformly in $y\in[\alpha, \beta]$. Hence, for any $\varepsilon>0$, there exist $\delta=\delta(\varepsilon)>0$ and $t_0=t_0(\delta(\varepsilon), \varepsilon)$ such that

$$|y^{\theta}-c^{\theta}|<\frac{\varepsilon}{2}\quad\text{for}\quad |y-c|\leqslant\delta$$

and

$$\left|\frac{b(yt)}{b(t)}-y^{\theta}\right|<\frac{\varepsilon}{2}\quad\text{for all}\quad t\geqslant t_0\,.$$

Therefore

$$\left|\frac{b(yt)}{b(t)}-c^{\theta}\right|<\varepsilon\quad\text{for}\quad |y-c|\leqslant\delta\,.$$

This argument shows that $I_1\to 0$ as $t\to\infty$.

In proving the theorem, we may and do assume that the slowly varying function $L(x)$ in $a(x)$ is bounded on any bounded intervals. The reason is as follows. For any slowly varying function $L(x)$, we can find a slowly varying function $L_1(x)$ such that $L_1(x)$ is bounded on any bounded intervals and $L(x)/L_1(x)\to 1$ as $x\to\infty$. That is, for any $\varepsilon>0$,

$$(1-\varepsilon)L_1(n)\leqslant L(n)\leqslant(1+\varepsilon)L_1(n)$$

for all $n\geqslant n_0(\varepsilon)$, say. Since the first moment of F is finite, we have

$$\sum_{n-1}^{n_0} n^{\theta}L(n)F^{*n}((x, x+h])=o\left(\frac{1}{x}\right)$$

and

$$\sum_{n=1}^{n_0} n^{\theta}L_1(n)\,F^{*n}((x, x+h]=o\left(\frac{1}{x}\right)\,.$$

Therefore in case $\theta>-1$, if statement (1.5) is true for $L_1(n)$, then it is also true for $L(n)$. Also, in case $\theta \leqslant -1$, because of assumption (1.7) and Lemma 1,

$$\sum_{n=1}^{n_0} n^\theta L(n) F^{*n}((x, x+h]) = o(a(x))$$

and

$$\sum_{n=1}^{n_0} n^\theta L_1(n) F^{*n}((x, x+h]) = o(a(x)) .$$

We have used here that (1.7) is also satisfied in case $\theta = -1$ under condition (a) or (b) in statement (ii). So again, if (1.5) holds for $L_1(n)$, so it does for $L(n)$.

In the following proof, we assume that $L(x)$ is bounded on any bounded intervals.

We also note that the convergence of the series $V(I)$ for bounded intervals I is assured under our moment conditions.

3. Proof of Theorem 1 (i)

Our proof is based on the following result due to Kalma ([5], Theorem 1.15).

Lemma 3. *Let F be nonlattice and $U(x) = \sum_{n=0}^{\infty} F^{*n}(x)$. For $h > 0$ and $t \geqslant t_0$, t_0 being some positive number, define a family of integer-valued random variables $\xi(t)$ by*

$$P\{\xi(t) = m\} = \frac{F^{*m}((t, t+h])}{U((t, t+h])} .$$

Then as $t \to \infty$

$$\frac{\xi(t)}{t} \to \frac{1}{\mu} \quad \textit{in probability} .$$

By the definition of $\xi(t)$, we have

$$V((t, t+h]) = a(t) E\left[\frac{a(\xi(t))}{a(t)}\right] U((t, t+h]) , \tag{3.1}$$

so that by the ordinary Blackwell renewal theorem for $U(t)$

$$\lim_{t\to\infty} \frac{1}{a(t)} V((t, t+h]) = \frac{h}{\mu} \lim_{t\to\infty} E\left[\frac{a(\xi(t))}{a(t)}\right] . \tag{3.2}$$

It follows from Lemmas 2 and 3 that

$$\frac{a(\xi(t))}{a(t)} = \frac{a\left(\frac{\xi(t)}{t} \cdot t\right)}{a(t)} \to \frac{1}{\mu^\theta} \quad \text{in probability}$$

as $t\to\infty$. Hence the proof of the statement will be completed if on the right hand side of (3.2) we can interchange the expectation and the limit.

Since we have assumed that L is bounded on bounded intervals, for any $\gamma>0$, $y_0>0$ and $\varepsilon>0$, there exists $t_0=t_0(\varepsilon)$ such that

$$\frac{a(yt)}{a(t)}\leqslant(\varepsilon+y_0{}^{\gamma})y^{\theta-\gamma} \quad \text{for} \quad t\geqslant t_0,\ 0<y\leqslant y_0$$

and

$$\frac{a(yt)}{a(t)}\leqslant(\varepsilon+y_0{}^{-\gamma})y^{\theta+\gamma} \quad \text{for} \quad t\geqslant t_0,\ y\geqslant y_0\ ,$$

(see Taqqu [8] Lemma 4.1). Therefore we have

$$\frac{a(\xi(t))}{a(t)}\leqslant C\left\{\left(\frac{\xi(t)}{t}\right)^{\theta+\gamma}+\left(\frac{\xi(t)}{t}\right)^{\theta-\gamma}\right\} \quad \text{for} \quad t\geqslant t_0\ . \tag{3.3}$$

Here and below C denotes some absolute positive constant. Take γ such that $[\theta+\gamma]=[\theta]$ and $\theta-\gamma>-1$ (which is possible since $\theta>-1$).

If we apply Theorem A to case $a(n)=n^{\theta+\gamma}$ or $=n^{\theta-\gamma}$, we have

$$\sum_{n=1}^{\infty} n^{\theta+\gamma}F^{*n}((t,\ t+h])\sim\frac{t^{\theta+\gamma}}{\mu^{\theta+\gamma+1}}h \quad \text{as} \quad t\to\infty$$

and

$$\sum_{n=1}^{\infty} n^{\theta-\gamma}F^{*n}((t,\ t+h])\sim\frac{t^{\theta-\gamma}}{\mu^{\theta-\gamma+1}}h \quad \text{as} \quad t\to\infty\ ,$$

because of condition (1.2). This together with relation (3.1) and the ordinary Blackwell renewal theorem for U implies

$$E\left[\left(\frac{\xi(t)}{t}\right)^{\theta+\gamma}\right]\to\frac{1}{\mu^{\theta+\gamma}} \quad \text{as} \quad t\to\infty \tag{3.4}$$

and

$$E\left[\left(\frac{\xi(t)}{t}\right)^{\theta-\gamma}\right]\to\frac{1}{\mu^{\theta-\gamma}} \quad \text{as} \quad t\to\infty\ . \tag{3.5}$$

Note that if $0\leqslant X_n\leqslant Y_n$, $X_n\to X$ and $Y_n\to Y$ in probability respectively and if $EY_n\to EY<\infty$, then $EX_n\to EX$. Hence it follows from (3.3)–(3.5), Lemmas 2 and 3 that

$$\lim_{t\to\infty} E\left[\frac{a(\xi(t))}{a(t)}\right]=\frac{1}{\mu^{\theta}}\ .$$

The proof is thus completed.

4. Proof of Theorem 1 (ii), (iii), (iv)

Let $a(x)=x^{\theta}L(x)$, where $\theta\leqslant -1$. We prove the conclusion by induction method with respect to θ similarly to [1].

The following lemma was stated in [1] for F with $F((-\infty, 0))=0$, and is similarly proved for general F.

Lemma 4. *Let $Q(x)=\int_{-\infty}^{x} y\,dF(y)$. Then for $n\geqslant 1$ and all $h>0$,*

$$tF^{*n}((t, t+h])\leqslant nF^{*(n-1)}*Q((t, t+h])\leqslant (t+h)F^{*n}((t,t+h]) .$$

We have proved that (1.5) holds for $-1<\theta\leqslant 0$, under condition $\int_{-\infty}^{0}|x|^2 dF(x) <\infty$ when $\theta=0$. Let $\beta\leqslant 0$, and suppose (1.5) holds for $\theta=\beta$:

$$\sum_{n=1}^{\infty} n^{\beta}L(n)F^{*n}((t, t+h])\sim\frac{h}{\mu^{\beta+1}}t^{\beta}L(t) . \tag{4.1}$$

Then we shall prove (1.5) holds for $\theta=\beta-1$:

$$G((t, t+h])\equiv\sum_{n=1}^{\infty} n^{\beta-1}L(n)F^{*n}((t, t+h])\sim\frac{h}{\mu^{\beta}}t^{\beta-1}L(t) . \tag{4.2}$$

By Lemma 4,

$$G((t, t+h])\leqslant\frac{1}{t}\sum_{n=1}^{\infty} n^{\beta}L(n)F^{*(n-1)}*Q((t, t+h])$$

and

$$G((t,t+h])\geqslant\frac{1}{t+h}\sum_{n=1}^{\infty} n^{\beta}L(n)F^{*(n-1)}*Q((t, t+h]) .$$

Let

$$W(I)\equiv\sum_{n=1}^{\infty} n^{\beta}L(n)F^{*(n-1)}(I) .$$

Then to prove (4.2) it is enough to show that

$$W*Q((t, t+h])\sim\frac{h}{\mu^{\beta}}t^{\beta}L(t) \quad \text{as} \quad t\to\infty . \tag{4.3}$$

By Lemma 1 and by exactly the same argument as in [1], we get, under the induction hypothesis (4.1) that

$$W((t, t+h])\sim\frac{h}{\mu^{\beta+1}}t^{\beta}L(t) \quad \text{as} \quad t\to\infty . \tag{4.4}$$

First consider case $\beta<0$. We have

$$W*Q\,((t,t+h])=\left(\int_{-\infty}^{t/2}+\int_{t/2}^{\infty}\right)(W(t+h-s)-W(t-s))dQ(s) \tag{4.5}$$
$$\equiv I_1+I_2\ ,$$

say. In I_1, we have $t-s\geqslant t/2$ so that by (4.4) the integrand in I_1 is less than, for large x,

$$C(t-s)^{\beta}L(t-s)\leqslant C\left(\frac{t}{2}\right)^{\beta}L\left(\frac{t}{2}\right)\leqslant Ct^{\beta}L(t)\ .$$

Then it follows from Lebesgue's theorem that

$$\frac{I_1}{t^{\beta}L(t)}\to\int_{-\infty}^{\infty}\frac{h}{\mu^{\beta+1}}dQ(s)=\frac{t}{\mu^{\beta+1}}\quad\text{as}\quad t\to\infty\ . \tag{4.6}$$

In I_2, we have

$$W(t+h-s)-W(t-s)\leqslant CU((t-s,\ t+h-s])\leqslant Ch$$

so that

$$I_2\leqslant Ch\int_{t/2}^{\infty}dQ\,(s)=Ch\int_{t/2}^{\infty}sdF(s)\ .$$

Recall that $a(x)=x^{\beta-1}L(x)$, $\beta<0$, and condition (1.7), then

$$\begin{aligned}\frac{I_2}{t^{\beta}L(t)}&\leqslant\frac{Ch}{ta(t)}\int_{t/2}^{\infty}sdF(s)\\&=\frac{Ch}{ta(t)}\left\{\frac{t}{2}\left(1-F\left(\frac{t}{2}\right)\right)+o\left(\int_{t/2}^{\infty}a(s)ds\right)\right\}\\&\sim\frac{Ch}{ta(t)}\left\{o(ta(t))+o\left(\frac{\frac{t}{2}a\left(\frac{t}{2}\right)}{-\beta}\right)\right\}\to 0\quad\text{as}\quad t\to\infty\ .\end{aligned} \tag{4.7}$$

From (4.5)-(4.7) we conclude that (4.3) holds.

Next consider case $\beta=0$. Consider I_1 as in (4.5). Again by (4.4), in I_1,

$$\frac{W(t+h-s)-W(t-s)}{L(t-s)}$$

is bounded. Also note that

$$\frac{(t-s)^{-1}L(t-s)}{t^{-1}L(t)}$$

is bounded. Now

(4.8) $$I_1=\int_{-\infty}^{0}+\int_{0}^{t/2}\equiv I_{11}+I_{12}\ ,$$

say. Then

$$\frac{I_{11}}{L(t)}=\int_{-\infty}^{0}\frac{W(t+h-s)-W(t-s)}{L(t-s)}\cdot\frac{(t-s)^{-1}L(t-s)}{t^{-1}L(t)}\cdot\frac{t-s}{t}dQ(s)$$
$$=\int_{-\infty}^{0}f_t(s)dQ(s)\ ,$$

say. Since

$$0\leqslant f_t(s)\leqslant C\frac{t-s}{t}\leqslant C(1-s)\qquad\text{for large } t\ ,$$
$$\int_{-\infty}^{0}(1-s)dQ(s)<\infty\qquad\text{(because of (1.6))}$$

and

$$f_t(s)\to h\quad\text{as}\quad t\to\infty\ ,$$

we have

(4.9) $$\frac{I_{11}}{L(t)}\to h\int_{-\infty}^{0}dQ(s)\ .$$

Also in I_{12}, since

$$0\leqslant f_t(s)\leqslant C\frac{t-s}{t}\leqslant C\ ,$$

we have

(4.10) $$\frac{I_{12}}{L(t)}\to h\int_{0}^{\infty}dQ(s)\ .$$

Combining (4.8)-(4.10), we have

$$I_1\sim h\mu L(t)\quad\text{as}\quad t\to\infty\ .$$

Next consider I_2 in (4.5). First suppose (a) in (ii). Then we have

$$\frac{I_2}{L(t)}\leqslant\frac{Ch}{L(t)}\int_{t/2}^{\infty}sdF(s)\leqslant Ch\int_{t/2}^{\infty}\frac{s}{L(s)}dF(s)\to 0\quad\text{as}\quad t\to\infty\ .$$

Next suppose (b) in (ii). We have for fixed large $t_0>0$,

$$I_2=\int_{t/2}^{t-t_0}+\int_{t-t_0}^{\infty}\equiv I_{21}+I_{22}\ ,$$

say. As to I_{21}, for large t

$$\frac{I_{21}}{L(t)} \leqslant C\int_{t/2}^{t-t_0} \frac{L(t-s)}{L(t)} dQ(s) \leqslant Ct^{\delta/2} \int_{t/2}^{t-t_0} (t-s)^{\delta/2} dQ(s)$$

$$\leqslant Ct^{\delta} 2^{-\delta/2} \int_{t/2}^{t-t_0} dQ(s) \leqslant Ct^{1+\delta}\left(1-F\left(\frac{t}{2}\right)\right) \to 0 \quad \text{as} \quad t\to\infty .$$

As to I_{22}, for large t again

$$\frac{I_{22}}{L(t)} \leqslant \frac{C}{L(t)} \int_{t-t_0}^{t} dQ(s) \leqslant Ct^{\delta} \int_{t-t_0}^{\infty} s dF(s)$$

$$= Ct^{\delta}\left\{(t-t_0)(1-F(t-t_0)) + \int_{t-t_0}^{\infty} (1-F(s))ds\right\}$$

$$= Ct^{\delta}\left\{(t-t_0)(1-F(t-t_0)) + o\left(\int_{t-t_0}^{\infty} s^{-1-\delta} ds\right)\right\}$$

$$\to 0 \quad \text{as} \quad t\to\infty .$$

This proves (4.3) for $\beta=0$.

If $\theta<-1$ and is integer, condition (1.7) includes condition (b). So, in statements (iii) and (iv), we do not have to state condition (b) explicitly. The proof of the theorem is thus completed.

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