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THE BERRY-ESSEEN THEOREMS FOR U-STATISTICS GENERATED BY ABSOLUTELY REGULAR PROCESSES

By

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1. Introduction

Let $\{\xi_i\}$ be a strictly stationary, absolutely regular process, i.e., the process satisfying the condition

$$\beta(n)=E\{\sup_{A\in\mathscr{M}_n^\infty}|P(A\mid M_{-\infty}^0)-P(A)|\}\downarrow 0\quad(n\to\infty) \tag{1.1}$$

where $\mathscr{M}_a^b$ $(a\leqq b)$ is the σ-algebra of events generated by $\xi_a, \cdots, \xi_b$.

We denote the distribution function of ξ_i by $F(x)$, $x\in R^p$, the p-dimensional Euclidean space. Consider a functional

$$\theta(F)=\int\cdots\int_{R^{dp}} g(x_1, \cdots, x_d)dF(x_1)\cdots dF(x_d) \tag{1.2}$$

defined over $\mathscr{F}=\{F: |\theta(F)|<\infty\}$, where $g(x_1, \cdots, x_d)$ is symmetric in its d $(\geqq 1)$ arguments. As an estimator of $\theta(F)$, we define a U-statistic

$$U_n=\binom{n}{d}^{-1}\sum_{(i)}^{(n)} g(\xi_{i_1}, \cdots, \xi_{i_d})\ ,\quad n\geqq d \tag{1.3}$$

where the summation $\sum_{(i)}^{(n)}$ extends over all possible $1\leqq i_1<i_2<\cdots<i_d\leqq n$.

As in [6], for every c $(0\leqq c\leqq d)$ let

$$g_c(x_1, \cdots, x_c)=\int\cdots\int_{R^{(d-c)p}} g(x_1, \cdots, x_d)dF(x_{c+1})\cdots dF(x_d) \tag{1.4}$$

so that $g_0=\theta(F)$ and $g_d=g$. Let

$$\sigma^2=\sigma^2(F)=\{Eg_1{}^2(\xi_1)-\theta^2(F)\}+2\sum_{j=2}^{\infty}\{Eg_1(\xi_1)g_1(\xi_j)-\theta^2(F)\}\ . \tag{1.5}$$

In [1] Callaert and Janssen proved that the Berry-Esseen theorem for U-statistics constructed by i.i.d. random variables.

On the other hand, in [6] the author proved, among others, that under some conditions

$$P\left(\frac{n^{1/2}(U_n-\theta(F))}{d\sigma}<z\right)\to\Phi(z)\quad(n\to\infty) \tag{1.6}$$

where

$$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-t^2/2}dt\,. \tag{1.7}$$

As a continuation of the above result, in this paper we show the Berry-Esseen theorems for U-statistics generated by an absolutely regular processes, i.e., we estimate the convergence rate of the quantities $\{\Delta_n\}$ defined by

$$\Delta_n=\sup_z\left|P\left(\frac{n^{1/2}(U_n-\theta(F))}{d\sigma}<z\right)-\Phi(z)\right| \tag{1.8}$$

using Tikhomirov's method in [4] (Theorems 1–4). Theorems 3 and 4 are generalizations of Callaert-Janssen's result in [1] and Takahata's one in [3], respectively.

2. Conditions and auxiliary results

In what follows, we denote by the letter K, with or without subscript, various absolute constants, and whenever the integrations are over all of the space, we shall agree to omit the region of integration. Further, let $\|\zeta\|_r=\{E|\zeta|^r\}^{1/r}$ if the right-hand side exists for r (>1).

Let $\{\xi_i\}$ be a strictly stationary, absolutely regular process with mixing coefficient $\beta(n)$. Let

$$U_n^{(c)}=\frac{1}{n(n-1)\cdots(n-c+1)}\sum_{(i)}^{(n)}\int\cdots\int g_c(x_1,\cdots,x_c)\prod_{j=1}^{c}d[u(x_j-\xi_{i_j})-F(x_j)] \tag{2.1}$$

where $u(v)$ is equal to one when all the p components of v are nonnegative; otherwise $u(v)$ is equal to zero. It is easily obtained that

$$U_n=\theta(F)+\sum_{c=1}^{d}\binom{d}{c}U_n^{(c)}\,. \tag{2.2}$$

We consider the following condition;

(C) For any r (>2) there exists an absolute contstant M such that

$$\int\cdots\int|g(x_1,\cdots,x_d)|^r dF(x_1)\cdots dF(x_d)\leqq M \tag{2.3}$$

and

$$E|g(\xi_{i_1},\cdots,\xi_{i_d})|^r\leqq M\quad\text{for all}\quad 1\leqq i_1<\cdots<i_d\,. \tag{2.4}$$

Let $i_1<i_2<\cdots<i_k$ be arbitrary integers. For any j $(1\leqq j\leqq k-1)$ let

$$P_j^{(k)}(E^{(j)}\times E^{(k-j)})=P((\xi_{i_1},\cdots,\xi_{i_j})\in E^{(j)})P((\xi_{i_{j+1}},\cdots,\xi_{i_k})\in E^{(k-j)}) \tag{2.5}$$

and

$$P_0^{(k)}(E^{(k)})=P((\xi_{i_1},\cdots,\xi_{i_k})\in E^{(k)}) \tag{2.6}$$

where $E^{(i)}$ is a Borel set in R^{ip}.

The following lemma was proved in [6].

Lemma A (Lemma in [6]). *Let $\{\xi_i\}$ be an absolutely regular, (not necessarily strictly stationary) sequence with mixing coefficient $\beta(n)$. For any j $(0\leqq j\leqq k-1)$, let $h(x_1,\cdots,x_k)$ be a Borel function such that*

$$\int\cdots\int|h(x_1,\cdots,x_k)|^{1+\delta}dP_j^{(k)}\leqq M<\infty \tag{2.7}$$

for some $\delta>0$. Then

$$\left|\int\cdots\int h(x_1,\cdots,x_k)dP_0^{(k)}-\int\cdots\int h(x_1,\cdots,x_k)dP_j^{(k)}\right| \leqq 4M^{1/(1+\delta)}\{B(i_{j+1}-i_j)\}^{\delta/(1+\delta)}\ . \tag{2.8}$$

Next, for any random vector $(\xi_\alpha,\xi_\beta,\cdots,\xi_\gamma)$ and for any Borel set E define

$$P_{(\xi_\alpha,\xi_\beta,\cdots,\xi_\gamma)}(E)=P((\xi_\alpha,\xi_\beta,\cdots,\xi_\gamma)\in E)\ . \tag{2.9}$$

From Lemma A we obtain the following lemma which plays a fundamental role in this paper.

Lemma 2.1. *Let $\{\xi_i\}$ be an absolutely regular, (not necessarily strictly stationary) sequence with mixing coefficient $\beta(n)$. Let $i_1<i_2<\cdots<i_k$ be integers such that $\min\limits_{1\leqq j\leqq k}(i_{j+1}-i_j)\geqq m$ for some integer m. Put*

$$\zeta_j=(\xi_{i_j+1},\cdots,\xi_{i_{j+1}})\quad(j=1,\cdots,k)$$

and

$$z_j=(x_{i_j+1},\cdots,x_{i_{j+1}})\quad(j=1,\cdots,k)\ .$$

Furthermore, let $h(z_1,\cdots,z_k)$ be a Borel function such that

$$\int\cdots\int|h(z_1,\cdots,z_k)|^{1+\delta}dP_{(\zeta_1,\cdots,\zeta_j)}dP_{\zeta_{j+1}}\cdots dP_{\zeta_k}\leqq M<\infty \tag{2.10}$$

for some $\delta>0$, and for any j $(1\leqq j\leqq k)$. Then

$$\left|Eh(\zeta_1,\cdots,\zeta_k)-\int\cdots\int h(z_1,\cdots,z_k)dP_{\zeta_1}\cdots dP_{\zeta_k}\right|\leqq 4kM^{1/(1+\delta)}\{\beta(m)\}^{\delta/(1+\delta)}\ . \tag{2.11}$$

Corollary. *If η is $\mathscr{M}_{-\infty}^{k}$-measurable and ζ is $\mathscr{M}_{k+n}^{\infty}$-measurable* ($n\geqq 0$), *then*

$$E|\eta|^{r}<\infty\,, \qquad E|\zeta|^{s}<\infty\,, \quad r, s>1\,, \quad \frac{1}{r}+\frac{1}{s}<1$$

implies

$$|E\eta\zeta-E\eta E\zeta|\leqq 4\{\beta(n)\}^{1-1/r-1/s}\|\eta\|_{r}\|\zeta\|_{s}\,. \tag{2.12}$$

From now on, we put

$$\eta_{j}=g_{1}(\xi_{j})-\theta(F) \tag{2.13}$$

and

$$\zeta_{i,j}=G(\xi_{i}, \xi_{j})=g_{2}(\xi_{i}, \xi_{j})-g_{1}(\xi_{i})-g_{1}(\xi_{j})+\theta(F) \quad (i<j)\,. \tag{2.14}$$

Furthermore, let

$$S_{n}=\frac{n^{1/2}(U_{n}-\theta(F))}{d\sigma}\,. \tag{2.15}$$

Lemma 2.2. *Let $\{\xi_{j}\}$ be a strictly stationary, absolutely regular sequence. Suppose Condition* (C) (*with* $r=2+\delta$) *holds for some* δ ($0<\delta<2$). *If* $\beta(n)=O(n^{-(6+\delta)/\delta})$, *then the following relations hold*:

(i) *The series in* (1.5) *converges absolutely and*

$$E\left|\sum_{j=1}^{n}\eta_{j}\right|^{2}=n\sigma^{2}(1+O(n^{-1}))\,. \tag{2.16}$$

(ii) *For any i and j $(i<j)$*

$$|E\zeta_{i,j}|\leqq K\{\beta(j-i)\}^{\delta/(2+\delta)} \tag{2.17}$$

and so

$$\Big|\sum_{1\leqq i<j\leqq n} E\zeta_{i,j}\Big|\leqq Kn\,. \tag{2.18}$$

Furthermore, if $|i-\alpha|>m$ and $|j-\alpha|>m$, then

$$|E\zeta_{i,j}\eta_{\alpha}|\leqq K\{\beta(m)\}^{\delta/(2+\delta)}\,. \tag{2.19}$$

(iii) *For any positive integers m and l*

$$E\left|\sum_{p=1}^{n-l-m-1}\sum_{q=p+l+1}^{p+l+m}\zeta_{p,q}\right|^{2}=O(nm) \tag{2.20}$$

and

$$E\left|\sum_{p=1}^{n-1}\sum_{q=p+1}^{n}\zeta_{p,q}\right|^{2}=O(n^{2})\,. \tag{2.21}$$

Proof. (i) Since $E\eta_1=0$, so by Corollary to Lemma 2.1

$$\left|\frac{1}{n}E\left|\sum_{j=1}^{n}\eta_j\right|^2-\sigma^2\right|\leqq 2\left[\frac{1}{n}\sum_{j=1}^{n}j|E\eta_1\eta_{j+1}|+\sum_{j=n+1}^{\infty}|E\eta_1\eta_j|\right]$$

$$\leqq K\|\eta_1\|_{2+\delta}^2\left[\frac{1}{n}\sum_{j=1}^{n}j\{\beta(j)\}^{\delta/(2+\delta)}+\sum_{j=n+1}^{\infty}\{\beta(j)\}^{\delta/(2+\delta)}\right]\leqq\frac{K}{n}.$$

Hence, we have (i).

(ii) As

$$I_{i,j}=\int\cdots\int\{g_2(x_i,x_j)-g_1(x_i)-g_1(x_j)+\theta(F)\}dF(x_i)dF(x_j)=0,$$

so by Condition (C) and Lemma 2.1

$$|E\zeta_{i,j}|=|E\zeta_{i,j}-I_{i,j}|\leqq K\{\beta(j-i)\}^{(1+\delta)/(2+\delta)}$$

for all i and j $(i<j)$. Thus, we have

$$\sum_{1\leqq i<j\leqq n}|E\zeta_{i,j}|\leqq K\sum_{1\leqq i<j\leqq n}\{\beta(j-i)\}^{(1+\delta)/(2+\delta)}\leqq K.$$

Now, we note that by Condition (C) and Schwarz's inequality

$$\int\cdots\int|(g_1(x_\alpha)-\theta(F))G(x_i,x_j)|^{(2+\delta)/2}dP_{(\xi_\alpha,\xi_i,\xi_j)}$$

$$\leqq\left\{\int\cdots\int|g_1(x_\alpha)-\theta(F)|^{2+\delta}dF(x_\alpha)\right\}^{1/2}\left\{\int\cdots\int|G(x_i,x_j)|^{2+\delta}dP_{(\xi_i,\xi_j)}\right\}^{1/2}$$

$$\leqq K$$

and similarly

$$\int\cdots\int|(g_1(x_\alpha)-\theta(F))G(x_i,x_j)|^{(2+\delta)/2}dP_{\xi_\alpha}dP_{(\xi_i,\xi_j)}\leqq K$$

and

$$\int\cdots\int|(g_1(x_\alpha)-\theta(F))G(x_i,x_j)|^{(2+\delta)/2}dP_{\xi_i}dP_{(\xi_\alpha,\xi_j)}\leqq K.$$

So, by Lemma 2.1 and the fact $E\eta_\alpha=0$

$$|E\zeta_{i,j}\eta_\alpha|\leqq|E\eta_\alpha|\left|\int\cdots\int G(x_i,x_j)dF(x_i)dF(x_j)\right|+K\{\beta(m)\}^{\delta/(2+\delta)}\leqq K\{\beta(m)\}^{\delta/(2+\delta)}$$

if $|i-\alpha|>m$, $|j-\alpha|>m$ and $j-i>m$ and

$$|E\zeta_{i,j}\eta_\alpha|\leqq|E\eta_\alpha|\,|E\zeta_{i,j}|+K\{\beta(m)\}^{\delta/(2+\delta)}\leqq K\{\beta(m)\}^{\delta/(2+\delta)}$$

if $|i-\alpha|>m$, $|j-\alpha|>m$ and $0<j-i\leqq m$.

(iii) From the proof of Lemma 2 in [6] we have that if $\beta(n)=O(n^{-(2+\delta')/\delta'})$

for some δ' $(0<\delta'<\delta)$, then

$$E\left|\sum_{p=1}^{n-l-m-1}\sum_{q=p+l+1}^{p+l+m}\zeta_{p,q}\right|^2=O(nm)$$

and

$$E\left|\sum_{p=1}^{n-1}\sum_{q=p+1}^{n}\zeta_{p,q}\right|^2=O(n^2)\ .$$

Thus, (2.18) and (2.19) are obtained by putting $\delta'=\delta/3$.

Lemma 2.3. *Suppose conditions of Lemma* 2.2 *hold. Then*

$$E|U_n^{(c)}|^2=O(n^{-2})\quad(2\leqq c\leqq d)\ . \tag{2.22}$$

(For the proof of Lemma 2.3, see the proof of Lemma 2 in [6]).

Next, we shall say that $\{\xi_i\}$ satisfies the strong mixing condition if

$$\alpha(n)=\sup_{B\in\mathscr{M}_{-\infty}^0,\,A\in\mathscr{M}_n^\infty}|P(A\cap B)-P(A)P(B)|\downarrow 0\quad(n\to\infty)\ .$$

Since $\beta(n)\geqq\alpha(n)$, so if $\{\xi_i\}$ is absolutely regular then it is strong mixing. Hence, all results proved under the strong mixing condition are applicable to our problem. The following lemma was proved by Yokoyama in [4].

Lemma 2.4. *Let $\{\xi_j\}$ be a strictly stationary, strong mixing sequence with $E\xi_1=0$ and $E|\xi_1|^{2+\delta+\varepsilon}<\infty$ for some $\delta>0$ and $\varepsilon>0$. If*

$$\sum_{j=0}^{\infty}j^{\delta/2-1}\{\alpha(j)\}^{\delta/(2+\delta+\varepsilon)}<\infty \tag{2.23}$$

then there exists a constant M such that

$$E\left|\sum_{j=1}^{n}\xi_j\right|^{2+\delta}\leqq Mn^{1+\delta/2}\quad(n\geqq1)\ . \tag{2.24}$$

We often use the following lemma which is easily proved.

Lemma 2.5.

(i) (Lemma 1 in [2, p. 272]) *For $t\in(-\infty,\infty)$ and for any δ $(0\leqq\delta<1)$*

$$\left|e^{it}-\sum_{j=0}^{n}\frac{(it)^j}{j!}\right|\leqq\frac{2^{1-\delta}|t|^{n+\delta}}{(1+\delta)(2+\delta)\cdots(n+\delta)} \tag{2.25}$$

where the dominator of the right side of (2.25) *is unity for $n=0$.*

(ii) *For all $t\in(-\infty,\infty)$*

$$|e^{it}-1|\leqq2 \tag{2.26}$$

(iii) *For all complex number* z

$$|e^z-1|\leqq|z|e^{|z|} . \tag{2.27}$$

3. The Berry-Esseen theorems for U-statistics of order 2-(I)

In this section, we consider U-statistics of order 2 when $\beta(n)=O(e^{-\lambda n})$ for some $\lambda>0$.

Theorem 1. *Let* $\{\xi_i\}$ *be a strictly stationary, absolutely regular sequence. Suppose Condition* (C) *holds for* $r=2+\delta$ $(0<\delta\leqq 1)$ *and* $d=2$. *If* $\beta(n)=O(e^{-\lambda n})$ *for* $\lambda>0$ *and* σ^2, *defined by* (1.5), *is positive, then*

$$\Delta_n\leqq Kn^{-\delta/2}(\log n)^{1+\delta} . \tag{3.1}$$

To prove Theorem 1 we introduce some notations. Let

$$a_n=\frac{n^{1/2}}{\sigma n(n-1)} ,$$

$$T_n=\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n}\eta_j$$

and

$$V_n=\frac{n^{1/2}}{\sigma}U_n^{(2)}=a_n\sum_{1\leq\alpha<\beta\leq n}\zeta_{\alpha,\beta} .$$

Then

$$S_n=T_n+V_n .$$

Further, let $f_n(t)$ be the characteristic function of S_n and consider

$$\begin{aligned} f_n'(t)&=iES_ne^{itS_n} \\ &=i\{ET_ne^{itS_n}+EV_ne^{itS_n}\} . \end{aligned} \tag{3.2}$$

Firstly, we prove the following proposition.

Proposition 3.1. *Suppose conditions of Theorem* 1 *hold. Let* m *and* k *be integers such that* $\{\beta(m)\}^{\delta/(2+\delta)}=O(n^{-4})$, $k=O(\log n)$ *and* $m^{-(k-2)/4}\leqq K_0n^{-1}$. *Let* K_1 $(\geqq 1)$ *be an arbitrary positive constant. Then for all* t $(|t|<K_1n^{1/2}m^{-\delta/4})$

$$\begin{aligned} &|ET_ne^{itS_n}-itf_n(t)| \\ &\quad\leqq K[\{|t|n^{-1/2}m^{1/2}+|t|^{1+\delta}n^{-\delta/2}m^{1+\delta}+n^{-1}\}|f_n(t)| \\ &\qquad+\{(|t|^2n^{-1}m)^{(1+\delta)/(2+\delta)}m^{(1+\delta)/2}+|t|^{\delta/(2+\delta)}n^{-(1+\delta)/(2+\delta)}m^{\delta/(2(2+\delta))}+n^{-\delta})\}] . \end{aligned} \tag{3.3}$$

In the course of the proof of Proposition 3.1 we use the following notations; for any α $(1\leqq\alpha\leqq n)$ and j $(1\leqq j\leqq k)$

$$T_\alpha^{(0)}=T_n , \qquad V=V_n$$

$$V_{\alpha,0}^{(j)}=a_n \sum_{\substack{|p-\alpha|\leqq jm,\ |q-\alpha|\leqq jm \\ 1\leqq p<q\leqq n}} \zeta_{p,q}$$

$$V_{\alpha,1}^{(j)}=a_n \sum_{|p-\alpha|>jm,\ |q-\alpha|>jm} \zeta_{p,q}$$

$$V_{\alpha,2}^{(j)}=a_n \sum_{1\leqq p<\alpha-jm,\ \alpha-jm\leqq q\leqq\alpha+jm} \zeta_{p,q}$$

$$V_{\alpha,3}^{(j)}=a_n \sum_{\alpha-jm\leqq p\leqq\alpha+jm,\ \alpha+jm<q\leqq n} \zeta_{p,q}$$

$$T_\alpha^{(j)}=\frac{1}{\sigma\sqrt{n}} \sum_{|p-\alpha|>jm} \eta_{p,q} .$$

Then, by Lemma 2.2 the following relations are easily proved:

$$\|T_\alpha^{(j)}-T_\alpha^{(j+1)}\|_2\leqq C_0 n^{-1/2}m^{1/2} \quad (1\leqq\alpha\leqq n;\ 0\leqq j\leqq k-1) \tag{3.4}$$

$$\|V_{\alpha,0}^{(j)}\|_2\leqq Ka_n jm\leqq Kn^{-3/2}jm \quad (1\leqq\alpha\leqq n;\ 0\leqq j\leqq k-1) \tag{3.5}$$

$$\|V_{\alpha,l}^{(j)}\|_2\leqq Ka_n(nm)^{1/2}\leqq Kn^{-1}m^{1/2} \quad (1\leqq\alpha\leqq n;\ l=2, 3) \tag{3.6}$$

$$\|V_{\alpha,1}^{(1)}-V_{\alpha,1}^{(k)}\|_2\leqq Ka_n(nkm)^{1/2}\leqq Kn^{-1}k^{1/2}m^{1/2} \quad (1\leqq\alpha\leqq n) . \tag{3.7}$$

To prove Proposition 3.1 we need the following lemmas.

Lemma 3.1. *Suppose conditions of Proposition* 3.1 *hold. Then for any* α $(1\leqq\alpha\leqq n)$ *and for all* t

$$\|\eta_\alpha Z_\alpha^{(0)}\|_2\leqq K|t|^{\delta/(2+\delta)}n^{-(\delta/2(2+\delta))}m^{\delta/2(2+\delta)} \tag{3.8}$$

and

$$E\left|\eta_\alpha \prod_{j=0}^{r} Z_\alpha^{(j)}\right|\leqq\begin{cases}K[t^2n^{-1}m+n^{-2}] & (r=1)\\ K[\{C_0|t|n^{-1/2}m^{1/2}\}^r+n^{-2}] & (2\leqq r\leqq k)\end{cases} \tag{3.9}$$

where

$$Z_\alpha^{(0)}=\exp[it\{(T_\alpha^{(0)}-T_\alpha^{(1)})+V_{\alpha,0}^{(1)}\}]-1$$

and

$$Z_\alpha^{(j)}=\exp\{it(T_\alpha^{(j)}-T_\alpha^{(j+1)}\}-1 \quad (j=1, \cdots, k-1) .$$

Proof. We note first that by (2.25), (3.5) and (3.6)

$$\|Z_\alpha^{(0)}\|_2^2\leqq K|t|^2\{\|T_\alpha^{(0)}-T_\alpha^{(1)}\|_2^2+\|V_{\alpha,0}^{(1)}\|_2^2\}\leqq K|t|^2n^{-1}m$$

and

$$\|Z_\alpha^{(j)}\|_2^2\leqq t^2\|T_\alpha^{(j)}-T_\alpha^{(j+1)}\|_2^2\leqq C_0^2t^2n^{-1}m \quad (j=1, \cdots, k) .$$

By Hölder's inequalitiy and (2.26)

$$\begin{aligned}E|\eta_\alpha Z_\alpha^{(0)}|^2&\leqq\|\eta_\alpha\|_{2+\delta}^2\{E|Z_\alpha^{(0)}|^{2(2+\delta)/\delta}\}^{\delta/(2+\delta)}\leqq K\{E|Z_\alpha^{(0)}|^2\}^{\delta/(2+\delta)}\\ &\leqq K|t|^{2\delta/(2+\delta)}n^{-(\delta/(2+\delta))}m^{\delta/(2+\delta)} .\end{aligned}$$

So, (3.9) is obtained for $r=1$.

Next, for each j $(1 \leqq j \leqq r)$ let

$$H_\alpha^{(j)} = \int \cdots \int \left| \exp\left\{ i \frac{t}{\sigma\sqrt{n}} \sum_{jm < |l-\alpha| \leqq j(m+1)} x_l \right\} - 1 \right|^2 dP_{\tau_{1,j}} dP_{\tau_{2,j}}$$

where $\tau_{1,j} = (\xi_{\alpha-(j+1)m}, \cdots, \xi_{\alpha-jm-1})$ and $\tau_{2,j} = (\xi_{\alpha+jm+1}, \cdots, \xi_{\alpha+(j+1)m})$. Then, it is obvious that from (2.16) and (2.25) that

$$H_\alpha^{(j)} \leqq \frac{t^2}{\sigma^2 n} E \Big| \sum_{jm < |l-\alpha| \leqq (j+1)m} x_l \Big|^2 \leqq K t^2 n^{-1} m \quad (j=1, \cdots, r) .$$

Since η_α is $\mathscr{M}_\alpha^\alpha$-measurable and $Z_\alpha^{(1)}$ is $\mathscr{M}_{-\infty}^{\alpha-m} \times \mathscr{M}_{\alpha+m}^{\infty}$-measurable, so by Lemma 2.1, (2.26) and the definition of m

$$\begin{aligned} \|\eta_\alpha Z_\alpha^{(1)}\|_2^2 = E|\eta_\alpha Z_\alpha^{(1)}|^2 &\leqq \|\eta_\alpha\|_2^2 H_\alpha^{(1)} + K\{\beta(m)\}^{\delta/(2+\delta)} \|\eta_\alpha\|_{2+\delta}^2 \\ &\leqq K[t^2 n^{-1} m + K\{\beta(m)\}^{\delta/(2+\delta)}] \leqq K[t^2 n^{-1} m + n^{-4}] . \end{aligned}$$

Hence, by Schwarz's inequality

$$E|\eta_\alpha Z_\alpha^{(0)} Z_\alpha^{(1)}| \leqq \|\eta_\alpha Z_\alpha^{(1)}\|_2 \|Z_\alpha^{(0)}\|_2 \leqq K[t^2 n^{-1} m + n^{-2}] .$$

Now, we proceed to prove (3.9) in the case $r \geqq 2$. By Schwarz's inequality

$$E\Big| \eta_\alpha \prod_{j=0}^{r} Z_\alpha^{(j)} \Big| \leqq \|\eta_\alpha \textstyle\prod_j' Z_\alpha^{(j)}\|_2 \|\prod_j'' Z_\alpha^{(j)}\|_2$$

where $\prod_j'$ is the product over all even number j such that $2 \leqq j \leqq r$ and $\prod_j''$ the product over all odd number j such that $1 \leqq j \leqq r$. Since for each j $(2 \leqq j \leqq k)$ $Z_\alpha^{(j)}$ is $\mathscr{M}_{\alpha-(j+1)m}^{\alpha-jm} \times \mathscr{M}_{\alpha+jm}^{\alpha+(j+1)m}$-measurable, so from Lemma 2.1, (2.25) and (3.5)

$$\begin{aligned} E|\eta_\alpha \textstyle\prod_j' Z_\alpha^{(j)}|^2 &\leqq E|\eta_\alpha Z_\alpha^{(0)}|^2 \textstyle\prod_j' H_\alpha^{(j)} + Kr2^r \|\eta_\alpha\|_{2+\delta}^2 \{\beta(m)\}^{\delta/(2+\delta)} \\ &\leqq K[\{C_0 t^2 n^{-1} m\}^{[r/2]} + r2^r n^{-4}] . \end{aligned}$$

Similarly

$$E|\textstyle\prod_j'' Z_\alpha^{(j)}| \leqq K[\{C_0 t^2 n^{-1} m\}^{[(r-1)/2]} + r2^r n^{-4}] .$$

Hence, by the definition of k we have (3.19) for r $(2 \leqq r \leqq k)$ and so the proof is completed.

Lemma 3.2. *Suppose conditions of Proposition* 3.1 *hold. Then*

$$(3.10) \qquad \begin{aligned} &\Big| \sum_{\alpha=1}^{n} E\eta_\alpha \sum_{j=0}^{r-1} Z_\alpha^{(j)} \exp\{it(T_\alpha^{(r)} + V_{\alpha,1}^{(k)})\} \\ &\qquad - \frac{1}{n} \sum_{\alpha=1}^{n} E\eta_\alpha \prod_{j=0}^{r-1} Z_\alpha^{(j)} \sum_{p=1}^{n} E\exp\{it(T_p^{(r)} + V_{p,1}^{(k)})\} \Big| \\ &\qquad\qquad \leqq \begin{cases} K(t^2 n^{-1} m)^{\delta/2(2+\delta)} & (r=1) \\ K[t^2 n^{-1} m + n^{-2}] & (r=2) \\ K[\{C_0 |t| n^{-1/2} m^{1/2}\}^r + n^{-2}] & (3 \leqq r \leqq k) \end{cases} \end{aligned}$$

for all t ($|t| < C_0^{-1} n^{1/2} m^{-1/2}$).

Proof. For brevity, we put

$$X_\alpha^{(r)} = \eta_\alpha \prod_{j=0}^{r-1} Z_\alpha^{(j)} \quad (1 \leqq \alpha \leqq n,\ 1 \leqq r \leqq k-1)$$

and

$$W_\alpha^{(r)} = \exp\{it(T_\alpha^{(r+1)} + V_{\alpha,1}^{(k)})\} \quad (1 \leqq \alpha \leqq n,\ 1 \leqq r \leqq k-1)\,.$$

Since for each r ($3 \leqq r \leqq k-1$) $X_\alpha^{(r)}$ is $\mathscr{M}_{\alpha-rm}^{\alpha+rm}$-measurable and $W_\alpha^{(r)}$ is $\mathscr{M}_{-\infty}^{\alpha-(r+1)m} \times \mathscr{M}_{\alpha+(r+1)m}^{\infty}$-measurable, so by Lemma 3.1, (2.26) and the definition of m

$$\begin{aligned} (3.11) \qquad & |EX_\alpha^{(r)} W_\alpha^{(r)} - EX_\alpha^{(r)} EW_\alpha^{(r)}| \\ & \leqq K\{\beta(m)\}^{\delta/(2+\delta)} \|X_\alpha^{(r)}\|_{(2+\delta)/2} \leqq K n^{-4} \left\| \eta_\alpha \prod_{\alpha=1}^{r-1} Z_\alpha^{(j)} \right\|_{(2+\delta)/2} \\ & \leqq K n^{-4} 2^r \|\eta\|_{2+\delta} \leqq K 2^r n^{-4}\,. \end{aligned}$$

Further, since for each α ($rm \leqq \alpha \leqq n-rm$) $X_\alpha^{(r)}$ has the same distribution as that of $X_{\alpha_0}^{(r)}$ where $\alpha_0 = rm+1$, which implies

$$EX_\alpha^{(r)} = EX_{\alpha_0}^{(r)} \quad (rm \leqq \alpha \leqq n-rm)\,,$$

so

$$\begin{aligned} (3.12) \qquad \sum_{\alpha=1}^{n} EX_\alpha^{(r)} EW_\alpha^{(r)} &= \left\{ \sum_{\alpha=1}^{\alpha_0-1} + \sum_{\alpha=n-\alpha_0+1}^{n} \right\} EX_\alpha^{(r)} EW_\alpha^{(r)} + EX_{\alpha_0}^{(r)} \sum_{\alpha=\alpha_0}^{n-\alpha_0} EW_\alpha^{(r)} \\ &= EX_{\alpha_0}^{(r)} \sum_{\alpha=1}^{n} EW_\alpha^{(r)} + \left\{ \sum_{\alpha=1}^{\alpha_0-1} + \sum_{\alpha=n-\alpha_0+1}^{n} \right\} EX_\alpha^{(r)} EW_\alpha^{(r)} \\ &\quad - \left\{ \sum_{\alpha=1}^{\alpha_0-1} + \sum_{\alpha=n-\alpha_0+1}^{n} \right\} EX_{\alpha_0}^{(r)} EW_\alpha^{(r)} \end{aligned}$$

and

$$(3.13) \qquad \sum_{\alpha=1}^{n} EX_\alpha^{(r)} = \{n-2rm\} EX_{\alpha_0}^{(r)} + \left\{ \sum_{\alpha=1}^{\alpha_0-1} + \sum_{\alpha=n-\alpha_0+1}^{n} \right\} EX_\alpha^{(r)}\,.$$

From (3.11)-(3.13) it follows that

$$\begin{aligned} & \sum_{\alpha=1}^{n} EX_\alpha^{(r)} EW_\alpha^{(r)} - \frac{1}{n} \sum_{\alpha=1}^{n} EX_\alpha^{(r)} \sum_{p=1}^{n} EW_p^{(r)} \\ & = \left\{ \sum_{\alpha=1}^{\alpha_0-1} + \sum_{\alpha=n-\alpha_0+1}^{n} \right\} EX_\alpha^{(r)} \left[EW_\alpha^{(r)} - \frac{1}{n} \sum_{p=1}^{n} EW_p^{(r)} \right] \\ & \quad - \left\{ \sum_{\alpha=1}^{\alpha_0-1} + \sum_{\alpha=n-\alpha_0+1}^{n} \right\} EX_{\alpha_0}^{(r)} \left[EW_\alpha^{(r)} - \frac{1}{n} \sum_{p=1}^{n} EW_p^{(r)} \right]. \end{aligned}$$

Thus, by Lemma 3.1 we have the lemma for r ($3 \leqq r \leqq k$). The proofs in the cases $r=1$ and 2 are analogous and so is omitted.

Lemma 3.3. *Suppose conditions of Proposition* 3.1 *hold. Then for each* r $(1\leqq r\leqq k-1)$

$$\left|\frac{1}{n}\sum_{p=1}^{n} E\exp\{it(T_p{}^{(r+1)}+V_{p,1}^{(k)})\}-f_n(t)\right|\leqq K[|t|n^{-1}m+n^{-2}+|t|n^{-1/2}m|f_n(t)|]\ . \tag{3.14}$$

Proof. We remark that for all t

$$\begin{aligned}\text{LHS of (3.14)}&\leqq\frac{1}{n}\sum_{p=1}^{n}|E\exp\{it(T_p{}^{(r+1)}+V_{p,1}^{(k)})\}-E\exp\{it(T_p{}^{(r+1)}+V_n)\}|\\&\quad+\left|\frac{1}{n}\sum_{p=1}^{n}E\exp\{it(T_p{}^{(r+1)}+V_n)\}-Ee^{itS_n}\right|\\&=I_1+I_2\ ,\quad \text{(say)}\ .\end{aligned} \tag{3.15}$$

Since by Lemma 2.4, (3.5) and (3.6)

$$\begin{aligned}&|E\exp\{it(T_p{}^{(r+1)}+V_{p,1}^{(k)})\}-E\exp\{it(T_p{}^{(r+1)}+V_n)\}|\\&\qquad\leqq E|\exp\{it(V_n-V_{p,1}^{(k)})\}-1|\leqq|t|E|V_n-V_{p,1}^{(k)}|\leqq|t|E\{|V_{p,0}^{(k)}|+|V_{p,2}^{(k)}|+|V_{p,3}^{(k)}|\}\\&\qquad\leqq|t|\{\|V_{p,0}^{(k)}\|_2+\|V_{p,2}^{(k)}\|_2+\|V_{p,3}^{(k)}\|_2\}\leqq K|t|n^{-1}m\ ,\end{aligned}$$

so

$$I_1\leqq K|t|n^{-1}m\ . \tag{3.16}$$

Next, let

$$Y_p{}^{(r)}=1-\exp\{-it(T_p{}^{(0)}-T_p{}^{(r+1)})\}\quad(1\leqq p\leqq n)\ .$$

Then

$$\begin{aligned}I_2&\leqq\left|Ee^{itS_n}\frac{1}{n}\sum_{p=1}^{n}EY_p{}^{(r)}+E\left[\frac{1}{n}\sum_{p=1}^{n}(Y_p{}^{(r)}-EY_p{}^{(r)})e^{itS_n}\right]\right|\\&\leqq|f_n(t)|\frac{1}{n}\sum_{p=1}^{n}|t|E|T_p{}^{(0)}-T_p{}^{(r+1)}|+\frac{1}{n}\left\|\sum_{p=1}^{n}(Y_p{}^{(r)}-EY_p{}^{(r)})\right\|_2\ .\end{aligned} \tag{3.17}$$

By (3.4)

$$\frac{1}{n}\sum_{p=1}^{n}E|T_p{}^{(0)}-T_p{}^{(r+1)}|\leqq\frac{1}{n}\sum_{p=1}^{n}\|T_p{}^{(0)}-T_p{}^{(r+1)}\|_2\leqq Kn^{-1/2}(mk)^{1/2}\leqq Kn^{-1/2}m\ . \tag{3.18}$$

On the other hand

$$\begin{aligned}E\left|\sum_{p=1}^{n}(Y_p{}^{(r)}-EY_p{}^{(r)})\right|^2&=\sum_{p=1}^{n}E|Y_p{}^{(r)}-EY_p{}^{(r)}|^2\\&\quad+2\sum_{1\leqq p<q\leqq n}|E(Y_p{}^{(r)}-EY_p{}^{(r)})(Y_q{}^{(r)}-EY_q{}^{(r)})|\ .\end{aligned}$$

As $Y_p{}^{(r)}$ is $\mathscr{M}_{p-rm}^{p+rm}$-measurable, so by Corollary to Lemma 2.1

$$\begin{aligned}&\sum_{\substack{1\leqq p<q\leqq n\\ q-p>(2r+1)m}}|E(Y_p{}^{(r)}-EY_p{}^{(r)})(Y_q{}^{(r)}-EY_q{}^{(r)})|\\&\qquad\leqq Kn^2\{\beta(m)\}^{\delta/(2+\delta)}\|Y_p{}^{(r)}-EY_p{}^{(r)}\|_{2+\delta}^2\leqq Kn^{-2}\ .\end{aligned}$$

Further, by Schwarz's inequality

$$|E(Y_p^{(r)}-EY_p^{(r)})(Y_q^{(r)}-EY_q^{(r)})|\leqq \max_{1\leqq p\leqq n}\|Y_p^{(r)}-EY_p^{(r)}\|_2^2\leqq 4\max_{1\leqq p\leqq n}\|Y_p^{(r)}\|_2^2$$
$$\leqq 4|t|^2\max_{1\leqq p\leqq n}\|T_p^{(0)}-T_p^{(r+1)}\|_2^2\leqq K|t|^2n^{-1}rm$$

and so

$$\sum_{\substack{1\leqq p<q\leqq n\\ q-p\leqq(2r+1)m}}|E(Y^{(r)}-EY_p^{(r)})(Y_q^{(r)}-EY_q^{(r)})|\leqq K|t|^2r^2m^2 .$$

Hence, using the relations

$$\sum_{p=1}^{n}E|Y_p^{(r)}-EY_p^{(r)}|^2\leqq 4n\max_{1\leqq p\leqq n}\|Y_p^{(r)}\|_2^2\leqq K|t|^2rm ,$$

we have

$$E\left|\sum_{p=1}^{n}(Y_p^{(r)}-EY_p^{(r)})\right|^2\leqq K[|t|^2r^2m^2+n^{-2}] . \tag{3.19}$$

Thus, by (3.17)-(3.19)

$$I_2\leqq K[|t|n^{-1/2}m|f_n(t)|+n^{-1}\{|t|rm+n^{-1}\}] . \tag{3.20}$$

Now, (3.14) follows from (3.15), (3.16) and (3.20) and the proof is completed.

Lemma 3.4. *Suppose conditions of Proposition* 3.1 *hold. Then*

$$\left|\frac{1}{\sigma\sqrt{n}}\sum_{\alpha=1}^{n}\eta_\alpha Z_\alpha^{(0)}-it\right|\leqq K\{|t|n^{-1}m+n^{-1}+|t|^{1+\delta}n^{-\delta/2}m^{1+\delta}\} . \tag{3.21}$$

Proof. By (2.25) and Hölder's inequality

$$\begin{aligned}&|E\eta_\alpha Z_\alpha^{(0)}-itE\eta_\alpha\{(T_\alpha^{(0)}-T_\alpha^{(1)})+V_{\alpha,0}^{(1)}\}|\\ &\quad\leqq K|t|^{1+\delta}E|\eta_\alpha|\,|(T_\alpha^{(0)}-T_\alpha^{(1)})+V_{\alpha,0}^{(1)}|^{1+\delta}\\ &\quad\leqq K|t|^{1+\delta}\{E|\eta_\alpha|\,|T_\alpha^{(0)}-T_\alpha^{(1)}|^{1+\delta}+E|\eta_\alpha|\,|V_{\alpha,0}^{(1)}|^{1+\delta}\}\\ &\quad\leqq K|t|^{1+\delta}\{\|\eta_\alpha\|_{2+\delta}\|T_\alpha^{(0)}-T_\alpha^{(1)}\|_{2+\delta}^{1+\delta}+\|\eta_\alpha\|_{2+\delta}\|V_{\alpha,0}^{(1)}\|_{2+\delta}^{1+\delta}\} .\end{aligned}$$

Since by Minkowski's inequality

$$\|T_\alpha^{(0)}-T_\alpha^{(1)}\|_{2+\delta}\leqq\frac{1}{\sigma\sqrt{n}}\sum_{|j-\alpha|\leqq m}\|\eta_\alpha\|_{2+\delta}\leqq Kn^{-1/2}m$$

and

$$\|V_{\alpha,0}^{(1)}\|_{2+\delta}\leqq a_n\sum_{|p-\alpha|\leqq m,\ |q-\alpha|\leqq m}\|\zeta_{p,q}\|_{2+\delta}\leqq Kn^{-3/2}m^2$$

so

$$|E\eta_\alpha Z_\alpha^{(0)}-itE\eta_\alpha\{(T_\alpha^{(0)}-T_\alpha^{(1)})+V_{\alpha,0}^{(1)}\}|\leqq K|t|^{1+\delta}n^{-(1+\delta)/2}m^{1+\delta} .$$

We note here that by (2.12) and the fact $E\eta_\alpha=0$

$$|E\eta_\alpha T_\alpha{}^{(1)}| \leqq Kn^{-1/2}\{\sum_{1\leqq j<\alpha-m} + \sum_{\alpha+m<j\leqq n}\}E\eta_\alpha\eta_j$$
$$\leqq Kn^{-1/2}\sum_{l=m+1}^{\infty}\{\beta(l)\}^{\delta/(2+\delta)}\|\eta_1\|_{2+\delta}^2 \leqq Kn^{-1/2}\sum_{l=m+1}^{\infty}\{\beta(l)\}^{\delta/(2+\delta)} \leqq Kn^{-7/2}$$

and

$$|E\eta_\alpha V_{\alpha,0}^{(1)}| \leqq \|\eta_\alpha\|_2\|V_{\alpha,0}^{(1)}\|_2 \leqq Ka_n m \leqq Kn^{-3/2}m\ .$$

Hence

$$|E\eta_\alpha Z_\alpha{}^{(0)} - itE\eta_\alpha T_\alpha{}^{(0)}| \leqq |t|\{|E\eta_\alpha T_\alpha{}^{(1)}| + |E\eta_\alpha V_{\alpha,0}^{(1)}|\} + K|t|^{1+\delta}n^{-(1+\delta)/2}m^{1+\delta} \tag{3.22}$$
$$\leqq K\{|t|n^{-3/2}m + |t|^{1+\delta}n^{-(1+\delta)/2}m^{1+\delta}\}\ .$$

Now, (3.21) follows from (3.22) since by Lemma 2.2 (i)

$$\left|\frac{1}{\sigma\sqrt{n}}\sum_{\alpha=1}^{n}E\eta_\alpha T_\alpha{}^{(0)} - 1\right| = |ET_n{}^2 - 1| = \frac{1}{n\sigma^2}\left|E\left|\sum_{\alpha=1}^{n}\eta_\alpha\right|^2 - n\sigma^2\right| \leqq Kn^{-1}\ .$$

Proof of Proposition 3.1. Firstly, we rewrite $ET_ne^{itS_n}$ as follows:

$$ET_ne^{itS_n} = \frac{1}{\sigma\sqrt{n}}\Big[\sum_{\alpha=1}^{n}E\eta_\alpha\exp\{it(T_\alpha{}^{(1)} + V_{\alpha,1}^{(1)})\}Z_\alpha{}^{(0)}[\exp\{it(V_{\alpha,2}^{(1)} + V_{\alpha,3}^{(1)})\} - 1] \tag{3.23}$$
$$+ \sum_{\alpha=1}^{n}E\eta_\alpha Z_\alpha{}^{(0)}\exp\{it(T_\alpha{}^{(1)} + V_{\alpha,1}^{(k)})\}[\exp\{it(V_{\alpha,1}^{(1)} - V_{\alpha,1}^{(k)})\} - 1]$$
$$+ \sum_{r=2}^{k-1}\sum_{\alpha=1}^{n}E\eta_\alpha\prod_{j=0}^{r-1}Z_\alpha{}^{(j)}\exp\{it(T_\alpha{}^{(r+1)} + V_{\alpha,1}^{(k)})\}$$
$$+ \sum_{\alpha=1}^{n}E\eta_\alpha\prod_{j=0}^{k-1}Z_\alpha{}^{(j)}\exp\{it(T_\alpha{}^{(k)} + V_{\alpha,1}^{(k)})\}$$
$$+ \sum_{\alpha=1}^{n}E\eta_\alpha Z_\alpha{}^{(0)}\exp\{it(T_\alpha{}^{(2)} + V_{\alpha,1}^{(k)})\}$$
$$+ \sum_{\alpha=1}^{n}E\eta_\alpha\exp\{it(T_\alpha{}^{(1)} + V_{\alpha,1}^{(1)})\}[\exp\{it(V_{\alpha,2}^{(1)} + V_{\alpha,3}^{(1)})\} - 1]$$
$$+ \sum_{\alpha=1}^{n}E\eta_\alpha\exp\{it(T_\alpha{}^{(1)} + V_{\alpha,1}^{(1)} + V_{\alpha,2}^{(1)} + V_{\alpha,3}^{(1)})\}\Big]$$
$$+ \sum_{j=1}^{7}I_j\ ,\quad \text{(say)}\ .$$

By Schwarz's inequality, (2.24), (3.6) and (3.8)

$$|I_1| \leqq \frac{1}{\sigma\sqrt{n}}\sum_{\alpha=1}^{n}E|\eta_\alpha Z_\alpha{}^{(0)}|\ |\exp\{it(V_{\alpha,2}^{(1)} + V_{\alpha,3}^{(1)})\} - 1| \tag{3.24}$$
$$\leqq \frac{1}{\sigma\sqrt{n}}\sum_{\alpha=1}^{n}\|\eta_\alpha Z_\alpha{}^{(0)}\|_2\|\exp\{it(V_{\alpha,2}^{(1)} + V_{\alpha,3}^{(1)})\} - 1\|_2$$
$$\leqq Kn^{-1/2}\sum_{\alpha=1}^{n}|t|^{\delta/(2+\delta)}n^{-\delta/2(2+\delta)}m^{\delta/2(2+\delta)}|t|\ \|V_{\alpha,2}^{(1)} + V_{\alpha,3}^{(1)}\|_2$$
$$\leqq Kn^{1/2}|t|^{2(1+\delta)/(2+\delta)}m^{\delta/2(2+\delta)}n^{-\delta/2(1+\delta)}a_n(nm)^{1/2}$$
$$\leqq K|t|^{2(1+\delta)/(2+\delta)}n^{-(1+\delta)/(2+\delta)}m^{(1+\delta)/(2+\delta)}$$

and similarly

$$(3.25)\qquad \begin{aligned}|I_2| &\leqq \frac{1}{\sigma\sqrt{n}}\sum_{\alpha=1}^{n} E|\eta_\alpha Z_\alpha^{(0)}|\,|\exp\{it(V_{\alpha,1}^{(1)}-V_{\alpha,1}^{(k)})\}-1|\\ &\leqq Kn^{-1/2}\sum_{\alpha=1}^{n}\|\eta_\alpha Z_\alpha^{(0)}\|_2\|\exp\{it(V_{\alpha,1}^{(1)}-V_{\alpha,1}^{(k)})\}-1\|_2\\ &\leqq K|t|n^{-(1+\delta)/(2+\delta)}|t|^{\delta/(2+\delta)}m^{\delta/(2+\delta)}na_n(nkm)^{1/2}\\ &\leqq K|t|^{2(1+\delta)/(2+\delta)}n^{-(1+\delta)/(2+\delta)}m^{2(1+\delta)/(2+\delta)}\,.\end{aligned}$$

By Lemma 2.1 and the fact $E\eta_\alpha=0$

$$(3.26)\qquad |I_6|\leqq Kn^{1/2}\{\beta(m)\}^{(1+\delta)/(2+\delta)}\|\eta_\alpha\|_{2+\delta}\leqq Kn^{1/2}n^{-8}\leqq Kn^{-7}$$

and

$$(3.27)\qquad |I_7|\leqq Kn^{1/2}\{\beta(m)\}^{(1+\delta)/(2+\delta)}\|\eta_\alpha\|_{2+\delta}\leqq Kn^{-7}\,.$$

Let $X_\alpha^{(r)}$ and $W_\alpha^{(r)}$ be as in the proof of Lemma 3.2.

By Lemmas 3.1–3.4 we have that for all t $(|t|<n^{1/2}m^{-1/2})$

$$(3.28)\qquad \begin{aligned}|I_3| &\leqq \frac{1}{\sigma\sqrt{n}}\sum_{r=2}^{k-1}\left|\sum_{\alpha=1}^{n} EX_\alpha^{(r)}W_\alpha^{(r)}\right|\\ &\leqq \frac{1}{\sigma\sqrt{n}}\sum_{r=2}^{k-1}\sum_{\alpha=1}^{n}|EX_\alpha^{(r)}|\left|\frac{1}{n}\sum_{p=1}^{n}EW_p^{(r)}\right|\\ &\quad +Kn^{-1/2}\left[\sum_{r=2}^{k-1} rm(C_0|t|n^{-1/2}m^{1/2})^r+2^rn^{-4}\right]\\ &\leqq Kn^{-1/2}\sum_{r=2}^{k-1}\sum_{\alpha=1}^{n}|EX_\alpha^{(r)}|\{(1+|t|n^{-1/2}m)|f_n(t)|+|t|n^{-1}m+n^{-2}\}\\ &\quad +Kn^{-1/2}[km(C_0|t|n^{-1/2}m^{1/2})^2+n^{-2}]\\ &\leqq Kn^{1/2}\left[\{t^2n^{-1}m+n^{-2}\}+\sum_{r=3}^{k-1}\{(C_0|t|n^{-1/2}m^{1/2})^r+n^{-2}\}\right]\\ &\quad \times\{(1+|t|n^{-1/2}m^{1/2})|f_n(t)|+|t|n^{-1}m+n^{-2}\}+K[t^2n^{-3/2}m^3+n^{-2}]\\ &\leqq K\{(t^2n^{-1/2}m+n^{-1})|f_n(t)|+t^2n^{-3/2}m^3+n^{-2}\}\,.\end{aligned}$$

Similarly, by Lemmas 3.1–3.3

$$(3.29)\qquad \begin{aligned}|I_4| &\leqq \frac{1}{\sigma\sqrt{n}}\sum_{\alpha=1}^{n}|EX_\alpha^{(k)}|\left|\frac{1}{n}\sum_{p=1}^{n}EW_\alpha^{(k-1)}\right|+Kn^{-1/2}km(C_0|t|n^{-1/2}m^{1/2})^k\\ &\leqq Kn^{1/2}(C_0|t|n^{-1/2}m^{1/2})^{k-1}[(1+|t|n^{-1/2}m)|f_n(t)|+|t|n^{-1/2}m^{1/2}+n^{-2}]\\ &\quad +Kn^{-1/2}km(C_0|t|n^{-1/2}m^{1/2})^k\,.\end{aligned}$$

Since by the definition of k

$$(C_0|t|n^{-1/2}m^{1/2})^{k-2}\leqq Kn^{-1}\quad\text{for all}\quad t\ (|t|<K_1n^{1/2}m^{-3/4})$$

so for all t $(|t|<K_1n^{1/2}m^{-3/4})$

$$\text{(3.30)}\qquad \begin{aligned}|I_4| &\leqq K[|t|n^{-1}m^{1/2}\{(1+|t|n^{-1/2}m)|f_n(t)|+|t|n^{-1/2}m^{1/2}+n^{-2}\}+t^2n^{-5/2}km^2] \\ &\leqq K[|t|n^{-1}m^{1/2}|f_n(t)|+|t|n^{-1}m^{1/2}] .\end{aligned}$$

Finally, by Lemmas 3.1–3.4 and (3.9)

$$\text{(3.31)}\qquad \begin{aligned}|I_5-itf_n(t)| &\leqq \left|I_5-\frac{1}{\sigma\sqrt{n}}\sum_{\alpha=1}^{n}EX_\alpha{}^{(1)}\frac{1}{n}\sum_{p=1}^{n}EW_p{}^{(1)}\right| \\ &\quad+\left|\frac{1}{\sigma\sqrt{n}}\sum_{\alpha=1}^{n}EX_\alpha{}^{(1)}\right|\left|\frac{1}{n}\sum_{p=1}^{n}EW_p{}^{(1)}-f_n(t)\right| \\ &\quad+\left|\frac{1}{\sigma\sqrt{n}}\sum_{\alpha=1}^{n}EX_\alpha{}^{(1)}-it\right||f_n(t)| \\ &\leqq K[|t|^{\delta/(2+\delta)}n^{-(1+\delta)/(2+\delta)}m^{\delta/2(2+\delta)}+\{|t|+n^{-1}+|t|^{1+\delta}n^{-\delta/2}m^{1+\delta}\} \\ &\quad\times\{|t|n^{-1}m+n^{-2}+|t|n^{-1/2}m|f_n(t)|\} \\ &\quad+\{|t|n^{-1}m+n^{-1}+|t|^{1+\delta}n^{-\delta/2}m^{1+\delta}\}|f_n(t)|] \\ &\qquad\qquad\text{for all}\quad t\ \ (|t|<K_1n^{1/2}m^{-3/4}) .\end{aligned}$$

Hence, (3.3) follows from (3.23)–(3.31).

Proposition 3.2. *Suppose conditions of Proposition* 3.1 *hold. Then, for all* t $(|t|<K_1n^{1/2}m^{-3/4})$

$$\text{(3.32)}\qquad \begin{aligned}|EV_ne^{itS_n}| &\leqq K[\{|t|n^{-1/2}m^{1/2}+|t|^{1+\delta}n^{-\delta/2}m^{1+\delta}+n^{-1}\}|f_n(t)| \\ &\quad+\{(t^2n^{-1}m)^{(1+\delta)/(2+\delta)}m^{(1+\delta)/2}+|t|^{\delta/(2+\delta)}n^{-(1+\delta)/(2+\delta)}m^{\delta/2(2+\delta)}+n^{-2}\}] .\end{aligned}$$

To prove Proposition 3.2, we introduce the following notations:

$$T_{\alpha,\beta}^{(0)}=T_n$$

$$T_{\alpha,\beta}^{(j)}=\frac{1}{\sigma\sqrt{n}}\sum_{\substack{|p-\alpha|>m,\ |p-\beta|>jm\\ 1\leqq p\leqq n}}\eta_p\quad (j=1,2,\cdots,k)$$

$$V_{\alpha,\beta}^{(0)}=a_n\sum_{\substack{\max\{|p-\alpha|,\ |p-\beta|,\ |q-\alpha|,\ |q-\beta|\}\leqq m\\ 1\leqq p<q\leqq n}}\zeta_{p,q}$$

$$V_{\alpha,\beta}^{(j)}=a_n\sum_{\substack{\max\{|p-\alpha|,\ |p-\beta|,\ |q-\alpha|,\ |q-\beta|\}>jm\\ 1\leqq p<q\leqq n}}\zeta_{p,q}\quad (j=1,k) .$$

Then by Lemma 2.2 (i) and (ii) the following inequalities are easily proved:

$$\text{(3.33)}\qquad \|T_{\alpha,\beta}^{(j)}-T_{\alpha,\beta}^{(j+1)}\|_2\leqq C_0n^{-1/2}m^{1/2}\quad (j=0,1,\cdots,k-1) ,$$

$$\text{(3.34)}\qquad \|V_n-V_{\alpha,\beta}^{(k)}\|_2\leqq Kn^{-1}k^{1/2}m^{1/2} ,$$

$$\text{(3.35)}\qquad \|V_{\alpha,\beta}^{(1)}-V_{\alpha,\beta}^{(k)}\|_2\leqq Kn^{-1}k^{1/2}m^{1/2} .$$

Here, α and β are arbitrary integers such that $1\leqq\alpha\leqq\beta\leqq n$ and C_0 is the constant in (3.4).

As before, we need some lemmas.

Lemma 3.5. *Suppose conditions of Proposition* 3.2 *hold. Then, for all t and for all α and β* $(1\leqq\alpha<\beta\leqq n)$

$$\|\zeta_{\alpha,\beta}Z_{\alpha,\beta}^{(0)}\|_2\leqq K(t^2n^{-1}m)^{\delta/2(2+\delta)} \tag{3.36}$$

and

$$E\left|\zeta_{\alpha,\beta}\prod_{j=0}^{r}Z_{\alpha,\beta}^{(j)}\right|\leqq\begin{cases}K[t^2n^{-1}m+n^{-2}] & (r=1)\\ K[(C_0|t|n^{-1/2}m^{1/2})^r+n^{-2}] & (2\leqq r\leqq k-1)\end{cases} \tag{3.37}$$

where

$$Z_{\alpha,\beta}^{(0)}=\exp[it\{(T_{\alpha,\beta}^{(0)}-T_{\alpha,\beta}^{(1)})+V_{\alpha,\beta}^{(0)}\}]-1$$

and

$$Z_{\alpha,\beta}^{(j)}=\exp\{it(T_{\alpha,\beta}^{(j)}-T_{\alpha,\beta}^{(j+1)})\}-1 \quad (j=1,\cdots,k-1)\,.$$

The proof of Lemma 3.5 is identical to that of Lemma 3.1 and so is omitted.

Lemma 3.6. *Suppose conditions of Proposition* 3.1 *hold. Then*

$$\begin{aligned}\Bigg|\sum_{1\leqq\alpha<\beta\leqq n}&E\zeta_{\alpha,\beta}\prod_{j=0}^{r-1}Z_{\alpha,\beta}^{(j)}\exp\{it(T_{\alpha,\beta}^{(r)}+V_{\alpha,\beta}^{(k)})\}\\ &-\frac{1}{n(n-1)}\sum_{1\leqq\alpha<\beta\leqq n}E\zeta_{\alpha,\beta}\prod_{j=0}^{r-1}Z_{\alpha,\beta}^{(j)}\sum_{1\leqq p<q\leqq n}E\exp\{it(T_{p,q}^{(r)}+V_{p,q}^{(k)})\}\Bigg|\\ &\leqq\begin{cases}K(t^2n^{-1}m)^{\delta/2(2+\delta)} & (r=1)\\ K[t^2n^{-1}m+n^{-2}] & (r=2)\\ K[(C_0|t|n^{-1/2}m^{1/2})^r+n^{-2}] & (3\leqq r\leqq k)\end{cases}\end{aligned} \tag{3.38}$$

for all t $(|t|<K_1n^{1/2}m^{-3/4})$.

Proof. We note that for each r $(1\leqq r\leqq k-1)\zeta_{\alpha,\beta}\prod_{j=0}^{r-1}Z_{\alpha,\beta}^{(j)}$ is $M_{\alpha-rm}^{\alpha+rm}\times M_{\beta-rm}^{\beta+rm}$-measurable and $T_{\alpha,\beta}^{(r+1)}+V_{\alpha,\beta}^{(k)}$ is $M_{-\infty}^{\alpha-(r+1)m}\times M_{\alpha+(r+1)m}^{\beta-(r+1)m}\times M_{\beta+(r+1)m}^{\infty}$-measurable where $M_{\alpha+(r+1)m}^{\beta-(r+1)m}$ must be dropped if $\beta-(r+1)m<\alpha+(r+1)m$. Hence, the proof is identical to that of Lemma 3.2 and so is omitted.

Lemma 3.7. *Suppose conditions of Proposition* 3.2 *hold. Then for each r* $(1\leqq r\leqq k-1)$ *and t*

$$\begin{aligned}\left|\frac{1}{n(n-1)}\sum_{1\leqq p<q\leqq n}E\exp\{it(T_{\alpha,\beta}^{(r+1)}+V_{\alpha,\beta}^{(k)})\}-f_n(t)\right|\\ \leqq K\{|t|n^{-1}m+n^{-2}+|t|n^{-1/2}m|f_n(t)|\}\,.\end{aligned} \tag{3.39}$$

The proof is obtained by the same method as that of Lemma 3.3.

Lemma 3.8. *Suppose conditions of Proposition* 3.1 *hold. Then*

$$|a_n\sum_{1\leqq\alpha<\beta\leqq n}E\zeta_{\alpha,\beta}Z_{\alpha,\beta}^{(0)}|\leqq K\{|t|n^{-1/2}m+|t|^{1+\delta}n^{-\delta/2}m^{1+\delta}\}\,. \tag{3.40}$$

Proof. By (2.24) and Hölder's inequality

$$\begin{aligned}
&|E\zeta_{\alpha,\beta}Z^{(0)}_{\alpha,\beta}-itE\zeta_{\alpha,\beta}\{(T_n-T^{(1)}_{\alpha,\beta})+V^{(0)}_{\alpha,\beta}\}| \\
&\qquad\leqq K|t|^{1+\delta}E|\zeta_{\alpha,\beta}|\,|(T_n-T^{(1)}_{\alpha,\beta})+V^{(0)}_{\alpha,\beta}|^{1+\delta} \\
&\qquad\leqq K|t|^{1+\delta}\{E|\zeta_{\alpha,\beta}|\,|T_n-T^{(1)}_{\alpha,\beta}|^{1+\delta}+E|\zeta_{\alpha,\beta}|\,|T_n-T^{(1)}_{\alpha,\beta}|^{1+\delta}\} \\
&\qquad\leqq K|t|^{1+\delta}\{\|\zeta_{\alpha,\beta}\|_{2+\delta}\|T_n-T^{(1)}_{\alpha,\beta}\|^{1+\delta}_{2+\delta}+\|\zeta_{\alpha,\beta}\|_{2+\delta}\|T_n-T^{(1)}_{\alpha,\beta}\|^{1+\delta}_{2+\delta}\}\,.
\end{aligned}$$

Since by Minkowski's inequality

$$\|T_n-T^{(1)}_{\alpha,\beta}\|_{2+\delta}\leqq Kn^{-1/2}m$$

and

$$\|V^{(0)}_{\alpha,\beta}\|_{2+\delta}\leqq Kn^{-3/2}m^2\,,$$

so

$$|E\zeta_{\alpha,\beta}Z^{(0)}_{\alpha,\beta}-itE\zeta_{\alpha,\beta}\{(T_n-T^{(1)}_{\alpha,\beta})+V^{(0)}_{\alpha,\beta}\}|\leqq K|t|^{1+\delta}n^{-(1+\delta)/2}m^{1+\delta}\,.$$

Further, by (2.19)

$$\begin{aligned}
|E\zeta_{\alpha,\beta}T^{(1)}_{\alpha,\beta}|&\leqq Kn^{-1/2}\{\sum_{|p-\alpha|>m,\ |p-\beta|>m}|E\zeta_{\alpha,\beta}\eta_p|\}\leqq Kn^{-1/2}\sum_{|p-\alpha|>m,\ |p-\beta|>m}\{\beta(m)\}^{\delta/(2+\delta)} \\
&\leqq Kn^{-1/2}n^2n^{-4}\leqq Kn^{-5/2}
\end{aligned}$$

and

$$|E\zeta_{\alpha,\beta}V^{(0)}_{\alpha,\beta}|\leqq\|\zeta_{\alpha,\beta}\|_2\|V^{(0)}_{\alpha,\beta}\|_2\leqq Ka_nm\leqq Kn^{-3/2}m\,.$$

Hence

$$|E\zeta_{\alpha,\beta}Z^{(0)}_{\alpha,\beta}-itE\zeta_{\alpha,\beta}T_n|\leqq K\{|t|n^{-3/2}m+|t|^{1+\delta}n^{-(1+\delta)/2}m^{1+\delta}\}\,. \tag{3.41}$$

On the other hand, by Lemma 2.2 (i) and (ii)

$$|a_n\sum_{1\leqq\alpha<\beta\leqq n}E\zeta_{\alpha,\beta}T_n|\leqq a_n\|\sum_{1\leqq\alpha<\beta\leqq n}\zeta_{\alpha,\beta}\|_2\|T_n\|_2\leqq Kn^{-3/2}n=Kn^{-1/2}\,. \tag{3.42}$$

Thus, (3.40) follows from (3.41) and (3.42).

Proof of Proposition 3.2. We can rewrite $EV_ne^{itT_n}$ as follows:

$$\begin{aligned}
&EV_ne^{itS_n} \\
&=a_n\Big[\sum_{1\leqq\alpha<\beta\leqq n}E\zeta_{\alpha,\beta}Z^{(0)}_{\alpha,\beta}\exp\{it(T^{(1)}_{\alpha,\beta}+V^{(1)}_{\alpha,\beta})\}[\exp\{it(V_n-V^{(1)}_{\alpha,\beta}-V^{(0)}_{\alpha,\beta})\}-1] \\
&\quad+\sum_{1\leqq\alpha<\beta\leqq n}E\zeta_{\alpha,\beta}Z^{(0)}_{\alpha,\beta}\exp\{it(T^{(1)}_{\alpha,\beta}+V^{(k)}_{\alpha,\beta})[\exp\{it(V^{(1)}_{\alpha,\beta}-V^{(k)}_{\alpha,\beta})\}-1] \\
&\quad+\sum_{r=2}^{k-1}\sum_{1\leqq\alpha<\beta\leqq n}E\zeta_{\alpha,\beta}\prod_{j=0}^{r-1}Z^{(j)}_{\alpha,\beta}\exp\{it(T^{(r+1)}_{\alpha,\beta}+V^{(k)}_{\alpha,\beta})\} \\
&\quad+\sum_{1\leqq\alpha<\beta\leqq n}E\zeta_{\alpha,\beta}\prod_{j=0}^{k-1}Z^{(k)}_{\alpha,\beta}\exp\{it(T^{(k)}_{\alpha,\beta}+V^{(k)}_{\alpha,\beta})\} \\
&\quad+\sum_{1\leqq\alpha<\beta\leqq n}E\zeta_{\alpha,\beta}Z^{(0)}_{\alpha,\beta}\exp\{it(T^{(2)}_{\alpha,\beta}+V^{(k)}_{\alpha,\beta})\}
\end{aligned} \tag{3.43}$$

$$+\sum_{1\leq\alpha<\beta\leq n} E\zeta_{\alpha,\beta}\exp\{it(T^{(1)}_{\alpha,\beta}+V^{(1)}_{\alpha,\beta})\}[\exp\{it(V_n-V^{(1)}_{\alpha,\beta}-V^{(0)}_{\alpha,\beta})\}-1]$$

$$\left.+\sum_{1\leq\alpha<\beta\leq n} E\zeta_{\alpha,\beta}\exp\{it(T^{(1)}_{\alpha,\beta}+V^{(1)}_{\alpha,\beta})\}\right].$$

So, by the method used in the proof of Proposition 3.1 (using Lemmas 3.5–3.8 instead of Lemmas 3.1–3.4 and noting (2.18)) we have

$$|EV_n e^{itS_n}-ita_n\sum_{1\leq\alpha<\beta\leq n}(E\zeta_{\alpha,\beta}T_n)f_n(t)| \tag{3.44}$$
$$\leq K[\{n^{-1}+|t|^{1+\delta}n^{-\delta/2}m^{1+\delta}\}|f_n(t)|+(t^2n^{-1}m)^{(1+\delta)/2(2+\delta)}m^{1/2}$$
$$+|t|^{\delta/(2+\delta)}n^{-(1+\delta)/(2+\delta)}m^{\delta/2(2+\delta)}].$$

Hence, (3.32) follows from (3.42)–(3.44).

Proof of Theorem 1. In what follows, we denote by $\theta_j(t)$ some function such that $|\theta_j(u)|\leq 1$ and by C_j an absolute constant.

Let $m=[(4(2+\delta)/\lambda\delta)\log n]$ and $k=[(4/3)\log n/\log\log n]$ where $[s]$ denotes the largest integer not exceeding s. Then, by Propositions 3.1 and 3.2 we can rewrite (3.2) as

$$f_n'(t)=\{-t+h_n(t)\}f_n(t)+Q_n(t) \tag{3.45}$$

for all t ($|t|<K_1n^{1/2}m^{-3/4}$) where

$$h_n(t)=C_1\theta_1(t)|t|^{1+\delta}n^{-\delta/2}m^{1+\delta}+C_2\theta_2(t)|t|n^{-1/2}m^{1/2}+C_3\theta_3(t)n^{-1}$$

and

$$Q_n(t)=C_4\theta_4(t)\{(t^2n^{-1}m)^{(1+\delta)/(2+\delta)}m^{(1+\delta)/2}+|t|^{\delta/(2+\delta)}n^{-(1+\delta)/(2+\delta)}m^{\delta/2(2+\delta)}\}+C_5\theta_5(t)n^{-2}.$$

Solving the differential equation (3.45) we have

$$f_n(t)=\exp\left\{-\frac{t^2}{2}+\int_0^t h_n(z)dz\right\}\left[1+\int_0^t Q_n(u)\exp\left\{\frac{u^2}{2}-\int_0^u h_n(z)dz\right\}du\right] \tag{3.46}$$

for all t ($|t|<K_1n^{1/2}m^{-3/4}$).

Since there exists a constant K_8 such that for all t ($K_8n^{-1/2}<|t|<K_1n^{1/2}m^{-3/4}$)

$$\left|\mathrm{Re}\left(\int_0^t h_n(z)dz\right)\right|\leq\frac{t^2}{4},$$

so

$$\left|\exp\left(-\frac{t^2}{2}+\int_0^t h_n(z)dz\right)\right|\leq e^{-t^2/4}. \tag{3.47}$$

Further, if $|u|\leq|t|$

$$\left|\int_u^t \theta_1(z)|z|^{1+\delta}dz\right|\leq|t|^\delta\int_{|u|}^{|t|}|z|dz\leq|t|^\delta(t^2-u^2)$$

and if $n^{-1/2} \leqq |u| \leqq |t|$

$$\left| \int_u^t \theta_3(z) n^{-1} dz \right| \leqq n^{-1} \int_{|u|}^{|t|} dz = n^{-1}(|t| - |u|) \leqq n^{-1/2}(t^2 - u^2) .$$

Thus, there exist positive constants K_4 and K_5 such that for all t and u ($K_4 n^{-1/2} \leqq |u| \leqq |t| \leqq K_5 n^{1/2} m^{-(1+\delta)/\delta}$)

$$\left| \int_u^t h_n(z)\, dz \right| \leqq \frac{1}{4}(t^2 - u^2) ,$$

which implies

$$\operatorname{Re}\left\{ -\frac{t^2}{2} + \frac{u^2}{2} + \int_u^t h_n(z) dz \right\} \leqq -\frac{t^2}{2} + \frac{u^2}{2} + \frac{1}{4}(t^2 - u^2) = -\frac{t^2}{4} + \frac{u^2}{4} . \tag{3.48}$$

Hence, if $K_4 n^{-1/2} \leqq |t| \leqq K_5 n^{1/2} m^{-(1+\delta)/\delta}$, then by (3.47) and (3.48)

$$\begin{aligned} B_1(t) &= \left| \exp\left\{ -\frac{t^2}{2} + \int_0^t h_n(z) dz \right\} \right| \left| \int_0^t Q_n(u) \exp\left\{ \frac{u^2}{2} - \int_0^u h_n(z) dz \right\} du \right| \\ &\leqq K e^{-t^2/4} \int_0^{K_4 n^{-1\ 2}} n^{-1} du + \int_{K_4 n^{-1/2}}^{|t|} |Q_n(u)| \left| \exp\left\{ -\frac{t^2}{2} + \frac{u^2}{2} + \int_u^t h_n(z) dz \right\} \right| du \\ &\leqq K \Big[n^{-3/2} e^{-t^2/4} + \int_0^t \{ n^{-(1+\delta)/(2+\delta)} m^{(4+3\delta)/2(2+\delta)} (|u|^{2(1+\delta)/(2+\delta)} + |u|^{\delta/(2+\delta)} + n^{-2}) \} \\ &\qquad \times e^{-t^2/4 + u^2/4} du \Big] \\ &\leqq K [n^{-3/2} e^{-t^2/4} + n^{-(1+\delta)/(2+\delta)} m^{(4+3\delta)/2(2+\delta)} \min(|t|^{\delta/(2+\delta)}, |t|^{(4+3\delta)/(2+\delta)}) + n^{-2}|t|] , \end{aligned}$$

since

$$\left| \int_0^t u^\gamma e^{u^2/4} du \right| = K\theta(t) e^{t^2/4} \min(|t|^{\gamma+1}, |t|^{\gamma-1}) \quad (\gamma > 0) .$$

On the other hand, if $0 < |t| < K_4 n^{-1/2}$, then

$$B_1(t) = K n^{-1} m^{(1+\delta)/\delta} |t| . \tag{3.49}$$

Finally, we remark that there exists a constant K_6 such that for all t ($|t| < K_6 n^{1/2} m^{-(1+\delta)/\delta}$)

$$\left| \int_0^t h_n(z) dz \right| \leqq C_6 |t|^{2+\delta} n^{-\delta/2} m^{1+\delta} + C_7 |t|^2 n^{-1/2} m + C_8 |t| n^{-1} \leqq \frac{1}{6} t^2 + C_8 |t| n^{-1} .$$

Thus, by (2.26) we have that for all t ($|t| < K_6 n^{1/2} m^{-(1+\delta)/\delta}$)

$$\begin{aligned} A(t) &= \left| \exp\left(-\frac{t^2}{2} \right) - \exp\left(-\frac{t^2}{2} + \int_0^t h_n(z) dz \right) \right| \\ &\leqq e^{-t^2/2} \left| \exp\left(\int_0^t h_n(z) dz \right) - 1 \right| \end{aligned} \tag{3.50}$$

$$\leqq e^{-t^2/2}\left|\int_0^t h_n(z)dz\right|\exp\left|\int_0^t h_n(z)dz\right|$$
$$\leqq Ke^{-t^2/2}\{|t|^{2+\delta}n^{-\delta/2}m^{1+\delta}+|t|^2n^{-1}m^{1/2}+|t|n^{-1}\}\exp\left\{\frac{t^2}{6}+C_8\frac{|t|}{n}\right\}$$
$$\leqq K\{|t|^{2+\delta}n^{-\delta/2}m^{1+\delta}+|t|^2n^{-1}m^{1/2}+|t|n^{-1}\}\exp\left\{-\frac{t^2}{3}+C_8\frac{|t|}{n}\right\}.$$

Now, let

$$L=\min(K_5, K_6)n^{1/2}m^{-(1+\delta)/\delta}.$$

Then, by Lemma 2 in [2, p. 297] and (3.46)-(3.50) we have that for δ $(0<\delta\leqq 1)$

$$\Delta_n\leqq\int_0^L\frac{|f_n(t)-e^{-t^2/2}|}{|t|}dt+KL^{-1}$$
$$\leqq\int_0^L\frac{A(t)}{|t|}dt+\left\{\int_0^{K_4n^{-1/2}}+\int_{K_4n^{-1/2}}^L\right\}\frac{B_1(t)}{|t|}dt+KL^{-1}$$
$$\leqq K\Bigg[n^{-\delta/2}m^{1+\delta}+\Bigg\{\int_0^{K_4n^{-1/2}}n^{-1}m^{(1+\delta)/2}dt$$
$$+\int_{K_4n^{-1/2}}^L\frac{1}{|t|}(n^{-3/2}e^{-t^2/4}+n^{-(1+\delta)/(2+\delta)}m^{(4+3\delta)/2(2+\delta)}|t|^{\delta/(2+\delta)}+n^{-2}|t|)dt\Bigg\}\Bigg]+KL^{-1}$$
$$\leqq K\max\{n^{-1/2}m^{1+\delta}, n^{-1/2}m^{(1+\delta)/\delta}\}$$
$$\leqq Kn^{-\delta/2}m^{1+\delta}$$

since $L=O(n^{1/2}m^{-(1+\delta)/\delta})$ and $m=O(\log n)$. Thus, the proof is completed.

4. The Berry-Esseen theorems for U-statistics of order 2-(II)

Theorem 2. *Let $\{\xi_i\}$ be a strictly stationary, absolutely regular sequence with mixing coefficient $\beta(n)$. Suppose Condition* (C) *holds with $r=2+\delta$ $(0<\delta\leqq 1)$ and $d=2$. If $\beta(n)=O(n^{-4(2+\delta)(1+\delta)(\gamma+1)/\delta^2})$ for some $\gamma>1$, then*

$$\Delta_n\leqq Kn^{-(\delta/2)((\gamma-1)/(\gamma+1))}. \tag{4.1}$$

Proof. Let $m=[n^{\delta/(1+\delta)(\gamma+1)}]$ and $k=[5(1+\delta)(\gamma+1)/\delta]$. Then it is obvious that $\{\beta(m)\}^{\delta/(2+\delta)}=O(n^{-4})$ and $(m^{1/4})^k=O(n^{-1})$. So, from Propositions 3.1 and 3.2 (4.1) follows.

The following theorem is a generalization of Callaert-Janssen's results in [1].

Theorem 3. *Let $\{\xi_i\}$ be a strictly stationary sequence of m-dependent random variables. Suppose Condition* (C) *holds with $r=2+\delta$ $(0<\delta\leqq 1)$ and $d=2$. If $\sigma_0^2>0$, then*

$$\Delta_n\leqq Kn^{-\delta/2}, \tag{4.2}$$

where

$$\sigma_0^2=\{Eg_1^2(\xi_1)-\theta^2(F)\}+2\sum_{j=1}^{m-1}(Eg_1(\xi_1)g_1(\xi_{j+1})-\theta^2(F)) .$$

The proof is obvious from the proof of Theorem 1 in the preceding section. The following theorem is a generalization of Takahata's result in [3].

Theorem 4. *Let $\{\xi_i\}$ be a strictly stationary, absolutely regular sequence with mixing coefficient $\beta(n)$. Suppose Condition* (C) *holds with $r=3+\gamma$ ($\gamma>0$) and $d=2$. If $\beta(n)=O(e^{-\lambda n})$ for some $\lambda>0$, then*

(4.3) $$\Delta_n\leqq Kn^{-1/2}\log n$$

and for all x

(4.4) $$\left|P_n\left(\frac{n^{1/2}(U_n-\theta(F))}{2\sigma}<x\right)-\Phi(x)\right|\leqq\frac{K\log^3 n}{\sqrt{n}(1+|x|^3)} .$$

Proof. Let $T_\alpha^{(j)}$ and $T_{\alpha,\beta}^{(j)}$ be the ones defined in the preceding section. Then by Lemma 2.4

$$\|T_\alpha^{(0)}-T_\alpha^{(1)}\|_3^2\leqq Kn^{-1}m$$

and

$$\|T_{\alpha,\beta}^{(0)}-T_{\alpha,\beta}^{(1)}\|_3^2\leqq Kn^{-1}m .$$

So, corresponding to Lemmas 3.4 and 3.8 the following holds; under the conditions of Theorem 4

$$\left|\frac{1}{\sigma\sqrt{n}}\sum_{\alpha=1}^{n}\eta_\alpha Z_\alpha^{(0)}-it\right|\leqq K\{|t|n^{-1}m+t^2n^{-1/2}m\}$$

and

$$|a_n\sum_{1\leqq\alpha<\beta\leqq n}E\zeta_{\alpha,\beta}Z_{\alpha,\beta}^{(0)}|\leqq K\{|t|n^{-1/2}m+t^2n^{-1/2}m\} .$$

Hence, (4.3) follows from the method of the proof of Theorem 1. (4.4) is obvious by (4.3) and the proof is completed.

5. Remarks.

Remark 5.1. Results in Sections 3 and 4 may be extended to the general U-statistics of order d ($\geqq 2$) using the same technique of proofs and Lemma 2.3.

Remark 5.2. For every c ($1\leqq c\leqq d$) let

$$V_n^{(c)}=\int\cdots\int g_c(x_1,\cdots,x_c)\prod_{j=1}^{c}d(F_n(x_j)-F(x_j))$$

where F_n is the empirical distribution. Then a von Mises' differentiable statistical functional $\theta(F_n)$ may be written as

$$\theta(F_n)=\int\cdots\int g(x_1,\cdots,x_d)dF_n(x_1)\cdots dF_n(x_d)$$
$$=\theta(F)+\sum_{c=1}^{d}\binom{d}{c}V_n^{(c)}\quad(n\geqq 1)\,.$$

Since under the conditions of Lemma 2.3

$$E(V_n^{(c)})=O(n^{-2})\quad(1\leqq c\leqq d)$$

(see, Lemma 4 in [6]), so we can prove the analogous results to theorems in Sections 3 and 4 by the same method.

Remark 5.3. By the way, we remark that Theorem 4 in [3] is modified slightly as follows:

Let $\{X_j\}$ be a strictly stationary, strong mixing sequence of random variables with $EX_i=0$ and $E|X_i|^{3+\varepsilon}<\infty$ for some $\varepsilon>0$. Suppose $\alpha(n)=O(e^{-\lambda n})$ for some $\lambda>0$. Then

$$\sup\left|P\left(\frac{1}{\sigma_n}\sum_{j=1}^{n}X_j<x\right)-\Phi(x)\right|\leqq Kn^{-1/2}\log n$$

where

$$\sigma_n^2=E\left(\sum_{j=1}^{n}X_j\right)^2\uparrow\infty\quad\text{as}\quad n\to\infty\,.$$

To prove this, it is only enough to remark that by Lemma 2.4

$$b_m^3=E\left|\sum_{j=1}^{n}X_j\right|^3\leqq Kn^{3/2}$$

under the assumptions described above. The rest of the proof of Theorem 4 is identical and so is omitted.

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