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FUNCTIONS OF POWER SERIES

By

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1. Introduction

Recently, a number of authors investigated the following problem. Let $(a_n)_N \in l^1$ and Φ be a function, analytic in an open region containing the set $A(z) := \{\sum_{n=0}^{\infty} a_n z^n \mid |z| \leqq 1\}$. Then the equation $\Phi(A(z)) = B(z) := \sum_{n=0}^{\infty} b_n z^n$, $|z| \leqq 1$ defines a sequence $(b_n)_N \in l^1$. In many applications, the question arises how one can relate asymptotic properties of $(a_n)_N$ and $(b_n)_N$, given the behaviour of Φ.

In tackling this problem, there are two possible points of view: either asymptotic inequalities (o or O results) or asymptotic equivalence ($\sim$results). As usual, for any two sequences x_n and y_n, $x_n \sim y_n$ means $\lim_{n\to\infty}(x_n/y_n)=1$, if not stated explicitely. The present paper aims at bringing together the most important existing results, most of them published independently, generalise them in a natural way and discuss some applications scattered throughout the literature.

2. Review of results

2.1 *O and o-results*

For some Φ and $(a_n)_N$, let $(b_n)_N$ be defined as in the introduction. The purpose of this section is to unify the known sufficient conditions on the sequence $\alpha = (\alpha_n)_N$ of positive reals such that Theorem A below holds:

Theorem A. *If* $|a_n| = O(\alpha_n)$ $(n\to\infty)$, *then* $|b_n| = O(\alpha_n)$ $(n\to\infty)$.

(i) Rogozin [20], [21] and Borovkov [2], [3, Appendix 3] take $\alpha_n = n^{-\beta} L(n)$ where $\beta > 1$ and L is some slowly varying function (s.v.), i.e. L is a positive Lebesgue measurable function such that $L(tx) \sim L(t)$ $(t\to\infty)$ for all $x>0$. See [7], [22] for more details. The standard way for proving Theorem A is by showing that the set $R(\alpha)$ of all l^1 sequences $(x_n)_N$ for which $|x_n| = O(\alpha_n)$ forms a Banach algebra with respect to convolution. The maximal ideal space for $R(\alpha)$ can be represented as follows: for each $x = (x_n)_N \in R(\alpha)$, the associated maximal ideals correspond to $\sum_{n=0}^{\infty} x_n t^n$, $|t| \leqq 1$ and Theorem A follows.

(ii) In a similar way, Rogozin [21] shows that Theorem A remains valid for sequences α such that

$$\sup_{n\geq 1}\sup_{k\geq n/2}\alpha_k/\alpha_n<\infty\ .$$

(iii) In [17] Grübel complements (ii) by proving Theorem A for sequences α such that α is nonincreasing, $\log\alpha_n=o(n)$ $(n\to\infty)$ and either

$$\limsup_{n\to\infty}\sup_{m\geq 2n}\frac{1}{\alpha_m}\sum_{k=n}^{m-n}\alpha_k\alpha_{m-k}=0 \tag{2.1a}$$

or

$$\sup_{m\geq 0}\frac{1}{\alpha_m}\sum_{k=0}^{m}\alpha_k\alpha_{m-k}<\infty\ . \tag{2.1b}$$

(iv) Finally, it is not difficult to see that Theorem A remains valid when $O(\alpha_n)$ is replaced by $o(\alpha_n)$.

2.2 *Asymptotic equivalence*

Below, we list a number of sufficient conditions on the sequence $a=(a_n)_N$ to ensure that

$$b_n\sim a_n\Phi'(A(1))\quad(n\to\infty)\ , \tag{2.2}$$

which turns out to be the natural asymptotic relationship.

(i) Borovkov [3, Appendix 3] defines the set $\mathscr{B}$ as those l^1 sequences $(x_n)_N$ for which $x_{n+1}\sim x_n$ $(n\to\infty)$ and for which there exist positive numbers A, $B(A\leqq B)$ such that

$$0<An^{-\alpha}L(n)\leqq|x_n|\leqq Bn^{-\alpha}L(n)$$

for all $n\geqq 1$ and some $\alpha>1$, L s.v.. Then if $a\in\mathscr{B}$, (2.2) holds.

(ii) Later, Eggermont and Luxemburg [8], [9], [18] defined a set $\mathscr{A}$ of sequences as follows. An l^1 sequence $(x_n)_N\in\mathscr{A}$ if there exists some continuous and positive function L on the positive reals such that $x_n\sim cL(n)$ $(n\to\infty)$ for some $c\in\boldsymbol{R}$ and such that

$$\lim_{x\to\infty}\frac{L(x+t)}{L(x)}=1\qquad\text{for all}\quad t\in\boldsymbol{R}$$

$$\max_{x<t\leqq 2x}\frac{L(t)}{L(2x)}\leqq\lambda_L<\infty\qquad\text{for all}\quad x>0\ .$$

Again (2.2) holds if $a\in\mathscr{A}$.

(iii) To our opinion, the most natural setting for the the problem was given in a paper by Chover, Ney and Wainger [5]. They introduced the class $\mathscr{S}$ of subexponential sequences, taken to be probability measures for convenience. A probability measure $p=(p_n)_N$ belongs to $\mathscr{S}$ if $p_{n+1}\sim p_n$ $(n\to\infty)$ and if

(2.3) $$\lim_{n\to\infty}\frac{p_n^{(2)}}{p_n}=2\ .$$

Here and in the sequel $p^{(k)}$ denotes the k^{th} convolution of p with itself. The name subexponential follows from the trivial property that for $p\in\mathscr{S}$ always $\lim_{n\to\infty} e^{\varepsilon n}p_n=\infty$, for all $\varepsilon>0$. Related properties of $\mathscr{S}$ can be found in [5], [6], [10] and [11].

The main theorem in [5] is that (2.2) still holds for $a\in\mathscr{S}$. Note however that compared to (2.1b), the stronger assumption (2.3) gives the stronger result (2.2).

(iv) A closer examination of the proof of [5, Theorem 1] yields the following theorem, generalising the asymptotic results stated above.

Theorem C. *Suppose $a=(a_n)_N$ belongs to the class $\mathscr{H}$ of sequences satisfying $a\in l^1$,*

(a) $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1\ ;$$

(b) $$\lim_{n\to\infty}\frac{a_n^{(2)}}{a_n}=2\sum_{n=0}^{\infty}a_n=2A(1)\ ;$$

(c) $$\sup_{n\geq 0}\frac{|a_n|^{(2)}}{|a_n|}<\infty\ .$$

Then if Φ is analytic on an open region containing $\{A(z)\mid |z|\leqq 1\}$, there exists a sequence $(b_n)_N\in l^1$ such that $B(z)=\Phi(A(z))$, $|z|\leqq 1$ and such that

$$b_n\sim a_n\Phi'(A(1))\quad (n\to\infty)\ .$$

It is obvious that $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{S}$ are subsets of $\mathscr{H}$.

Some typical examples in $\mathscr{H}$ are

(a) $a_n\sim e^{n^\alpha}$ $(n\to\infty)$, $0<\alpha<1$;

(b) $a_n\sim\exp(-n(\log n)^{-\beta})$ $(n\to\infty)$, $\beta>0$;

(c) all positive sequences satisfying (2.1) and $a_{n+1}\sim a_n$ $(n\to\infty)$.

A final extension which is natural is formulated in the next theorem.

Theorem D. *A sequence $a=(a_n)_N$ belongs to $\mathscr{H}(r)$ (with $r\geqq 1$) if $a\in l^1$ and*

(a) $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=r^{-1}\ ;$$

(b) $$\lim_{n\to\infty}\frac{a_n^{(2)}}{a_n}=2A(r)\ ;$$

(c) $$\sup_{n\geq 0}\frac{|a_n|^{(2)}}{|a_n|}<\infty\ .$$

If Φ is analytic on an open region containing $\{A(z)\mid |z|\leqq r\}$ then there exists a sequence $(b_k)_N$ in l^1 such that $B(z)=\Phi(A(z))$, $|z|\leqq r$ and

$$b_n\sim a_n\Phi'(A(r))\quad (n\to\infty)\ .$$

The proof of Theorem D follows from Theorem C upon taking $a_n'=r^n a_n$. In some way, these extensions are best possible, however we shall not go into further detail on this, the interested reader is referred to [10].

3. Applications.

The classes of regularly varying and subexponential sequences have been used in various problems in probability theory, for instance:

—renewal theory: Rogozin [20, 21], Greenwood—Omey—Teugels [16], Grübel [17], Frenk [15]

—queueing theory: Borovkov [3]

—fluctuation theory of random walks: Embrechts—Hawkes [10, Theorem 3], Greenwood—Omey—Teugels [16]

—discrete infinite divisibility: Embrechts—Hawkes [10, Theorem 1]

—network theory: Brown—Pollett [4, Lemma 5].

3.1 *Discrete infinite divisible (i.d.) probability measures.*

In [10], the authors examine the relation between the asymptotic behaviour of an i.d. probability measure $p=(p_n)_N$ and that of its Lévy measure $\alpha=(\alpha_n)_{N_0}$, i.e. the measure such that

$$\sum_{n=0}^{\infty}p_n z^n=\exp\{-\lambda(1-\sum_{n=1}^{\infty}\alpha_n z^n)\}\ .$$

The following theorem holds:

Theorem 3.1 [10]. *The following three conditions are equivalent*:

(i) $p\in\mathscr{S}$;

(ii) $\alpha\in\mathscr{S}$;

(iii) $p_n\sim\lambda\alpha_n$ $(n\to\infty)$ *and* $p_{n+1}\sim p_n$ $(n\to\infty)$.

Going from (i) to (ii) one has $\Phi(z)=e^{-\lambda(1-z)}$ hence Φ is an entire function and therefore Theorem C is relevant. We refer the reader to the above mentioned paper for further details and related references.

3.2 *Renewal theory*

In this paragraph we shall apply the results of paragraph 2.2 to renewal theory. Let X be a random variable concentrated on the nonnegative integers, and let $p_n=P\{X=n\}$, $n\geqq 0$. We suppose that the distribution of X is aperiodic, i.e. $\gcd\{n \mid p_n>0\}=1$. Further assume X has a positive, finite mean μ. The renewal squence $(u_n)_N$ associated with $(p_n)_N$ can be defined as

$$u_n=\sum_{k=0}^{\infty} p_n^{(k)}\,, \quad n\geqq 0$$

where $(p_n^{(0)})_N$ denotes unit mass at 0. The classical renewal theorem states that (see [14])

$$\lim_{n\to\infty} u_n=\frac{1}{\mu} \tag{3.1}$$

Over the recent years, a lot of papers have been written on the speed of convergence in (3.1). See for instance Stone—Wainger [24], Grübel [17], Frenk [15] and many others. For instance, Grübel [17, p. 118] states that for $\gamma>0$,

$$E(X^{1+\gamma})<\infty \Leftrightarrow \sum_{n=1}^{\infty} n^{\gamma}\,|u_n-u_{n-1}|<\infty\,,$$

hence both imply

$$\sum_{n=1}^{\infty} n^{\gamma-1}\left|u_n-\frac{1}{\mu}\right|<\infty\,.$$

The theory of paragraph 2.2 enables us to get asymptotic estimates involving $\sim$ rather than o or O estimates. For this, let $r_n=(1/\mu)\sum_{k=n+1}^{\infty} p_k$ the associated equilibrium measure. As always $C(z)$ denotes the generating function of a complex sequence $(c_n)_N$. Since $\mu=\sum np_n$ is finite,

$$\lim_{z\uparrow 1} R(z)=\lim_{z\uparrow 1}\frac{1-P(z)}{\mu(1-z)}=1\,.$$

Also for $|z|\leqq 1$ and $z\neq 1$, $1-P(z)\neq 0$ and $R(1)=1\neq 0$. By Wiener's theorem [3, p. 258] it follows that for $|z|\leqq 1$,

$$\frac{1}{R(z)}=\sum_{n=0}^{\infty}\lambda_n z^n \quad \text{with} \quad \sum_{n=0}^{\infty}|\lambda_n|<\infty\,. \tag{3.2}$$

(At this point we essentially needed $\mu<\infty$). Hence $\sum\lambda_n=1$. Furthermore, since $U(z)=(1-P(z))^{-1}$, it is clear that

$$\frac{1}{R(z)}=\mu(1-z)U(z)$$
$$=\mu\left(1+\sum_{n=1}^{\infty}(u_n-u_{n-1})z^n\right).$$

So

(3.3) $$\lambda_n=\mu(u_n-u_{n-1}), \quad n\geqq 1,$$

(3.4) $$u_n-\frac{1}{\mu}=-\frac{1}{\mu}\sum_{k=n+1}^{\infty}\lambda_k, \quad n\geqq 0.$$

Using these identities, we can prove the following result.

Theorem 3.2. *The three following statements are equivalent as* $n\to\infty$:

(i) $$r_n^{(2)}\sim 2r_n;$$

(ii) $$(u_{n-1}-u_n)_N\in\mathscr{H};$$

(iii) $$u_{n-1}-u_n\sim\frac{1}{\mu}r_n \quad \textit{and} \quad r_{n+1}\sim r_n.$$

Either of them implies

(iv) $$u_n-\frac{1}{\mu}\sim\frac{1}{\mu}\sum_{k=n+1}^{\infty}r_k.$$

Proof.

Part 1: (i) implies (iii) and (iv). Since $(r_n)_N$ is monotone, we certainly have $\liminf_{n\to\infty} r_{n-1}/r_n\geqq 1$. From this, (i) and [10, Lemma 2] it follows that $r\in\mathscr{S}$. Hence using Theorem C in (3.2) with $\Phi(z)=z^{-1}$ it follows that

(3.5) $$r_n\sim-\lambda_n \quad (n\to\infty).$$

Using (3.3) and (3.4), (iii) and (iv) follow from (3.5).

Part 2: (iii) implies (i).

For $N>0$ fixed we can write

$$r_n^{(2)}=\left(\sum_{k=0}^{N}+\sum_{k=N+1}^{n}\right)r_{n-k}r_k\equiv I_1+I_2.$$

Use $r_{n+1}\sim r_n$ $(n\to\infty)$ and dominated convergence to see that

(3.6) $$\lim_{n\to\infty}\frac{I_1}{r_n}=\sum_{k=0}^{N}r_k.$$

To handle I_2, first observe that (3.2) implies that

(3.7) $$\sum_{k=0}^{n} \lambda_k r_{n-k}=0\,, \quad n\geqq 1\,.$$

Using (3.5) we have that for $\varepsilon>0$ and $N\geqq N_0(\varepsilon)$,

$$-(1-\varepsilon)\sum_{k=N+1}^{n} \lambda_k r_{n-k} \leqq I_2 \leqq -(1+\varepsilon)\sum_{k=N+1}^{n} \lambda_k r_{n-k}\,.$$

Hence by (3.7)

$$(1-\varepsilon)\sum_{k=0}^{N} \lambda_k r_{n-k} \leqq I_2 \leqq (1+\varepsilon)\sum_{k=0}^{N} \lambda_k r_{n-k}\,.$$

Again using $r_{n+1}\sim r_n$ and dominated convergence, we obtain:

(3.8) $$(1-\varepsilon)\sum_{k=0}^{N} \lambda_k \leqq \liminf_{n\to\infty} \frac{I_2}{r_n} \leqq \limsup_{n=\infty} \frac{I_2}{r_n} \leqq (1+\varepsilon)\sum_{k=0}^{N} \lambda_k\,.$$

Now combine (3.6) and (3.8). First let $N\to\infty$ and then $\varepsilon\downarrow 0$ to obtain (i).

Part 3: (i) implies (ii).

Since (i) implies (iii) and $(r_n)_N\in\mathscr{S}$, it follows that $v_n\equiv u_n-u_{n-1}$ satisfies $v_{n+1}\sim v_n$ $(n\to\infty)$ and that v_n is positive for n large. To prove (ii), it therefore remains to show that

(3.9) $$v_n^{(2)}\sim\left(2\sum_{k=0}^{\infty} v_k\right)v_n \quad (n\to\infty)\,.$$

To this end, for some N large enough, we write

$$v_n^{(2)}=\left(2\sum_{k=0}^{N}+\sum_{k=N+1}^{n-N-1}\right)v_{n-k}v_k\equiv I_1+I_2\,.$$

Using $v_{n+1}\sim v_n$ $(n\to\infty)$ and dominated convergence it follows that:

(3.10) $$\lim_{n\to\infty}\frac{I_1}{v_n}=2\sum_{k=0}^{N} v_k\,.$$

To handle I_2, use (iii) to see that for $\varepsilon>0$ and $N\geqq N_0(\varepsilon)$,

$$\frac{1}{\mu^2}(1-\varepsilon)\sum_{k=N+1}^{n-N-1} r_{n-k}r_k \leqq I_2 \leqq \frac{1}{\mu^2}(1+\varepsilon)\sum_{k=N+1}^{n-N-1} r_{n-k}r_k\,.$$

Now taking (i) into account, we have:

$$\lim_{n\to\infty}\frac{\sum_{k=N+1}^{n-N-1} r_{n-k}r_k}{r_n}=2\sum_{k=N+1}^{\infty} r_k\,.$$

Hence

(3.11)
$$\frac{1}{\mu^2}(1-\varepsilon)2\sum_{k=N+1}^{\infty} r_k \leqq \liminf_{n\to\infty} \frac{I_2}{r_n}$$
$$\leqq \limsup_{n\to\infty} \frac{I_2}{r_n} \leqq \frac{1}{\mu^2}(1+\varepsilon)2\sum_{k=N+1}^{\infty} r_k$$

Now combine (3.10) and (3.11). Let first $N\to\infty$ and then $\varepsilon\downarrow 0$, hence (3.9) holds.

Part 4: (ii) implies (i).
Use Theorem D to see that (ii) implies (iii). The arguments used in Part 3 now also apply to prove (i).

Remarks.

a) For arbitrary $r \geqq 1$, the above theorem can be reformulated as follows. Equivalent are, as $n\to\infty$

(i) $$(r_n)_N \in \mathscr{H}(r)\ ;$$

(ii) $$u_{n-1}-u_n \sim \frac{1}{\mu R^2(r)} r_n \quad \text{and} \quad r_{n+1} \sim \frac{1}{r} r_n\ .$$

Either of them implies

(iii) $$u_n - \frac{1}{\mu} \sim \frac{1}{\mu R^2(r)} \sum_{k=n+1}^{\infty} r_k\ .$$

Hence our theorem provides estimates for $u_{n-1}-u_n$ and $u_n-(1/\mu)$ directly in terms of the tail of the underlying probability measure.

b) Theorem 3.2 should be compared with [24, Theorem 1] and [17, p. 30]. Indeed, in [24] it is proved that, for some $0<\rho<1$:

$$E(X \exp X^\rho) < \infty \ \Rightarrow\ u_n - \frac{1}{\mu} = o(\exp(-n^\rho))\ .$$

This result was improved, using Banach algebra methods in [17] to:

$$p\{X>n\} = o(n^{-1}\exp(-n^\rho)) \ \Rightarrow\ u_n - \frac{1}{\mu} = o(n^{-\rho}\exp(-n^\rho))\ .$$

If we strengthen the above conditions to say $p\{X=n\} \sim cn^{-\beta}\exp(-n^\rho)$ where $\beta>2$ and $o<\rho<1$ then it follows from Theorem 3.2. that

$$u_n - \frac{1}{\mu} \sim c' n^{-\rho+1-\beta}\exp(-n^\rho) \quad (n\to\infty)\ .$$

This situation is typical.

Theorem 3.2 also complements well known results of Rogozin [20], Stone [23], Ney [19] and Frenk [15, Theorem 3.1.6. and 3.1.7., see also Remark p. 134].

There remains the question to what extent a converse of Theorem 3.2 (i) implies (iv) holds. To this end, we need an extra definition.

Given a slowly varying function L, a sequence of (eventually) positive reals $(a_n)_N$ belongs to $\Pi(L)$ if the function $a(x)=a_{[x]}$ satisfies for $x>0$,

$$\lim_{t\to\infty}\frac{a(tx)-a(t)}{L(t)}=\log x\,.$$

Regularly varying and Π-varying sequences have shown to be useful in all sorts of asymptotic problems. Relevant references are Bojanic-Seneta [1] and de Haan [7]. The following theorem has partly been proved in [16, Lemma 4.1] and Frenk [15, Theorems 1.7 and 3.3.6.].

Theorem 3.3. *For any $\alpha\geqq 1$, and any slowly varying function L, the following statements are equivalent* $(n\to\infty)$:

(i) $$r_n\sim n^{-\alpha}L(n)\,;$$

(ii) $$u_{n+1}-u_n\sim\frac{1}{\mu}n^{-\alpha}L(n)\,;$$

(iii) a) *if* $\alpha>1$, $$u_n-\frac{1}{\mu}\sim\frac{1}{\mu(\alpha-1)}n^{1-\alpha}L(n)\,;$$

b) *if* $\alpha=1$, $$\left(u_n-\frac{1}{\mu}\right)_N\in\Pi(L)\,.$$

Proof. (i)⇒(ii)⇒(iii) follows easily from Theorem 3.1. as every regularly varying sequence is subexponential. It therefore only remains to show that (iii) implies (i). The proof for $1<\alpha<2$ follows from a more general result in [11]. There we study the asymptotic behaviour of sequences $(c_n)_N$, $(b_n)_N$ which are related by

$$c_n=\sum_{k=0}^{n}u_kb_{n-k}\,.$$

Since $\sum_{k=0}^{n}(u_k-1/\mu)=\sum_{k=0}^{n}u_{n-k}\sum_{m=k}^{\infty}r_m$, [11, Corollary 2] applies. If $\alpha=1$, or $\alpha=2$, the proof can be found in Frenk [15]. Finally, whenever $\alpha>2$, it follows from (iii) and (3.4) that $\lambda_n=o(n^{1-\alpha}L(n))$, $(n\to\infty)$. Moreover, from (3.2) and Theorem A it follows that

$$r_n=o(n^{1-\alpha}L(n))\,,\quad n\to\infty\,. \tag{3.12}$$

Use generating functions to prove that

(3.13) $$\sum_{m=n+1}^{\infty} r_m = \mu \sum_{m=0}^{n} \left(u_{n-m} - \frac{1}{\mu}\right) r_m .$$

It is not difficult to prove from (iii), (3.12) and (3.13) that

$$\sum_{m=n+1}^{\infty} r_m \sim \frac{n^{1-\alpha}L(n)}{\alpha-1}, \quad n\to\infty .$$

Since $(r_n)_N$ is monotone, (i) follows.

This result should be compared with the following corollary to Theorem A:

for all $\alpha>1$,

$$u_{n+1}-u_n = o(n^{-\alpha}L(n)) \Leftrightarrow r_n = o(n^{-\alpha}L(n))$$

and

$$|u_{n+1}-u_n| = O(n^{-\alpha}L(n)) \Leftrightarrow r_n = O(n^{-\alpha}L(n)) .$$

See also [17, Korollar 1.22].

3.3 *Harmonic renewal measures*

Suppose $(p_n)_N$ is a probability measure, its generating function $P(z)$ can always be written as

(3.14) $$1-P(z) = \exp(-G(z)), \quad |z| \leqq 1$$

where $G(z)$ is the generating function of a sequence $(g_n)_N$ defined by

(3.15) $$g_n = \sum_{m=1}^{\infty} \frac{1}{m} p_n^{(m)} \quad (n \geqq 1) .$$

Functions of the form $\sum_{m=0}^{\infty} a_m p_n^{(m)}$ are called generalized renewal sequences. If $a_m=1$ for all $m\geqq 0$ we have $g_n=u_n$, the renewal sequence. If $a_0=0$ and $a_m=m^{-1}$ for $m\geqq 1$ we get the harmonic renewal sequence (3.15).

Lemma 3.2. *For any probability measure $(p_n)_N$ with finite mean μ,*

$$\sum_{m=1}^{\infty} \left(\frac{1}{m} - g_m\right) = \log \mu .$$

Proof. From (3.14) it follows that

$$\lim_{s\uparrow 1} \sum_{n=1}^{\infty} \left(\frac{1}{n} - g_n\right) s^n = \log \mu .$$

Since $1-ng_n = 1-\sum_{m=1}^{n} m p_m u_{n-m}$ it is easy to prove that $\lim_{n\to\infty}(1-ng_n)=0$. Hence applying the Tauberian theorem for generating functions (Feller [14, p. 473]), the lemma is proved.

In the sequel, we are concerned with the problem of linking the asymptotic behaviour of $(p_n)_N$ to that of $(1/n-g_n)_{N_0}$. We shall assume that $\mu>1$.

From (3.14) it follows that

(3.16) $$R(z)=\exp(-\lambda(1-Q(z))$$

were $r=(r_n)_N$ is as before, $\lambda=\log\mu$ and $q=(q_n)_{N_0}=((1/\lambda)(1/n-g_n))_{N_0}$. Note that if r is i.d. we can identify its Lévy measure with the measure q. In this case, the relevant theorem on the asymptotic behaviour of these two measures was stated in section 3.1. When r is not necessarily i.d. the result remains valid in the following form.

Theorem 3.3. *The following statements are equivalent*

(i) $$r\in\mathscr{S};$$

(ii) $$\frac{1}{n}-g_n\sim r_n \quad \text{and} \quad r_{n+1}\sim r_n \quad \text{as} \quad n\to\infty .$$

Proof. The proof goes along the same lines as that of Theorem 1 in [10]. Note that (ii) implies that $(1/n)-g_n$ is positive for n large.

The special case where $(q_n)_{N_0}$ is regularly varying was treated in [16], where also applications to random walk theory are discussed. See also [11, Theorem 3] for the random walk analogue of theorem 3.3 above.

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Note Added in Proof. Some recent interesting work on discrete renewal theory is given in R. Grübel, Functions of discrete probability measures: Rates of convergence in the renewal theorem, Z. Wahrscheinlichkeitstheorie verw. Gebiete 64, 341–357, (1983).