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EXPLOSION AND TRANSFORMATION OF DAMPING OF SECOND ORDER ITO PROCESSES

By

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§ 1. Introduction

In this paper we consider the response of the oscillator

(1.1) $$\ddot{y}+g(t, y, \dot{y})\dot{y}+a(t)f(y)=h(t, y, \dot{y})\dot{w}$$

with the damping term g and the restoring force f to the random disturbance of so-called white noise $\dot{w}$, where by $\cdot$ we mean the symbolic derivative d/dt. According to the recent paper [6], the following results are known;

(I) a sufficient condition for the existence of the global solution of (1.1),

(II) a sufficient condition for the non existence of the global solution of (1.1) when g is strictly negative.

But the investigation of the following problems are left undone;

(III) a necessary condition for the non existence of the global solution of (1.1),

(IV) a construction of the non global solution of (1.1) when g is not strictly negative.

Our purpose of this paper is to investigate the above problems (III) and (IV). For (IV) we use the method of the transformation of the damping term g. For (III) we consider the relation between the non existence of global solution of (1.1) and the convergence of the integral

$$\int_0^{\infty}(1+F(u))^{-1/2}du \quad \left(\text{or} \int_0^{-\infty}(1+F(u))^{-1/2}du\right)$$

under the assumption that $uf(u)>0$ for $u \neq 0$, where $F(u)=\int_0^u f(s)\,ds$. This investigation corresponds to an analogue of Burton-Grimmer theorem [1] where the random disturbance is not considered.

For these purposes we treat the two dimensional nonlinear Ito equation and investigate the explosion criteria for the solution of the stochastic differential equation.

Let $(\Omega, \boldsymbol{F}, P)$ be a probability space with an increasing family $\{\boldsymbol{F}_t;\ t \geqq 0\}$ of sub-σ-algebras of $\boldsymbol{F}$ and $w(t)$ be a one dimensional Brownian motion process adapted to $\boldsymbol{F}_t$. Then we consider the stochastic differential equation

$$(1.2)\quad \begin{cases} dX_1(t)=X_2(t)dt\,, \\ dX_2(t)=\{-g(t, X_1(t), X_2(t))X_2(t)-a(t)f(X_1(t))\}dt+h(t, X_1(t), X_2(t))dw(t)\,. \end{cases}$$

Throughout this paper we assume the following conditions; a: $[0, \infty)\to(-\infty, \infty)$ is continuously differentiable, f: $(-\infty, \infty)\to(-\infty, \infty)$ is continuously differentiable, g and h: $[0, \infty)\times(-\infty, \infty)\times(-\infty, \infty)\to(-\infty, \infty)$ have continuous first partials with respect to $t\in[0, \infty)$, $x_1\in(-\infty, \infty)$ and $x_2\in(-\infty, \infty)$.

The system (1.2) is one of the formulations such that $X_1(t)$ may correspond to the response of the harmonic oscillator (1.1) (see [2]). In general, the solution $X(t)=(X_1(t), X_2(t))$ of (1.2) with the initial condition $X(t_0)=x_0\in R^2(t_0\geqq 0)$ is defined up to the random time $e(t_0, x_0)$, where $e(t_0, x_0)=\lim_{n\uparrow\infty} e_n(t_0, t_0)$ and $e_n(t_0, x_0)=n\wedge \inf\{t;\ |X(t)]\geqq n\}$ (here and hereafter $R^2=(-\infty, \infty)\times(-\infty, \infty)$ and $a\wedge b$ stands for the smaller of a and b). This random time $e(t_0, x_0)$ is called *the explosion time* of the solution $X(t)$ of (1.2) with the initial condition $X(t_0)=x_0$. The next remark enables us to understand the meaning of the explosion time (see [3], [4] and [5]).

Remark 1.1. $\lim_{t\uparrow e(t_0,x_0)} |X(t)|=\infty$ for $e(t_0, x_0)<\infty$, almost surely. Hence, if $e(t_0, x_0)<\infty$, then *the explosion occurs.*

The problem introduced at the beginning of this section can be restated in terms of the explosion for (1.2). Namely, we construct explosive solutions of (1.2) where the function $g(t, x_1, x_2)$ is not necessarily negative by the method of the transformation of the function $g(t, x_1, x_2)$ and also we give a necessary condition for the explosive solution of (1.2).

Throughout this paper, we shall use the differential generator

$$(1.3)\qquad L=\frac{\partial}{\partial t}+x_2\frac{\partial}{\partial x_1}-\{g(t, x_1, x_2)x_2+a(t)f(x_1)\}\frac{\partial}{\partial x_2}+\frac{1}{2}h^2(t, x_1, x_2)\frac{\partial^2}{\partial x_2{}^2}$$

associated with the system (1.2). Introduce the function

$$(1.4)\qquad V(t, x)=a(t)F(x_1)+x_2{}^2/2\,,$$

where $t\geqq 0$, $x=(x_1, x_2)\in R^2$ and $F(x_1)=\int_0^{x_1} f(s)ds$. Then we note that

$$(1.5)\qquad LV(t, x)=a'(t)F(x_1)-g(t, x_1, x_2)x_2{}^2+h^2(t, x_1, x_2)/2$$

for all $t\geqq 0$ and $x=(x_1, x_2)\in R^2$. Moreover, let $U(t, x)$ be a scalar function which is twice continuously differentiable with respect to $x\in R^2$ and once with respect to $t\geqq 0$, and let $K(u)$ be a twice continuously differentiable function on $(-\infty, \infty)$.

Then we notice that

$$LK(U(t, x))=(LU(t, x))K'(U(t, x)) + \frac{1}{2}h^2(t, x_1, x_2)\left(\frac{\partial U(t, x)}{\partial x_2}\right)^2 K''(U(t, x)) \tag{1.6}$$

for $t\geqq 0$ and $x=(x_1, x_2)\in R^2$.

We shall use (1.5) and (1.6) in the proof of our theorem.

§ 2. An analogue of Burton-Grimmer theorem

Here we discuss the relation between the finite explosion time and the restriction on the growth of the coefficient f of (1.2). First of all consider the deterministic nonlinear second order differential equation

$$\ddot{y}+a(t)f(y)=0\ , \tag{2.1}$$

where a is continuous, f is continuous and $yf(y)>0$ for $y\neq 0$, and suppose that $a(t)<0$ on an interval $t_0\leqq t<t_1$ with $a(t_1)\leqq 0$. Then it follows from the theorem of Burton and Grimmer [1] that (2.1) has a solution $y(t)$ defined for $t=t_0$ satisfying $\lim_{t\uparrow T}|y(t)|=\infty$ for some $T\in(t_0, t_1]$ if and only if either

$$\int_0^\infty (1+F(u))^{-1/2}du<\infty \tag{2.2}$$

or

$$\int_0^{-\infty} (1+F(u))^{-1/2}du>-\infty \tag{2.3}$$

holds, where $F(u)=\int_0^u f(s)ds$.

For the stochastic case of (1.1), consider the system (1.2), and let $e(t_0, x_0)$ be the the explosion time of the solution $X(t)$ of (1.2) with the initial condition $X(t_0)=x_0\in R^2$. Then we investigate the problem whether one of the conditions (2.2) and (2.3) plays a role on the occurrence of the explosion or does not. A sufficient condition for the finite explosion time is given by [6] as follows.

Theorem 2.1. *Suppose that the following conditions hold;*

(i) $a(t)\leqq -m$ *and* $a'(t)\geqq 0$ *for all* $t\geqq 0$ *with a constant* $m>0$,

(ii) $x_1f(x_1)>0$ *for all* $x_1\neq 0$,

(iii) $g(t, x_1, x_2)\leqq -\varepsilon_0 h^2(t, x_1, x_2)$ *for all* $t\geqq 0$ *and* $(x_1, x_2)\in R^2$ *with a constant* $\varepsilon_0>0$,

(iv) $h^2(t, x_1, x_2)\geqq k(t)$ *for all* $t\geqq 0$ *and* $(x_1, x_2)\in R^2$

with a nonnegative and continuous function $k(t)$ *satisfying*

$$\int_0^\infty k(t)dt=\infty .$$

Further, suppose that both (2.2) *and* (2.3) *hold. Then,* $P(e(t_0, x_0)<\infty)>0$ *for all* $t_0\geqq 0$ *and* $x_0\in R^2$.

In the next theorem we show that one of the conditions (2.2) and (2.3) becomes a necessary condition for the finite explosion time under certain conditions of the other coefficients.

Theorem 2.2. *Suppose that the following conditions hold;*

(i) $-M\leqq a(t)$ *and* $a'(t)\leqq 0$ *for all* $t\geqq 0$ *with a constant* $M>0$,

(ii) $x_1 f(x_1)>0$ *for all* $x_1\neq 0$,

(iii) $g(t,x_1,x_2)\geqq \varepsilon_0 k(t)$ *for all* $t\geqq 0$ *and* $(x_1, x_2)\in R^2$ *with a constant* $\varepsilon_0>0$ *and a continuous function* $k(t)\geqq 0$,

(iv) $h^2(t, x_1, x)\leqq k(t)$ *for all* $t\geqq 0$ *and* $(x_1, x_2)\in R^2$ *with the same function* $k(t)$ *given in* (iii). *Further, suppose that* (1.2) *has a solution* $X(t)=(X_1(t), X_2(t))$ *with the initial condition* $X(t_0)=x_0\in R^2$ *and that the explosion time satisfies*

$$P(e(t_0,x_0)<\infty)>0 .$$

Then, either (2.2) *or* (2.3) *holds.*

Proof. For the preparation of the proof we choose a positive number p and a continuous function $c(t)$ such that

$$0<p\leqq 2\varepsilon_0 \quad \text{and} \quad c(t)\geqq k(t)/2 ,$$

and then we set $K(u)=\exp(pu)$ for $u\in(-\infty, \infty)$ and

$$U(t, x)=V(t, x)-\int_0^t c(s)ds$$

for $t\geqq 0$ and $x\in R^2$ with the function $V(t, x)$ defined by (1.4). Now, let $X(t)=(X_1(t), X_2(t))$ be any solution of (1.2) with the initial condition $X(t_0)=x_0\in R^2$ and let $e(t_0, x_0)$ be its explosion time. Then, under the conditions (i), (ii), (iii) and (iv) we first show that

$$P(\sup_{t_0<t<e} U(t, X(t))<\infty)=1$$

and that

(2.4) $$P\left(X_2^2(t)/2\leqq\alpha+\int_0^t c(s)ds+MF(X_1(t))\quad\text{for all}\quad t_0\leqq t<e\right)=1\ ,$$

where $e=e(t_0, t_0)$, $\alpha=\sup_{t_0\leq t<e} U(t, X(t))$ and M is the constant appearing in (i). In fact, by L we denote the differential generator defined by (1.3). Then we see by (1.5) and (1.6) that

$$\begin{aligned}LK(U(t, x))=&[a'(t)F(x_1)-\{g(t, x_1, x_2)-ph^2(t, x_1, x_2)/2\}x_2^2\\&-\{c(t)-h^2(t, x_1, x_2)/2\}]pK(U(t, x))\\\leqq&-[\{\varepsilon_0-p/2\}x_2^2k(t)+\{c(t)-k(t)/2\}]pK(U(t, x))\end{aligned}$$

for $t\geqq 0$ and $x=(x_1, x_2)\in R^2$, since (i), (ii), (iii) and (iv) hold by the assumption. Accordingly, it follows from the choice of p and $c(t)$ that $LK(U(t, x))\leqq 0$ for all $t\geqq 0$ and $x\in R^2$. Therefore, Ito's formula concerning stochastic differentials impiles that for arbitrarily fixed n

$$\{K(U(t\wedge e_n, X(t\wedge e_n)));\ t\geqq 0\}$$

is a positive super-martingals, where $e_n=n\wedge\inf\{t;\ |X(t)|\geqq n\}$. Hence the super-martingale inequality yields that for any number $N>0$

$$P(\sup_{t_0<t<e_n} K(U(t, X(t)))>N)\leqq N^{-1}K(U(t_0, x_0))$$

Let n tend to infinity in the above equation and notice that $e_n\uparrow e$ for $n\uparrow\infty$, where $e=e(t_0, x_0)$. Then we get that

$$P(\sup_{t_0<t<e} K(U(t, X(t)))>N)\leqq N^{-1}K(U(t_0, x_0))\ ,$$

from which follows

$$P(\sup_{t_0<t<e} K(U(t, X(t)))<\infty)=1\ .$$

Since $K(u)=\exp(pu)$ with $p>0$, we obtain that

$$P(\sup_{t_0<t<e} U(t\ X(t))<\infty)=1\ .$$

Thus, the definition of $U(t, x)$ implies (2.4). Next, let $X(t)=(X_1(t), X_2(t))$ be the solution of (1.2) with the initial condition $X(t_0)=x_0\in R^2$ and suppose that $P(e(t_0, x_0)<\infty)>0$. In the following we take a sample such that $e(t_0, x_0)<\infty$, and for notational simplicity we put $e=e(t_0, x_0)$ once again. Then we show that

(2.5) $$\sup_{t_0<t<e}|X_1(t)|=\infty\quad\text{for}\quad e<\infty\ .$$

In fact, if it were that $\sup_{t_0<t<e}|X_1(t)|<\infty$ for $e<\infty$, then we would have by (2.4) that

$$X_2^2(t)/2\leqq\alpha+\int_0^e c(s)ds+M\bar{F}$$

for all $t_0\leqq t<e$, where

$$\bar{F}=\max\{\ F(\inf_{t_0\leqq t<e}X_1(t))\ ,\quad F(\sup_{t_0\leqq t\leqq e}X_1(t))\ \}\ .$$

Namely both $X_1(t)$ and $X_2(t)$ stay bounded as $t\uparrow e$, which is a contradiction since $|X(e-)|=\infty$ by Remark 1.1. Hence we get (2.5).

Now consider a sample such that $e<\infty$. Then (2.4) implies that

$$X_2^2(t)/2\leqq\beta+MF(X_1(t))$$

for all $t_0\leqq t<e$, where $\beta=\alpha+\int_0^e c(s)ds+b$ with a positive number b chosen so large that $\beta>0$. Accordingly, $X_1(t)$ and $X_2(t)$ satisfy

$$-(2M)^{1/2}\leqq(\delta+F(X_1(t)))^{-1/2}X_2(t)\leqq(2M)^{1/2}$$

for all $t_0\leqq t<e$, where $\delta=\beta/M>0$. Integrate both sides of the above equation from t_0 to t $(<e)$ and notice that $(2M)^{1/2}(t-t_0)\leqq(2M)^{1/2}(e-t_0)$ for all $t_0\leqq t<e$. Then we get that

$$-(2M)^{1/2}(e-t_0)\leqq\int_{X_1(t_0)}^{X_1(t)}(\delta+F(u))^{-1/2}du\leqq(2M)^{1/2}(e-t_0) \tag{2.6}$$

for all $t_0\leqq t<e$, since $dX_1(t)=X_2(t)dt$ and since $dY(X_1(t))=Y'(X_1(t))dX_1(t)$ for

$$Y(x_1)=\int_0^{x_1}(\delta+F(u))^{-1/2}du\ .$$

We notice (2.5) and discuss according to each of two cases;

$$\sup_{t_0<t<e}X_1(t)=\infty\ ,\quad \inf_{t_0<t<e}X_1(t)=-\infty\ .$$

Let us consider that $\sup_{t_0<t<e}X_1(t)=\infty$. Then we take the superior limit as t tends to e in the equation (2.6), so that

$$-(2M)^{1/2}(e-t_0)\leqq\int_{X_1(t_0)}^{\infty}(\delta+F(u))^{-1/2}du\leqq(2M)^{1/2}(e-t_0)\ . \tag{2.7}$$

If $\delta\leqq1$, then we have that

$$(\delta+F(u))^{-1/2}\geqq(1+F(u))^{-1/2}$$

and so (2.2) holds. If $\delta>1$, then we have that

$$\delta^{-1/2}\int_{x_1(t_0)}^{\infty}(1+\delta^{-1}F(u))^{-1/2}du>\delta^{-1/2}\int_{x_1(t_0)}^{\infty}(1+F(u))^{-1/2}du .$$

Since the first integral of the above equation converges by (2.7), so does the second and hence (2.2) holds. Last, let us consider that $\inf_{t_0<t<e} X_1(t)=-\infty$. Then we take the inferior limit as t tends to e in the equation (2.6) and a similar argument can be carried out, showing (2.3). Hence the proof is completed.

Remark 2.1. Suppose that $F(u)\geqq Cu^{\alpha}$ for $u>0$ with some constants $C>0$ and $\alpha>2$. Then $F(u)$ satisfies (2.2).

Remark 2.2. Suppose that $uf(u)>0$ for $u\neq 0$ and that $f^2(u)\geqq\beta F(u)^{\gamma}$ for $u>0$ with some constants $\beta>0$ and $\gamma>1$. Then $F(u)$ satisfies (2.2). In fact, we see that

$$\begin{aligned}\int_1^{\infty}\frac{du}{(1+F(u))^{1/2}}&=\int_1^{\infty}\frac{f(u)du}{f(u)(1+F(u))^{1/2}}\\&\leqq\beta^{-1/2}\int_1^{\infty}\frac{f(u)du}{F(u)^{\gamma/2}(1+F(u))^{1/2}}\\&=\beta^{-1/2}\int_{F(1)}^{F(\infty)}\frac{dv}{v^{\gamma/2}(1+v)^{1/2}}\\&\leqq\beta^{-1/2}\int_{F(1)}^{\infty}\frac{dv}{v^{(\gamma+1)/2}}\\&<\infty .\end{aligned}$$

Remark 2.3. During the last several years a number of exceptionally sharp results regarding uniqueness, continuation, and oscillation of solutions of the nonlinear second order differential equation

$$\ddot{y}+a(t)f(y)=0$$

have been obtained. Our works for (1.1) are influenced by these deterministic cases.

§ 3. Transformation of damping

Here we consider the relation between the explosions of the solutions of the following two systems of the stochastic differential equations;

$$\text{(3.1)}\qquad \begin{cases}dX_1(t)=X_2(t)dt ,\\ dX_2(t)=\{-g(t)X_2(t)-a(t)f(X_1(t))\}dt+h(t)dw(t) ,\end{cases}$$

$$(3.2)\qquad \begin{cases} dY_1(t)=Y_2(t)dt\ , \\ dY_2(t)=-A(t)f(Y_1(t))dt+H(t)d\tilde{w}(t)\ , \end{cases}$$

where $g(t)$, $a(t)$, $f(x_1)$, $h(t)$, $A(t)$ and $H(t)$ are continuously differentiable functions, and $w(t)$ and $\tilde{w}(t)$ are one dimensional Brownian motion processes.

Both $X_1(t)$ and $Y_1(t)$ correspond to the responses of the oscillators

$$(3.1)\qquad \ddot{y}+g(t)\dot{y}+a(t)f(y)=h(t)\dot{w}$$

and

$$(3.2)\qquad \ddot{y}+A(t)f(y)=H(t)\dot{\tilde{w}}$$

to the formal white noise $\dot{w}$ and $\dot{\tilde{w}}$, respectively. Roughly speaking, we will show that

if (3.1) has an explosive solution corresponding to the family $\{g(t)<0, a(t), f(x_1), h(t); w(t)\}$, then (3.2) has an explosive solution corresponding to some family $\{A(t), f(x_1), H(t); \tilde{w}(t)\}$ (Theorem 3.1)

and that

if (3.2) has an explosive solution corresponding to the family $\{A(t), f(x), H(t); \tilde{w}(t)\}$, then (3.1) has an explosive solution corresponding to some family $\{g(t)>0, a(t), f(x_1), h(t), w(t)\}$ (Theorem 3.2).

By these results we can know that there exist explosive solutions of (1.2) with the nonnegative damping function g. Therefore, combining Theorem 2.1 with Theorem 3.1 and Theorem 3.2, we can construct explosive solutions of (1.2) with any damping function g which depends only on the time variable $t\geqq 0$.

For the continuous function $g(t)$ let

$$(3.3)\qquad r(t)=\exp\left(\int_0^t g(u)du\right)$$

and let

$$(3.4)\qquad s(t)=\int_0^t 1/r(u)du=\int_0^t \exp\left(-\int_0^u g(v)\,dv\right)du\ .$$

Then we notice that

$$r(t)>0\ ,\quad r'(t)=g(t)r(t)\quad \text{and}\quad s'(t)=1/r(t)\ .$$

If

$$(3.5)\qquad s(\infty)=\infty\ ,$$

then by $\phi(t)$ we denote the inverse function of $s(t)$, that is $\phi(t)=s^{-1}(t)$. We also notice that $\phi'(t)=r(\phi(t))>0$. By the time substitution rule we transform the damping term g and construct explosive solutions. For the sake of convenience, we consider the solutions of (3.1) and (3.2) with the initial time $t_0=0$.

Theorem 3.1. *Suppose that the following conditions hold;*

(i) $g(t)<0$ *for all* $t\geqq 0$,

$r(t)$ *and* $s(t)$ *are defined accordingly as in* (3.3) *and* (3.4),

(ii) $h(t)\neq 0$ *for all* $t\geqq 0$,

(iii) *there exists a point* $x_0\in R^2$ *such that* $P(e(0, x_0)<\infty)>0$ *holds, where* $e(0, x_0)$ *stands for the explosion time of the solution* $X(t)=(X_1(t), X_2(t))$ *of* (3.1) *with the initial condition* $X(0)=x_0$.

Define $Y(t)=(Y_1(t), Y_2(t))$ *by*

$$Y_1(t)=X_1(\phi(t)) \quad \textit{and} \quad Y_2(t)=X_2(\phi(t))\phi'(t)\ ,$$

where $\phi(t)$ *is the inverse function of* $s(t)$.

Then, $Y(t)$ *is a solution of* (3.2) *with the initial condition* $Y(0)=x_0\in R^2$ *corresponing to the coefficients*

$$A(t)=r^2(\phi(t))a(\phi(t))\ ,\quad H(t)=r^{3/2}(\phi(t))h(\phi(t))\ ,\quad f(x_1)$$

and some Brownian motion process $\tilde{w}(t)$.

Moreover, let $\tilde{e}(0, x_0)$ *be the explosion time of the above process* $Y(t)$ *with the initial condition* $Y(0)=x_0$. *Then,*

$$P(\ e(0, x_0)<\tilde{e}(0, x_0)=s(e(0, x_0))<\infty\ )>0\ .$$

Proof. Let $X(t)=(X_1(t), X_2(t))$ be the solution of (3.1) with the initial condition $X(0)=x_0\in R^2$ such that $P(e(0, x_0)<\infty)>0$. Then, in the following we consider a sample such that $e(0, x_0)<\infty$. For simplicity of the notation, we set $e=e(0, x_0)$. Since $g(t)<0$ by the condition (i), it follows from (3.3) and (3.4) that $r(t)<1$ and $s(t)>t$ for all $t>0$. This implies (3.5), and hence the inverse function $\phi(t)$ of $s(t)$ is well defined for all $t\geqq 0$, satisfying

$$\phi(t)<t \quad \text{for all} \quad t>0\ . \tag{3.6}$$

Set

$$Y_1(t)=X_1(\phi(t)) \quad \text{and} \quad Y_2(t)=X_2(\phi(t))\phi'(t) \quad (\ =X_2(\phi(t))r(\phi(t))\ )$$

for all $0\leqq t<e$. This definition is possible since $X(t)$ is defined for all $0\leqq t<e$ and since (3.6) holds. First we see that

$$Y_1(t)=X_1(0)+\int_0^{\phi(t)} X_2(u)du$$

$$=X_1(0)+\int_0^t X_2(\phi(u))\phi'(u)du$$
$$=X_1(0)+\int_0^t Y_2(u)du$$

for all $0\leqq t<e$. Notice that $dr(t)=g(t)r(t)dt$ and use Ito's formula concerning stochastic differentials. Then, we also see that

$$\begin{aligned}d(X_2(t)r(t))&=X_2(t)dr(t)+r(t)dX_2(t)\\&=X_2(t)g(t)r(t)dt+r(t)[\{-g(t)X_2(t)-a(t)f(X_1(t))\}dt+h(t)dw(t)]\\&=-r(t)a(t)f(X_1(t))dt+r(t)h(t)dw(t)\end{aligned}$$

and hence

$$\begin{aligned}Y_2(t)&=X_2(\phi(t))r(\phi(t))\\&=X_2(0)-\int_0^{\phi(t)} r(u)a(u)f(X_1(u))du+\tilde{M}(t)\\&=X_2(0)-\int_0^t r(\phi(u))a(\phi(u))f(X_1(\phi(u)))\phi'(u)du+\tilde{M}(t)\\&=X_2(0)-\int_0^t r^2(\phi(u))a(\phi(u))f(Y_1(u))du+\tilde{M}(t)\end{aligned}$$

for all $0\leqq t<e$, where $\tilde{M}(t)=M(\phi(t))$ and

$$M(t)=\int_0^t r(u)h(u)dw(u) .$$

Since $r(u)$ and $h(u)$ are continuously differentiable by the assumption, $M(t)$ *is* a continuous and square integrable martingale with its increasing process

$$\langle M\rangle(t)=\int_0^t r^2(u)h^2(u)du .$$

This yields that $\tilde{M}(t)$ is a contintinuous and square integrable martingale with its increasing process $\langle\tilde{M}\rangle(t)$, for which

$$\begin{aligned}\langle\tilde{M}\rangle(t)&=\int_0^{\phi(t)} r^2(u)h^2(u)du\\&=\int_0^t r^2(\phi(u))h^2(\phi(u))\phi'(u)du\\&=\int_0^t r^3(\phi(u))h^2(\phi(u))du .\end{aligned}$$

Set

$$\tilde{w}(t)=\int_0^t \frac{1}{r^{3/2}(\phi(u))h(\phi(u))}d\tilde{M}(u) ,$$

which is well defined since $h(t)\neq 0$ for all $t\geqq 0$ by the condition (ii), so that $\tilde{w}(t)$ is a continuous and square integrable martingale with its increasing process

$$\langle\tilde{w}\rangle(t)=\int_0^t \frac{1}{r^3(\phi(u))\,h^2(\phi(u))}d\langle\tilde{M}\rangle(u)=t .$$

This implies that $\tilde{w}(t)$ is a Brownian motion process and that

$$\tilde{M}(t)=\int_0^t r^{3/2}(\phi(u))h(\phi(u))d\tilde{w}(u) .$$

Therefore, $Y(t)=(Y_1(t),\ Y_2(t))$ satisfies

$$Y_1(t)=X_1(0)+\int_0^t Y_2(u)du ,$$

$$Y_2(t)=X_2(0)-\int_0^t r^2(\phi(u))a(\phi(u))f(Y_1(u))du$$

$$+\int_0^t r^{3/2}(\phi(u))h(\phi(u))d\tilde{w}(u)$$

for all $0\leqq t<e$. Accordingly, $Y(t)$ is a solution of (3.2) with the initial condition $Y(0)=x_0\in R^2$ corresponding to the coefficients

$$A(t)=r^2(\phi(t))a(\phi(t)) ,\quad H(t)=r^{3/2}(\phi(t))h(\phi(t)) ,\quad f(x_1)$$

and the Brownian motion process $\tilde{w}(t)$.

On the other hand, let $\tilde{e}(0,\ x_0)$ be the explosion time of $Y(t)$ with the initial condition $Y(0)=x_0\in R^2$, where $Y(t)$ is given by the preceding definition. Consider that

$$|Y(t)|^2=Y_1^2(t)+Y_2^2(t)=X_1^2(\phi(t))+X_2^2(\phi(t))r^2(\phi(t))$$

and hence

$$|Y(s(t))|^2=X_1^2(t)+X_2^2(t)r^2(t)$$

for all $0\leqq t<e$. Then, since $r(t)<1$ for $t>0$, the above equation yields that

$$|Y(s(t))|^2/r^2(t)=X_1^2(t)/r^2(t)+X_2^2(t)\geqq|X(t)|^2$$

for all $0\leqq t<e$. Let t tend to e on the both sides of the above equation and notice that $\lim_{t\uparrow e}|X(t)|=\infty$ on $e<\infty$ since Remark 1.1 holds. Then we obtain that

$$\lim_{t\uparrow e}|Y(s(t))|=\infty .$$

In fact, if it were that $\liminf_{t\uparrow e} |Y(s(t))|<\infty$, we would have that

$$\begin{aligned}\infty &> \liminf_{t\uparrow e} |Y(s(t))|^2/r^2(e)\\ &=\liminf_{t\uparrow e} |Y(s(t))|^2/r^2(t)\\ &\geqq \liminf_{t\uparrow e} |X(t)|^2\\ &=\infty ,\end{aligned}$$

which is absurd. Therefore, we get that

$$\lim_{t\uparrow s(e)} |Y(t)|=\infty ,$$

where $e=e(0, x_0)$. Combining this with the definition of the explosion time $\tilde{e}(0, x_0)$ of $Y(t)$, we see that $\tilde{e}(0, x_0)\leqq s(e)$. Last, we show that $\tilde{e}(0, x_0)=s(e)$. Assume that $\tilde{e}(0, x_0)<s(e)$ to the contrary. Then we see that $\phi(\tilde{e}(0, x_0))<e$. Since $r(u)<1$ for all>0, we also see that

$$|Y(t)|^2=X_1^2(\phi(t))+X_2^2(\phi(t))r^2(\phi(t))\leqq X_1^2(\phi(t))+X_2^2(\phi(t))$$

and hence

$$|Y(t)|^2\leqq|X(\phi(t))|^2$$

for all $0\leqq t<e$. Let t tend to $\tilde{e}(0, x_0)$ in the both sides of the above equation. Then, Remark 1.1 and the fact that $\phi(\tilde{e}(0, x_0))<e$ yield

$$\infty=\lim_{t\uparrow \tilde{e}(0,x_0)} |Y(t)|^2\leqq|X(\phi(\tilde{e}(0, x_0)))|^2<\infty ,$$

which is a contradiction. Hence we obtain that $\tilde{e}(0, x_0)=s(e)>e$ and the proof is completed.

Theorem 3.2. *Suppose that the following conditions hold;*

(i) $g(t)>0$ *for all* $t\geqq0$,

$r(t)$ *and* $s(t)$ *are defined accordingly as in* (3.3) *and* (3.4), *satisfying* $s(\infty)=\infty$,

(ii) $H(t)\neq0$ *for all* $t\geqq0$,

(iii) *there exists a point* $x_0\in R^2$ *such that* $P(\tilde{e}(0, x_0)<\infty)>0$ *holds, where* $\tilde{e}(0, x_0)$ *stands for the explosion time of the solution* $Y(t)=(Y_1(t), Y_2(t))$ *of* (3.2) *with the initial condition* $Y(0)=x_0$.

Define $X(t)=(X_1(t), X_2(t))$ *by*

$$X_1(t)=Y_1(s(t)) \quad and \quad X_2(t)=Y_2(s((t))/r(t) .$$

Then, $X(t)$ *is a solution of* (3.1) *with the initial condition* $X(0)=x_0\in R^2$ *corresponding to the coefficients*

$$g(t)>0\ ,\quad a(t)=A(s(t))/r^2(t)\ ,\quad h(t)=H(s(t))/r^{3/2}(t)\ ,\quad f(x_1)$$

and some Brownian motion process $w(t)$.

Moreover, let $e(0, x_0)$ *be the explosion time of the above process* $X(t)$ *with the initial condition* $X(0)=x_0$. *Then,*

$$P(\ \tilde{e}(0, x_0)<e(0, x_0)=\phi(\tilde{e}(0, x_0))<\infty\ )>0\ ,$$

where $\phi(t)$ *is the inverse function of* $s(t)$.

Proof. Let $Y(t)=(Y_1(t), Y_2(t))$ be the solution of (3.2) with the initial condition $Y(0)=x_0 \in R^2$ such that $P(\tilde{e}(0, x_0)<\infty)>0$. Then, in the following we consider a sample such that $\tilde{e}(0, x_0)<\infty$. For simplicity of the notation, we set $\tilde{e}=\tilde{e}(0, x_0)$. Since $s(\infty)=\infty$ by the condition (i), the inverse function $\phi(t)$ of $s(t)$ is well defined for all $t\geqq 0$. Since $g(t)>0$ for all $t\geqq 0$ by the condition (i), we notice that

$$s(t)<t \quad \text{for all} \quad t>0\ . \tag{3.7}$$

Set

$$X_1(t)=Y_1(s(t)) \quad \text{and} \quad X_2(t)=Y_2(s(t))/r(t)$$

for all $0\leqq t<\tilde{e}$. This definition is possible since $Y(t)$ is well defined for all $0\leqq t<\tilde{e}$ and since (3.7) holds. First we see that

$$\begin{aligned}
X_1(t)&=Y_1(0)+\int_0^{s(t)} Y_2(u)du\\
&=Y_1(0)+\int_0^t Y_2(s(u))s'(u)du\\
&=Y_1(0)+\int_0^t Y_2(s(u))/r(u)du\\
&=Y_1(0)+\int_0^t X_2(u)du
\end{aligned}$$

for all $0\leqq t<\tilde{e}$. Now put $q(t)=1/r(\phi(t))$ with the inverse function $\phi(t)$ of $s(t)$, so that $q(t)=\exp\left(-\int_0^{\phi(t)} g(u)\,du\right)$ and

$$\begin{aligned}
q'(t)&=-\phi'(t)g(\phi(t))\exp\left(-\int_0^{\phi(t)} g(u)du\right)\\
&=-r(\phi(t))g(\phi(t))/r(\phi(t))\\
&=-g(\phi(t))\ ,
\end{aligned}$$

that is $dq(t)=-g(\phi(t))dt$. Thus, by Ito's formula concerning stochastic differentials we see that

$$\begin{aligned} d(Y_2(t)q(t)) &= Y_2(t)dq(\mathrm{t})+q(t)dY_2(t) \\ &= -g(\phi(t))Y_2(t)dt+q(t)[-A(t)f(Y_1(t))dt+H(t)d\tilde{w}(t)] \\ &= [-g(\phi(t))Y_2(t)-\{A(t)/r(\phi(t))\}f(Y_1(t))]dt+\{H(t)/r(\phi(t))\}d\tilde{w}(t) . \end{aligned}$$

Therefore we obtain that

$$\begin{aligned} X_2(t) &= Y_2(s(t))/r(t) \\ &= Y_2(0)+\int_0^{s(t)}\left[-g(\phi(u))Y_2(u)-\frac{A(u)}{r(\phi(u))}f(Y_1(u))\right]du+\tilde{N}(t) \\ &= Y_2(0)-\int_0^t\left[g(u)Y_2(s(u))+\frac{A(s(u))}{r(u)}f(Y_1(s(u)))\right]s'(u)du+\tilde{N}(t) \\ &= Y_2(0)-\int_0^t\left[g(u)Y_2(s(u))+\frac{A(s(u))}{r(u)}f(Y_1(s(u)))\right]\frac{1}{r(u)}du+\tilde{N}(t) \\ &= Y_2(0)-\int_0^t\left[g(u)X_2(u)+\frac{A(s(u))}{r^2(u)}f(X_1(u))\right]du+\tilde{N}(t) \end{aligned}$$

for all $0\leqq t<\tilde{e}$, where $\tilde{N}(t)=N(s(t))$ and

$$N(t)=\int_0^t\frac{H(u)}{r(\phi(u))}d\tilde{w}(u) .$$

It is easy to see that $N(t)$ is a continuous and square integrable martingale with its increasing process

$$\langle N\rangle(t)=\int_0^t\frac{H^2(u)}{r^2(\phi(u))}du .$$

Hence, $\tilde{N}(t)$ is a continuous and square integrable martingale with its increasing process $\langle\tilde{N}\rangle(t)$, for which

$$\begin{aligned} \langle\tilde{N}\rangle(t) &= \int_0^{s(t)}\frac{H^2(u)}{r^2(\phi(u))}du \\ &= \int_0^t\frac{H^2(s(u))}{r^2(u)}s'(u)du \\ &= \int_0^t\frac{H^2(s(u))}{r^3(u)}du . \end{aligned}$$

Set

$$w(t)=\int_0^t\frac{r^{3/2}(u)}{H(s(u))}d\tilde{N}(u) ,$$

which is well defined since $H(t)\neq 0$ for all $t\geqq 0$ by the condition (ii), so that $w(t)$ is a continuous and square integrable martingale with its increasing process

$$\langle w\rangle(t)=\int_0^t\frac{r^3(u)}{H^2(s(u))}d\langle\tilde{N}\rangle(u)=t .$$

This implies that $w(t)$ is a Brownian motion process and that

$$\tilde{N}(t)=\int_0^t \frac{H(s(u))}{r^{3/2}(u)}dw(u) .$$

Accordingly, $X(t)=(X_1(t), X_2(t))$ satisfies

$$X_1(t)=Y_1(0)+\int_0^t X_2(u)du ,$$

$$X_2(t)=Y_2(0)-\int_0^t\left[g(u)X_2(u)+\frac{A(s(u))}{r^2(u)}f(X_1(u))\right]du$$

$$+\int_0^t \frac{H(s(u))}{r^{3/2}(u)}dw(u)$$

for all $0\leqq t<\tilde{e}$. Namely. $X(t)$ is a solution of (3.1) with the initial condition $X(0)=x_0\in R^2$ corresponding to the coefficients

$$g(t)>0 , \quad a(t)=A(s(t))/r^2(t) , \quad h(t)=H(s(t))/r^{3/2}(t) , \quad f(x_1)$$

and the Brownian motion process $w(t)$

Consider that

$$|X(t)|^2=X_1^2(t)+X_2^2(t)=Y_1^2(s(t))+X_2^2(s(t))/r^2(t)$$

and hence

$$|X(\phi(t))|^2=Y_1^2(t)+Y_2^2(t)/r^2(\phi(t))$$

for all $0\leqq t<\tilde{e}$. Since $r(t)>1$ for all $t>0$, the above equation implies that

$$r^2(\phi(t))|X(\phi(t))|^2=r^2(\phi(t))Y_1^2(t)+Y_2^2(t)\geqq|Y(t)|^2$$

for all $0\leqq t<\tilde{e}$. Let t tend to $\tilde{e}$ in the both sides of the above equation and notice that $\lim_{t\uparrow\tilde{e}}|Y(t)|=\infty$ on $\tilde{e}<\infty$ since Remark 1.1 holds. Then we obtain that

$$\lim_{t\uparrow\tilde{e}}|X(\phi(t))|=\infty .$$

In fact, if it were that $\liminf_{t\uparrow\tilde{e}}|X(\phi(t))|<\infty$, then we would have that

$$\begin{aligned}\infty> & \; r^2(\phi(\tilde{e}))\liminf_{t\uparrow\tilde{e}}|X(\phi(t))|^2\\ &=\liminf_{t\uparrow\tilde{e}} r^2(\phi(t))|X(\phi(t))|^2\\ &\geqq\liminf_{t\uparrow\tilde{e}}|Y(t)|^2\\ &=\infty ,\end{aligned}$$

which is absurd. Thus we have that

$$\lim_{t\uparrow\phi(\tilde{e})}|X(t)|=\infty\ ,$$

where $\tilde{e}=\tilde{e}(0, x_0)$. Combining this with the definition of the explosion time $e(0, x_0)$ of $X(t)$, we see that $e(0, x_0)\leqq\phi(\tilde{e})$. Last, we show that $e(0, x_0)=\phi(\tilde{e})$. Assume that $e(0, x_0)<\phi(\tilde{e})$ to the contrary. Then we get that $s(e(0, x_0))<\tilde{e}$. Since $r(t)>1$ for $t>0$ by the condition (i), we notice that

$$|X(t)|^2=Y_1^2(s(t))+Y_2^2(s(t))/r^2(t)\leqq Y_1^2(s(t))+Y_2^2(s(t))$$

and hence

$$|X(t)|^2\leqq|Y(s(t))|^2$$

for all $0\leqq t<\tilde{e}$. Let t tend to $e(0, x_0)$ in the both sides of the the above equation. Then Remark 1.1 and the fact that $s(e(0, x_0))<\tilde{e}$ yield

$$\infty=\lim_{t\uparrow e(0,x_0)}|X(t)|^2\leqq|Y(s(e(0, x_0)))|^2<\infty\ ,$$

which is a contradiction. Hence we obtain that $e(0, x_0)=\phi(\tilde{e})$ and proof is completed.

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