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CONVEX CLASS OF STARLIKE FUNCTIONS

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ABSTRACT. Let S denote the class of functions of the form $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ that are analytic and univalent in the unit disk U. Let $S(\alpha,\beta)$ and $K(\alpha,\beta)$ denote the subclasses of S consisting respectively, of starlike and close-to-convex functions of order $\alpha(0\leqq\alpha<1)$ and type $\beta(0<\beta\leqq 1)$. In this paper, we obtain a relationship between classes $S(\alpha,\beta)$ and $K(\alpha,\beta)$ by defining a subclass $B(\alpha,\beta)$ of $K(\alpha,\beta)$. Coefficient estimates, distortion and covering theorems are obtained for the class $B(\alpha,\beta)$. The class $B(\alpha,\beta)$ is convex.

1. Introduction

Let S denote the class of functions of the form $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ that are analytic and univalent in the unit disk $U=\{z\colon |z|<1\}$, We denote by $S(\alpha,\beta)$ and $C(\alpha,\beta)$, the subclassess of S consisting of functions which are, respectively, starlike of order $\alpha(0\leqq\alpha<1)$ and type $\beta(0<\beta\leqq 1)$, and convex of order α and type β.

A function $f\in S$ is in $S(\alpha,\beta)$ if and only if

$$\left|\left[\frac{zf'(z)}{f(z)}-1\right]\Big/\left[\frac{zf'(z)}{f(z)}+(1-2\alpha)\right]\right|<\beta\,,\quad z\in U$$

and is in $C(\alpha,\beta)$ if and only if

$$\left|\left[\frac{zf''(z)}{f'(z)}\right]\Big/\left[\frac{zf''(z)}{f'(z)}+2-2\alpha\right]\right|<\beta\,,\quad z\in U\,.$$

A function $f\in S$ is in $K(\alpha,\beta)$, the class of close-to-convex of order α and type β, if there exists a function $\varphi(z)\in S$ such that

$$\left|\left[\frac{zf'(z)}{\varphi(z)}-1\right]\Big/\left[\frac{zf'(z)}{\varphi(z)}+(1-2\alpha)\right]\right|<\beta\,,\quad z\in U\,.$$

It can easily be seen [1] that a function $f\in S$ is in $C(\alpha,\beta)$ if and only if $zf'(z)\in S(\alpha,\beta)$. In this paper, we shall obtain a relationship between the classes $S(\alpha,\beta)$ and $K(\alpha,\beta)$. We look for a subclass $B(\alpha,\beta)$ of $K(\alpha,\beta)$ that consists of members of $S(\alpha,\beta)$. For this class, results concerning coefficient estimates, distortion theorems and coverning theorems are obtained in this paper. The class $B(\alpha,\beta)$ is also proved to be convex.

2. Main Results.

Therem 1. *If $f \in K(\alpha, \beta)$, then*

$$\sum_{n=2}^{\infty} \{(1+\beta)n\,|a_n| - (1-\beta+2\alpha\beta)\,|b_n|\} \leqq 2\beta(1-\alpha)\,, \tag{2.1}$$

b_n's are the coefficients of the associated function belonging to class S.

Proof. By definition, if $f \in K(\alpha, \beta)$ then there exists a function $\varphi(z) = z - \sum_{n=2}^{\infty} |b_n|\, z^n$ in S that

$$\left| \frac{zf'(z) - \varphi(z)}{zf'(z) + (1-2\alpha)\varphi(z)} \right| < \beta\,, \quad z \in U\,.$$

This gives

$$\operatorname{Re} \left\{ \frac{\sum_{n=2}^{\infty} [n\,|a_n| - |b_n|] z^n}{2(1-\alpha)z - \sum_{n=2}^{\infty} [n\,|a_n| + (1-2\alpha)|b_n|] z^n} \right\} < \beta\,. \tag{2.2}$$

Sincee $|\operatorname{Re}(z)| \leqq |z|$ for all z, choose values of z on the real axis so that $[zf'(z)/\varphi(z)]$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1-$ through real values, we obtain the inequality

$$\sum_{n=2}^{\infty} [n|a_n| - |b_n|] \leqq \beta \left\{ 2(1-\alpha) - \sum_{n=2}^{\infty} [n\,|a_n| + (1-2\alpha)\,|b_n|] \right\}\,.$$

This, on simplification, gives the required coefficient inequality (2.1).

The inequality (2.1) and one more inequality give us the subclass of $K(\alpha, \beta)$, we are looking for.

The class $B(\alpha, \beta)$. A function $f(z) = z - \sum_{n=2}^{\infty} |a_n|\, z^n$ is said to be in $B(\alpha, \beta)$, if there exists a function $\varphi(z) = z - \sum_{n=2}^{\infty} |b_n|\, z^n$ in S such that

$$\sum_{n=2}^{\infty} \{(1+\beta)n\,|a_n| - (1-\beta+2\alpha\beta)\,|b_n|\} \leqq 2\beta(1-\alpha) \tag{i}$$

and

$$n\,|a_n| - |b_n| \geqq 0 \quad \text{for every} \quad n\,. \tag{ii}$$

Remark. In [2] it is shown that $\sum_{n=2}^{\infty} n\,|b_n| \leqq 1$ which gives further that $\sum_{n=2}^{\infty} |b_n| \leqq 1/2$. Hence, the inequality

$$\sum_{n=2}^{\infty} n\,|a_n| \leqq \frac{1+3\beta-2\alpha\beta}{2(1+\beta)} \tag{2.3}$$

would be a necessary condition for f to be in $B(\alpha, \beta)$. The inequality (2.3) would also be a sufficient condition if we drop the condition (ii) in the above definition.

Theorem 2. $B(\alpha, \beta) \subseteqq K(\alpha, \beta)$.

Proof. Taking (ii) into account, we have

$$\left|\frac{\frac{zf'(z)}{\varphi(z)}-1}{\frac{zf'(z)}{\varphi(z)}+(1-2\alpha)}\right| = \left|\frac{\sum_{n=2}^{\infty}[n|a_n|-|b_n|]z^n}{2(1-\alpha)z-\sum_{n=2}^{\infty}[n|a_n|+(1-2\alpha)|b_n|]z^n}\right|$$

$$\leqq \frac{\sum_{n=2}^{\infty}[n|a_n|-|b_n|]}{2(1-\alpha)-\sum_{n=2}^{\infty}[n|a_n|+(1-2\alpha)|b_n|]}$$

which is bounded above by β if (i) holds. Hence $B(\alpha, \beta) \subseteqq K(\alpha, \beta)$

Remarks 1. $S(\alpha, \beta) \subseteqq B(\alpha, \beta)$. It can be verified by taking $\varphi(z)=f(z)$ and applying

$$\sum_{n=2}^{\infty}\{(n-1)+\beta(n+1-2\alpha)\}|a_n| \leqq 2\beta(1-\alpha)$$

which is a necessary and sufficient condition for a function $f \in S$ to be in $S(\alpha, \beta)$, see [1, Theorem 1].

2. $B(\alpha_1, \beta_1) \subseteqq B(\alpha, \beta)$ for $0 \leqq \alpha < \alpha_1 < 1$ and $0 < \beta_1 < \beta \leqq 1$.

3. $B(0, 1)=S(0, 1)$, ([1, Theorem 1]).

Theorem 3 [Coefficient estimates]. *If* $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ *is in* $B(\alpha, \beta)$, *then*

$$|a_n| \leqq \frac{2\beta(1-\alpha)n+(1-\beta+2\alpha\beta)}{(1+\beta)n^2}$$

with equality only for functions of the form

$$f_n(z)=z-\frac{2\beta(1-\alpha)n+(1-\beta+2\alpha\beta)}{(1+\beta)n^2}z^n\,. \tag{2.4}$$

Proof. From (i) we have

$$\begin{aligned}(1+\beta)n|a_n| &\leqq 2\beta(1-\alpha)+(1-\beta+2\alpha\beta)|b_n| \\ &\leqq 2\beta(1-\alpha)+(1-\beta+2\alpha\beta)n^{-1}\end{aligned}$$

because for $\varphi(z)=z-\sum_{n=2}^{\infty}|b_n|z^n \in S$, $|b_n| \leqq 1/n$. This further gives that

$$|a_n| \leqq \frac{2\beta(1-\alpha)n+(1-\beta+2\alpha\beta)}{(1+\beta)n^2}$$

which is the required condition.

The function $f_n(z)$ defined in (2.4) is an extremal function with respect to $\varphi(z)=z-n^{-1}z^n \in S$.

Theorem 4 [Distortion Theorem]. *If* $f \in B(\alpha, \beta)$, *then for* $|z| \leqq r < 1$,

$$r-\frac{1+3\beta-2\alpha\beta}{4(1+\beta)}r^2 \leqq |f(z)| \leqq r+\frac{1+3\beta-2\alpha\beta}{4(1+\beta)}r^2 . \tag{2.5}$$

$$1-\frac{1+3\beta-2\alpha\beta}{2(1+\beta)}r \leqq |f'(z)| \leqq 1+\frac{1+3\beta-2\alpha\beta}{2(1+\beta)}r . \tag{2.6}$$

Proof. On account of (2.3), we get

$$2\sum_{n=2}^{\infty}|a_n| \leqq \sum_{n=2}^{\infty} n|a_n| \leqq \frac{1+3\beta-2\alpha\beta}{2(1+\beta)} .$$

Hence

$$\begin{aligned} |f(z)| &\leqq r+\sum_{n=2}^{\infty}|a_n|r^n \leqq r+r^2\sum_{n=2}^{\infty}|a_n| \\ &\leqq r+\frac{1+3\beta-2\alpha\beta}{4(1+\beta)}r^2 , \end{aligned}$$

and

$$|f(z)| \geqq r-r^2\sum_{n=2}^{\infty}|a_n| \geqq r-\frac{1+3\beta-2\alpha\beta}{4(1+\beta)}r^2 .$$

Thus (2.5) follows.

Further

$$\begin{aligned} |f'(z)| &\leqq 1+\sum_{n=2}^{\infty} n|a_n|\,|z|^{n-1} \\ &\leqq 1+r\sum_{n=2}^{\infty} n|a_n| \end{aligned}$$

and

$$|f'(z)| \leqq 1-r\sum_{n=2}^{\infty} n|a_n| .$$

Using (2.3), we get (2.6).

Remark. The bounds in (2.5) and (2.6) are sharp, since the equalities are attained for the functions

$$f(z)=z-\frac{1+3\beta-2\alpha\beta}{4(1+\beta)}z^2 , \quad z=\pm r . \tag{2.7}$$

Theorem 5. *Let $f \in B(\alpha, \beta)$. Then the disk $|z|<1$ is mapped onto a domain that contains the disk $|w|<(3+\beta+2\alpha\beta/4(1+\beta))$. The result is sharp with extremal function defined in* (2.7).

Proof. By letting $r \to 1$ in (2.5), the proof follows.

Theorem 6. *The class $B(\alpha, \beta)$ is convex.*

Proof. We shall prove the result by using the convexity of the class S. Let $f_1(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ and $f_2(z)=z-\sum_{n=2}^{\infty}|c_n|z^n$ be in $B(\alpha, \beta)$ with respect to functions $\varphi_1(z)=z-\sum_{n=2}^{\infty}|b_n|z^n$ and $\varphi_2(z)=z-\sum_{n=2}^{\infty}|d_n|z^n$ in S. For $0 \leqq \lambda \leqq 1$, we shall prove that $F(z)=\lambda f_1(z)+(1-\lambda)f_2(z)=z-\sum_{n=2}^{\infty}e_n(\lambda)z^n \in B(\alpha, \beta)$ with respect to $\Psi(z)=\lambda\varphi_1(z)+(1-\lambda)\varphi_2(z)=z-\sum_{n=2}^{\infty}t_n(\lambda)z^n \in S$. The function $F(z)$ will belong to $B(\alpha, \beta)$ if

(i)
$$\sum_{n=2}^{\infty}\{(1+\beta)ne_n(\lambda)-(1-\beta+2\alpha\beta)t_n(\lambda)\} \leqq 2\beta(1-\alpha)$$

and

(ii)
$$ne_n(\lambda)-t_n(\lambda) \geqq 0 \quad \text{for all} \quad n\,.$$

Since

$$n|a_n|-|b_n| \geqq 0 \quad \text{and} \quad n|c_n|-|d_n| \geqq 0$$

for all n, we easily get $ne_n(\lambda)-t_n(\lambda) \geqq 0$ for each n. Further

$$\begin{aligned}
&\sum_{n=2}^{\infty}\{n(1+\beta)e_n(\lambda)-(1-\beta+2\alpha\beta)t_n(\lambda)\} \\
&\quad =\lambda\sum_{n=2}^{\infty}\{(1+\beta)n|a_n|-(1-\beta+2\alpha\beta)|b_n|\} \\
&\qquad +(1-\lambda)\sum_{n=2}^{\infty}\{(1+\beta)n|c_n|-(1-\beta+2\alpha\beta)|d_n|\} \\
&\quad \leqq 2\beta(1-\alpha)\,.
\end{aligned}$$

The result follows.

References

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