YOKOHAMA MATHEMATICAL
JOURNAL VOL. 32, 1984

PROPERTIES OF SUBWEAKLY CONTINUOUS FUNCTIONS

By

C. W. BAKER

(Received August 24, 1983)

1. Introduction

The concept of subweak continuity was introduced by Rose in [7]. Weak continuity implies subweak continuity, but the converse implication does not hold (see Example 1). In [5] Noiri proved that if the range space of a weakly continuous function is Hausdorff, then the graph of the function is closed. In Section 3 we prove that the graph of a subweakly continuous function into a Hausdorff space is closed. In Section 4 additional properties of subweakly continuous functions are investigated.

2. Definitions and notation

Let U be a subset of a topological space X. The closure of U in X will be denoted by $Cl\,U$. If $U\subseteq A\subseteq X$, the closure of U in A will be denoted by $Cl_A\,U$.

Definition 1. (Levine [3]) A function $f\colon X\to Y$ is said to be weakly continuous if for each x in X and for each neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(U)\subseteq Cl\,V$.

The following theorem due to Noiri [6] and Rose [7] gives an alternate characterization of weak continuity.

Theorem 1. *A function $f\colon X\to Y$ is weakly continuous if and only if $Cl\,f^{-1}(V)\subseteq f^{-1}(Cl\,V)$ for each open subset V of Y.*

Definition 2. (Rose [7]) A function $f\colon X\to Y$ is said to be subweakly continuous if there is an open basis B for the topology on Y such that $Cl\,f^{-1}(V)\subseteq f^{-1}(Cl\,V)$ for all V in B.

Definition 3. (Gentry & Hoyle [2]) A function $f\colon X\to Y$ is said to be c-continuous if for each x in X and each open subset V of Y containing $f(x)$ and with compact complement, there exists an open subset U of X containing x such that $f(U)\subseteq V$.

3. Closed graph property

Let X and Y be topological spaces and $f: X \to Y$ a function. The graph of f will be denoted by $G(f)$ and is the set $\{(x, f(x)): x \in X\}$. We say the function f has a closed graph if the graph is closed as a subset of the product space $X \times Y$.

It is well known that the graph of a continuous function into a Hausdorff space is closed. In [5] Noiri proved that the graph of a weakly continuous function into a Hausdorff space is also closed. In the following theorem we prove that subweak continuity is sufficient for the graph of a function into a Hausdorff space to be closed.

Theorem 2. *If $f: X \to Y$ is a subweakly continuous function and Y is Hausdorff, then the graph of f is closed.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Let B be an open basis for the topology on Y such that $Cl\, f^{-1}(V) \subseteqq f^{-1}(Cl\, V)$ for all V in B. Then there exist disjoint open sets V and W in B such that $y \in V$ and $f(x) \in W$. Since $f(x) \in W$ which is open and disjoint from V, it follows that $f(x) \notin Cl\, V$. Thus $x \notin f^{-1}(Cl\, V)$. Since f is subweakly continuous and $V \in B$, $Cl\, f^{-1}(V) \subseteqq f^{-1}(Cl\, V)$. Then $(x, y) \in (X - Cl(f^{-1}(V))) \times V \subseteqq X \times Y - G(f)$. Hence $G(f)$ is closed.

Long and Hendrix [4] proved that the closed graph property implies c-continuity. Thus we have the following corollary.

Corollary 1. *If $f: X \to Y$ is subweakly continuous and Y is Hausdorff, then f is c-continuous.*

The following results are also implied by the closed graph property (Fuller [1]).

Corollary 2. *If $f: X \to Y$ is subweakly continuous and Y is Hausdorff, then for each compact subset C of Y, $f^{-1}(C)$ is closed in X.*

Corollary 3. *If $f: X \to Y$ is subweakly continuous and Y is Hausdorff, then for each compact subset C of X, $f(C)$ is closed in Y.*

4. Additional properties

Theorem 3. *If $f: X \to Y$ is subweakly continuous and A is an open subset of Y containing $f(X)$, then $f: X \to A$ is subweakly continuous.*

Proof. Let B be an open basis for the topology on Y such that $Cl\, f^{-1}(V) \subseteqq f^{-1}(Cl\, V)$ for all V in B. Then $C = \{V \cap A: V \in B\}$ is an open basis for the topology on A. Let $V \cap A \in C$. Then $Cl\, f^{-1}(V \cap A) = Cl\, f^{-1}(V) \subseteqq f^{-1}(Cl\, V) = f^{-1}((Cl\, V) \cap A)$. It remains to be shown that $(Cl\, V) \cap A \subseteqq Cl_A(V \cap A)$.

Let $y \in (Cl\, V) \cap A$. Let W be any subset of A that is open in A and contains y. Since A is open in Y, W is open in Y. Because $y \in Cl\, V$, $W \cap V \neq \phi$. Therefore $W \cap (V \cap A) = W \cap V \neq \phi$. Thus $y \in Cl_A(V \cap A)$ and $(Cl\, V) \cap A \subseteq Cl_A(V \cap A)$.

It follows that $Cl\, f^{-1}(V \cap A) \subseteq f^{-1}((Cl\, V) \cap A) \subseteq f^{-1}(Cl_A(V \cap A))$. Hence $f: X \to A$ is subweakly continuous.

Theorem 4. *If $f: X \to Y$ is subweakly continuous and A is a subset of X, then $f|_A: A \to Y$ is subweakly continuous.*

Proof. Let B be an open basis for the topology on Y such that $Cl\, f^{-1}(V) \subseteq f^{-1}(Cl\, V)$ for all V in B. Then for $V \in B$, $Cl_A(f|_A{}^{-1}(V)) = A \cap Cl\, (f|_A{}^{-1}(V)) = A \cap Cl\, (f^{-1}(V) \cap A) \subseteq A \cap Cl\, (f^{-1}(V)) \cap Cl A = A \cap Cl f^{-1}(V) \subseteq A \cap f^{-1}(Cl V) = f|_A{}^{-1}(Cl V)$. Thus $f|_A: A \to Y$ is subweakly continuous.

A space X is said to be an Urysohn space if for every pair of distinct points x and y in X there exist open sets U and V in X such that $x \in U$ and $y \in V$ and $(Cl\, U) \cap (Cl\, V) = \phi$. In [6] Noiri proved that if Y is an Urysohn space and $f: X \to Y$ is a weakly continuous injection, then X is Hausdorff. The following example due to Rose [7] shows that subweak continuity can not be substituted for weak continuity in this result.

Example 1. Let X be any set with a non-discrete T_1 topology, and let $Y = X$ have the discrete topology. Let $f: X \to Y$ be the identity mapping. Rose observed that this function is subweakly continuous but not weakly continuous. Note that Y is an Urysohn space and f is injective, but X need not be Hausdorff. The following related result is true for subweakly continuous functions.

Theorem 5. *If Y is Hausdorff and $f: X \to Y$ is a subweakly continuous injection, then X is T_1.*

Proof. Let x_1 and x_2 be distinct points in X. Then $f(x_1) \neq f(x_2)$. Let B be an open basis for the topology on Y such that $Cl f^{-1}(V) \subseteq f^{-1}(Cl\, V)$ for all V in B. Since Y is Hausdorff, there exist disjoint open sets U and V in Y such that $f(x_1) \in U$, $f(x_2) \in V$, and $V \in B$. Then since $f(x_1) \notin Cl\, V$, we have $x_1 \in X - f^{-1}(Cl\, V) \subseteq X - Cl\, f^{-1}(V)$. Therefore $X - Cl\, f^{-1}(V)$ is an open subset of X which contains x_1 but not x_2.

Theorem 6. *Let Y be a Hausdorff space, $f_1: X \to Y$ continuous, and $f_2: X \to Y$ subweakly continuous. Then $\{x \in X: f_1(x) = f_2(x)\}$ is a closed subset of X.*

Proof. Let $A = \{x \in X: f_1(x) = f_2(x)\}$. Let $x \in X - A$. Then $f_1(x) \neq f_2(x)$. Let B be an open basis for the topology on Y such that $Cl\, f_2{}^{-1}(V) \subseteq f_2{}^{-1}(Cl\, V)$ for all V

in B. Since Y is Hausdorff, there exist disjoint open sets V and W in Y such that $f_1(x) \in V, f_2(x) \in W$, and $V \in B$. Then $f_2(x) \notin Cl\, V$. Therefore $x \in X - f_2^{-1}(Cl\, V) \subseteq X - Cl\, f_2^{-1}(V)$. Hence $x \in f_1^{-1}(V) \cap (X - Cl\, f_2^{-1}(V)) \subseteq X - A$. Thus A is closed.

Corollary 1. *Let Y be Hausdorff, $f_1\colon X \to Y$ continuous, and $f_2\colon X \to Y$ subweakly continuous. If f_1 and f_2 agree on a dense subset of X, then $f_1 = f_2$.*

For a function $f\colon X \to Y$ the graph function is the map $g\colon X \to X \times Y$ given by $g(x) = (x, f(x))$.

Theorem 7. *If $f\colon X \to Y$ is subweakly continuous, then the graph function is subweakly continuous.*

Proof. Let $g\colon X \to X \times Y$ be the graph function for f. Let B be an open basis for the topology on Y such that $Cl\, f^{-1}(V) \subseteq f^{-1}(Cl\, V)$ for all V in B. Then $C = \{U \times V\colon U \subseteq X$ is open and $V \in B\}$ is an open basis for the topology on $X \times Y$. For $U \times V \in C$, $Cl\, g^{-1}(U \times V) = Cl(U \cap f^{-1}(V)) \subseteq (Cl\, U) \cap Cl\, f^{-1}(V) \subseteq (Cl\, U) \cap f^{-1}(Cl\, V) = g^{-1}((Cl\, U) \times Cl\, V) = g^{-1}(Cl(U \times V))$. Hence the graph function g is subweakly continuous.

In [6] Noiri proved that if A is a subset of X and $f\colon X \to A$ is a weakly continuous retraction of X onto A and X is Hausdorff, then A is a closed subset of X. The following somewhat weaker result is true for subweakly continuous functions. The proof is similar to Noiri's.

Theorem 8. *Let $A \subseteq X$ and let $f\colon X \to X$ be a subweakly continuous function such that $f(X) = A$ and $f|_A$ is the identity on A. Then if X is Hausdorff, A is a closed subset of X.*

Proof. Assume A is not closed. Let $x \in Cl\, A - A$. Let B be an open basis for the topology on X such that $Cl\, f^{-1}(V) \subseteq f^{-1}(Cl\, V)$ for all V in B. Since $x \notin A$, $x \neq f(x)$. Because X is Hausdorff, there exist disjoint open sets V and W such that $x \in V$, $f(x) \in W$ and $V \in B$. Let U be any open subset of X containing x. Then $x \in U \cap V$ which is an open subset of X. Since $x \in Cl\, A$, $(U \cap V) \cap A \neq \phi$. So there exists an element $y \in (U \cap V) \cap A$. Since $y \in A$, $f(y) = y \in V$. Hence $y \in f^{-1}(V)$. Thus $U \cap f^{-1}(V) \neq \phi$. It follows that $x \in Cl(f^{-1}(V))$. However $f(x) \in W$ which is open and disjoint from V. So $x \notin f^{-1}(Cl\, V)$. This contradicts the assumption that f is subweakly continuous. Hence A is closed.

Theorem 9. *Let $f_\alpha\colon X \to Y_\alpha$ be a subweakly continuous function for each α in A. Let $f\colon X \to \Pi Y_\alpha$ be given by $f(x) = (f_\alpha(x))$. Then f is subweakly continuous.*

Proof. For each α in A let B_α be an open basis for Y_α such that $Cl\, f_\alpha^{-1}(V_\alpha) \subseteq$

$f_\alpha^{-1}(Cl\,V_\alpha)$ for all V_α in B_α. Then $B=\{\Pi V_\alpha\colon V_\alpha=Y_\alpha$ for all but finitely many coordinates and if $V_\alpha\neq Y_\alpha$, then $V_\alpha\in B_\alpha\}$ is an open basis for ΠY_α. For $\Pi V_\alpha\in B$, $Cl\,f^{-1}(\Pi V_\alpha)=Cl\bigcap_\alpha f_\alpha^{-1}(V_\alpha)\subseteqq\bigcap_\alpha Cl\,f_\alpha^{-1}(V_\alpha)\subseteqq\bigcap_\alpha f_\alpha^{-1}(Cl\,V_\alpha)=f^{-1}(\Pi Cl\,V_\alpha)=f^{-1}(Cl\,\Pi V_\alpha)$. Thus f is subweakly continuous.

References

[1] R. V. Fuller, *Relations among continuous and various noncontinuous functions*, Pacific J. of Math., **25** (1968), 495–509.

[2] Karl R. Gentry and Hughes B. Hoyle, III, *C-continuous functions*, Yokohama Math. J., **18** (1970), 71–76.

[3] Norman Levine, *A decomposition of continuity in topological spaces*, Amer. Math. Monthly, **68** (1961), 44–46.

[4] Paul E. Long and Michael D. Hendrix, *Properties of c-continuous functions*, Yokohama Math. J., **22** (1974), 117–123.

[5] Takashi Noiri, *Between continuity and weak continuity*, Boll. U.M.I., **9** (1974), 647–654.

[6] Takashi Noiri, *On weakly continuous mappings*, Proc. Amer. Math. Soc., **46** (1974), 120–124.

[7] D. A. Rose, *Weak continuity and almost continuity*, International Journal of Mathematics and Mathematical Sciences, to appear.

Department of Mathematics
Indiana University Southeast
New Albany, Indiana 47150 USA