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CONVERGENT POWER SERIES EXPANSIONS FOR THE BIRKHOFF INVARIANTS OF MEROMORPHIC DIFFERENTIAL EQUATIONS

Part I. Definition of the coefficient functions

By

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0. Introduction

Solutions of the *standard differential equation of Poincaré rank one*

(0.1) $$zx'=(z\Lambda+A_1)x\ ,$$

where $\Lambda=\mathrm{diag}\,[\lambda_1, \cdots, \lambda_n]$ has all distinct diagonal entries, can be expressed in terms of convergent Laplace integrals of the form

$$x(z)=\int y(t)e^{zt}dt\ ,$$

and $y(t)$ is a solution of *the associated differential equation*

(0.2) $$(\Lambda-tI)y'=(I+A_1)y\ .$$

R. Schäfke [2] and, independently, W. Balser, W. B. Jurkat, and D. A. Lutz [1] have shown that the Stokes' multipliers of certain unique fundamental solutions of (0.1) can be explicitly calculated in terms of constants $c_{kj}(1\leqq j, k\leqq n)$, arising from (0.2) as follows: To every one of the points $\lambda_j(1\leqq j\leqq n)$ (which are regular singular points of (0.2)) there corresponds a unique solution $y_j(t)$ having a characteristic singular behavior at λ_j. A convergent expansion of $y_j(t)$, for $|t-\lambda_j|$ sufficiently small, is given in terms of the j^{th} column of a formal fundamental solution of (0.1), so that, given the formal solution, one may think of $y_j(t)$ to be known locally (i.e. at the point λ_j). Furthermore, $y_j(t)$ may be analytically continued to every other point $\lambda_k(1\leqq k\leqq n)$, and assuming this to be done, one may find c_{kj} mentioned above as the unique constant for which

(0.3) $$y_j(t)=y_k(t)c_{kj}+\mathrm{reg}\,(t-\lambda_k)$$

(by reg (t) we generically denote a vector or a matrix being analytic in a neighborhood of the origin). Hence, although in principle the constants c_{kj} (and from them, the Stokes' multipliers) may be calculated by means of analytic continuation of the functions $y_1(t), \cdots, y_n(t)$, an *explicit formula* for the analytic continuation would be helpful if not necessary, in order to make a calculation work effectively.

In this paper, we obtain such a formula by means of expanding the functions $y_j(t)$ as power series (in several variables) with respect to the off-diagonal elements of A_1 as variables, and we prove that the coefficients (which depend upon t, the points $\lambda_1, \cdots, \lambda_n$, and the diagonal elements of A_1 as parameters) are analytic functions of t (for $t \neq \lambda_1, \cdots, \lambda_n$) which can be *recursively calculated* from explicit integral formulas (Sections 1 and 2). In Section 3 we then show that the power series *converges for every* A_1, and that the convergence is uniform with respect to t, for t in every compact set avoiding the points $\lambda_1, \cdots, \lambda_n$. In Section 4 we analize the behavior of the coefficients at $\lambda_k (1 \leqq k \leqq n)$ and show that, in a sense, the convergence even is uniform for t close to λ_k. So, mainly because of the convergence for t near λ_k, the expansion we have obtained seems a useful means for calculating the constants c_{kj}, and in fact we show in Sections 4 and 5 that c_{kj} itself admits a convergent power series expansion with respect to the off-diagonal elements of A_1 as variables. The coefficients of the expansion of c_{kj} only depend upon the formal invariants (i.e. $\lambda_1, \cdots, \lambda_n$ and the diagonal entries of A_1) and may be obtained from the coefficients in the expansion of $y_j(t)$ by analysing their behavior at λ_k. While we do not yet have a recursion formula for the coefficients of c_{kj}, they still may be effectively calculated by means of the recursion formulas for the coefficients of $y_j(t)$ and "evaluating" them at λ_k.

As a consequence of this paper, we obtain that the constants c_{kj} are entire functions in the off-diagonal elements of A_1, which is not very surprising. However, based upon the results obtained here, we hope to analyse the nature of c_{kj} as a function of the elements $\lambda_1, \cdots, \lambda_n$, which should give more important information about the nature of these functions.

1. An integral equation for the associated functions

Throughout this paper, let $\lambda_1, \cdots, \lambda_n$ (for fixed $n \geqq 2$) always denote *mutually distinct* complex numbers, and let

$$\Lambda = \operatorname{diag}[\lambda_1, \cdots, \lambda_n]$$

denote the diagonal matrix with the (diagonal) entries $\lambda_1, \cdots, \lambda_n$. Furthermore, let A_1 be an arbitrary $n \times n$ matrix, which we like to decompose as

$$A_1 = \Lambda' + A \ ,$$

where

$$\Lambda' = \text{diag}\,[\lambda_1', \cdots, \lambda_n']$$

consists of the diagonal entries of A_1, and accordingly

$$A = [a_{jk}]\ , \qquad 1 \leqq j, k \leqq n\ , \quad a_{11} = \cdots = a_{nn} = 0\ ,$$

contains the off-diagonal entries of A_1.

Remark 1.1. Whenever we will later on make what we call *our basic assumptions,* it will be understood that n distinct numbers $\lambda_1, \cdots, \lambda_n$ and a matrix A_1 of size $n \times n$ are arbitrarily given, and that Λ, Λ', and A are as above.

Under our basic assumptions, it is easily seen (cf. e.g. [1], [2]) that the differential equation

$$(tI - \Lambda)y' = (\rho I - A_1)y\ , \tag{1.1}$$

for arbitrary complex ρ, has singularities only at the points $\lambda_1, \cdots, \lambda_n$, and ∞, which are all regular singular points. If we restrict ρ by

$$\rho \not\equiv \lambda_k' \bmod 1\ , \quad 1 \leqq k \leqq n\ , \tag{1.2}$$

then for every fixed k, $1 \leqq k \leqq n$, a fundamental solution of (1.1) is seen to exist which consists of $n-1$ solution vectors analytic at $t = \lambda_k$, and one of the form

$$y_k(t) = y_k(t;\,\rho) = (t - \lambda_k)^{\rho - \lambda_k'}\{\delta_k / \Gamma(1 + \rho - \lambda_k') + \text{reg}_0\,(t - \lambda_k)\}\ , \tag{1.3}$$

where δ_k is the k^{th} unit vector, and $\text{reg}_0\,(t)$ generically denotes, throughout, a vector or matrix of appropriate size whose components are analytic in some neighborhood of zero and vanish for $t = 0$.

Remark 1.2. For $\rho = -1$, the equation (1.1) coincides with the associated differential equation (0.2). The idea of introducing a parameter ρ is due to R. Schäfke [2] who also investigated the dependence of $y_k(t;\rho)$ upon ρ and considered, in particular, the extension of the definition of $y_k(t;\rho)$ to cases where $\rho - \lambda_k$ becomes an integer. We do, however, not consider these cases here.

Due to (1.2), the functions $y_1(t), \cdots, y_n(t)$ (which coincide, aside from scalar factors, with the *associated functions* introduced in [1]) are multi-valued and are, among all solutions of (1.1), uniquely characterized by (1.3) (because of the structure of fundamental solutions mentioned above). To have an unambiguous definition of these functions, we cut the t-plane along parallel rays

$$\arg\,(t - \lambda_k) = \eta\ , \quad 1 \leqq k \leqq n\ ,$$

for some fixed real value η for which no point $\lambda_j (j \neq k)$ lies on the cut $\arg\,(t - \lambda_k) = \eta$, $1 \leqq k \leqq n$. *For every t in the cut plane* (i.e. not on one of the cuts) we then select arg $(t-\lambda_k)$ to be in the interval $(\eta-2\pi, \eta)$, and define $\log(t-\lambda_k)$ and general powers of $t-\lambda_k$ consistent with this choice $(1\leq k\leq n)$. The functions $y_1(t), \cdots, y_n(t)$ then become single-valued analytic functions of t in the cut plane. If we fix Λ and Λ' (and the direction of the cuts), these functions depend upon ρ and the entries of A as parameters. To study this dependence, it is sufficient to concentrate on $y_1(t)$, e.g., since a permutation similarity of $[y_1(t), \cdots, y_n(t)]$ and the matrices Λ, Λ', A may be used to generalize results for $y_1(t)$ to $y_2(t), \cdots, y_n(t)$. For notational convenience, we therefore omit the index and write

$$y(t)=y(t;\rho, A)$$

instead of $y_1(t)$.

Lemma 1. *In addition to our basic assumptions, assume that* ρ *satisfies* (1.2) *and* $\mathrm{Re}(\rho-\lambda_1')>-1$. *Then the components*

$$y^{(1)}(t), \cdots, y^{(n)}(t)$$

of the vector solution $y(t)$ *satisfy the following system of integral equations:*

$$(1.4)\qquad \begin{cases} y^{(1)}(t)=(t-\lambda_1)^{\rho-\lambda_1'}\{1/\Gamma(1+\rho-\lambda_1')-\sum\limits_{j\neq 1} a_{1j}\displaystyle\int_{\lambda_1}^{t}(u-\lambda_1)^{\lambda_1'-\rho-1}y^{(j)}(u)du\}\,. \\ y^{(k)}(t)=-(t-\lambda_k)^{\rho-\lambda_k'}\sum\limits_{j\neq k} a_{kj}\displaystyle\int_{\lambda_1}^{t}(u-\lambda_k)^{\lambda_k'-\rho-1}y^{(j)}(u)du\,, \quad 2\leq k\leq n\,, \end{cases}$$

for t *in the cut plane.*

Proof. From (1.3) with $k=1$ we conclude for the components of $y(t)=y_1(t)$:

$$(1.5)\qquad (t-\lambda_1)^{\lambda_1'-\rho}y^{(1)}(t)=1/\Gamma(1+\rho-\lambda_1')+\mathrm{reg}_0(t-\lambda_1)\,,$$

$$(1.6)\qquad (t-\lambda_1)^{\lambda_1'-\rho-1}y^{(j)}(t)=\mathrm{reg}(t-\lambda_1)\,,\quad 2\leq j\leq n\,.$$

Furthermore, since $y(t)$ is a solution vector of (1.1), we obtain

$$(1.7)\qquad \frac{d}{dt}[(t-\lambda_k)^{\lambda_k'-\rho}y^{(k)}(t)]=-(t-\lambda_k)^{\lambda_k'-\rho-1}\sum_{j\neq k}a_{kj}y^{(j)}(t)\,,\quad 1\leq k\leq n\,.$$

For $k=1$, we conclude from (1.6) that the right hand side of (1.7) is analytic at $t=\lambda_1$, whereas for $k=2, \cdots, n$ it has an integrable singularity, due to (1.5), (1.6) and our assumption $\mathrm{Re}(\rho-\lambda_1')>-1$. Hence with suitable constants c_k, for $k=1, \cdots, n$:

$$(t-\lambda_k)^{\lambda_k'-\rho}y^{(k)}(t)=c_k-\int_{\lambda_1}^{t}(u-\lambda_k)^{\lambda_k'-\rho-1}\{\sum_{j\neq k}a_{kj}y^{(j)}(u)\}du\,,$$

and taking a limit as $t\to\lambda_1$ (and again using (1.5), (1.6)), we see

$$c_1=1/\Gamma(1+\rho-\lambda_1') , \qquad c_2= \cdots =c_n=0 .$$

In order to expand $y(t;\rho, A)$ as a power series in several variables (namely the components of A), we use the following notation:

An index matrix $P=[p_{jk}]$, $1\leqq j, k\leqq n$, is a matrix with non-negative integer entries, for which

$$p_{11}=p_{22}=\cdots=p_{nn}=0 .$$

For every such P, define

(1.8) $$A^P=\sum_{j\neq k} a_{jk}^{p_{jk}}$$

(i.e. A^P is a scalar!). Then for every two index matrices P and Q we have

$$A^P A^Q=A^{P+Q}=A^Q A^P ,$$

whatever the values of the entries of A are. The "modulus" of such an index matrix is defined by

(1.9) $$|P|=\sum_{j,k} p_{jk} .$$

Using these notations, we formally expand $y(t;\rho, A)$ as a power series

(1.10) $$y(t;\rho, A)=\prod_P g(t;\rho, P)A^P .$$

If E_{jk} (for $j\neq k$, $1\leqq j, k\leqq n$) denotes the index matrix with a single one in the (j, k) position and zeros elsewhere, then

$$a_{jk}=A^{E_{jk}} .$$

Inserting (1.10) into (1.4), formally interchanging summation and integration and comparing coefficients of like powers leads to recursion formulas for the components $g^{(1)}(t;\rho, P), \cdots, g^{(n)}(t;\rho, P)$ of the coefficient functions $g(t;\rho, P)$. If we interpret, in the formulas below, $g(t;\rho, P)\equiv 0$ if P happens to have some negative entry, then these relations may be written as

(1.11) $$\begin{cases} g^{(1)}(t;\rho, 0)=(t-\lambda_1)^{\rho-\lambda_1'}/\Gamma(1+\rho-\lambda_1') , \\ g^{(k)}(t;\rho, 0)\equiv 0 , \quad 2\leqq k\leqq n , \end{cases}$$

and for $P\neq 0$:

(1.12) $$g^{(k)}(t;\rho, P)=-(t-\lambda_k)^{\rho-\lambda_k'} \sum_{j\neq k} \int_{\lambda_1}^{t} (u-\lambda_k)^{\lambda_k'-\rho-1} g^{(j)}(u;\rho, P-E_{kj})du ,$$
$$1\leqq k\leqq n .$$

In the following section we show that the integrals in (1.12) always exist, provided $\operatorname{Re}(\rho-\lambda_1')>-1$, and for every such ρ, (1.11) and (1.12) define the *coefficient functions* $g^{(k)}(t;\rho,P)$ uniquely for every index matrix P. Obviously, the coefficient functions form a set of functions of t which (aside from the parameter ρ) only depend upon the *formal invariants* $\lambda_1,\cdots,\lambda_n$, and $\lambda_1',\cdots,\lambda_n'$, and it is a reasonable point of view if we consider them as "known" functions for the purpose of calculating the Stokes' multipliers. Nonetheless, they seem to be higher transcendental functions, and we are going to study some of their properties later.

Remark 1.3. One can see from (1.11), (1.12) that for many index matrices P, the corresponding coefficient function vanishes identically.

2. Analytic behavior of the coefficient functions

We are going to investigate the analytic behavior of $g(t;\rho,P)$ with respect to the variable t.

Proposition 1. *In addition to our basic assumptions, let* $\operatorname{Re}(\rho-\lambda_1')>-1$. *Then for every index matrix P, the function $g(t;\rho,P)$ is uniquely defined by* (1.11), (1.12), *analytic (with respect to t) in the cut plane, and may be analytically continued along arbitrary paths across the cuts which do not contain any of the points $\lambda_1,\cdots,\lambda_n$. Furthermore, for arbitrary non-negative integer q,*

$$g(t;\rho,P)=(t-\lambda_1)^{\rho-\lambda_1'+q+1}\operatorname{reg}(t-\lambda_1)\quad \textit{if}\quad |P|=2q+1\,, \tag{2.1}$$

resp.

$$g(t;\rho,P)=(t-\lambda_1)^{\rho-\lambda_1'+q}\{c_P\delta_1+\operatorname{reg}_0(t-\lambda_1)\}\quad \textit{if}\quad |P|=2q \tag{2.2}$$

(with a scalar constant $c_P=c_P(\rho)$).

Proof. Provided the integrals in (1.12) always exist, the statements upon the uniqueness, analyticity, and analytic continuation of $g(t;\rho,P)$ are immediate. Hence it suffices to prove (2.1), (2.2), because then the integrals exist, due to our assumption $\operatorname{Re}(\rho-\lambda_1')>-1$. We prove (2.1), (2.2) by induction with respect to $|P|$:

For $|P|=0$ (i.e. $P=0$), we see that (2.2) holds with $c_0=1/\Gamma(1+\rho-\lambda_1')$, due to (1.11). Hence we assume that (2.2) holds for some fixed $q\geqq 0$. Then for arbitrarily fixed P with $|P|=2q+1$, we have $|P-E_{kj}|=2q$ for every $k\neq j$, $1\leqq k$, $j\leqq n$, except if $p_{kj}=0$ in which case $P-E_{kj}$ is not an index matrix. But in the second case we defined $g^{(j)}(t;\rho,P-E_{kj})\equiv 0$; hence by induction hypothesis we have for every k, $1\leqq k\leqq n$:

$$g^{(j)}(t;\rho,P-E_{kj})=(t-\lambda_1)^{\rho-\lambda_1'+q+1}\operatorname{reg}(t-\lambda_1)\,,\quad j\neq k\,,\quad 2\leqq j\leqq n\,,$$

and (if $k\neq 1$)

$$g^{(1)}(t;\rho, P-E_{k1})=(t-\lambda_1)^{\rho-\lambda_1'+q}\operatorname{reg}(t-\lambda_1)\ .$$

For $k=1$, the integrands on the right of (1.12) are therefore of the form

$$(u-\lambda_1)^q \operatorname{reg}(u-\lambda_1)\ ,$$

hence

$$g^{(1)}(t;\rho, P)=(t-\lambda_1)^{\rho-\lambda_1'+q}\operatorname{reg}(t-\lambda_1)\ .$$

For $k=2, \cdots, n$, the integrands in (1.12) are of the form

$$(u-\lambda_1)^{\rho-\lambda_1'+q}\operatorname{reg}(u-\lambda_1)\ ,$$

and expanding the analytic function and integrating termwise, we find

$$g^{(k)}(t;\rho, P)=(t-\lambda_1)^{\rho-\lambda_1'+q+1}\operatorname{reg}(t-\lambda_1)\ ,\quad 2\leqq k\leqq n\ .$$

Therefore (2.1) holds for every P with $|P|=2q+1$, and in quite the same way one proves (2.2) (with $q+1$ in place of q) for every P with $|P|=2q+2$. This completes the proof.

Next we analyse the dependence of $g(t;\rho, P)$ upon ρ if ρ changes modulo one:

Proposition 2. *Under our basic assumptions, the function $g(t;\rho, P)$ (for every fixed index matrix P) is analytic with respect to ρ (for fixed t in the cut plane), in the halfplane*

$$\operatorname{Re}(\rho-\lambda_1')>-1\ . \tag{2.3}$$

Furthermore,

$$g(t;\rho, P)=\frac{d}{dt}g(t;\rho+1, P) \tag{2.4}$$

for t in the cut plane and every ρ with (2.3).

Proof. We proceed by induction with respect to $p=|P|$: For $p=0$, the statements are obviously true, due to (1.11), and we furthermore observe that $(\partial/\partial\rho)g^{(k)}(t;\rho, 0)$ has an absolutely integrable singularity (in t) at $t=\lambda_1$. Hence let $p\geqq 0$ be arbitrarily fixed, and suppose that the statements of the Proposition hold for every index matrix of "modulus" p, and that additionally the derivative with respect to ρ is always absolutely integrable at $t=\lambda_1$. Then for arbitrarily fixed P with $|P|=p+1$, we may differentiate (1.12) with respect to ρ to obtain that $g^{(k)}(t;\rho, P)$ (for arbitrarily fixed k, $1\leqq k\leqq n$) is analytic with respect to ρ, for ρ as in (2.3), and its derivative again is absolutely integrable at $t=\lambda_1$. Furthermore, we conclude from Proposition 1 and (2.3) that

$$(t-\lambda_1)^{\lambda_1'-\rho-1}g^{(j)}(t;\rho+1,P-E_{1j})\to 0 \quad \text{as} \quad t\to\lambda_1\,, \quad 2\leqq j\leqq n\,.$$

Hence we may integrate by parts to obtain (for $k=1,\cdots,n$, fixed)

$$\begin{aligned}(\rho+1-\lambda_k')\sum_{j\neq k}\int_{\lambda_1}^{t}(u-\lambda_k)^{\lambda_k'-\rho-2}g^{(j)}(u;\rho+1,P-E_{kj})du\\ =-\sum_{j\neq k}(t-\lambda_k)^{\lambda_k'-\rho-1}g^{(j)}(t;\rho+1,P-E_{kj})\\ +\sum_{j\neq k}\int_{\lambda_1}^{t}(u-\lambda_k)^{\lambda_k'-\rho-1}g^{(j)}(u;\rho,P-E_{kj})du\end{aligned}$$

(since by induction hypothesis, (2.4) holds for $P-E_{kj}$ in place of P). Then from (1.12), with $\rho+1$ instead of ρ, we conclude that

$$\begin{aligned}\frac{d}{dt}g^{(k)}(t;\rho+1,P)=&-(t-\lambda_k)^{-1}\sum_{j\neq k}g^{(j)}(t;\rho+1,P-E_{kj})\\ &-(\rho+1-\lambda_k')(t-\lambda_k)^{\rho-\lambda_k'}\sum_{j\neq k}\int_{\lambda_1}^{t}(u-\lambda_k)^{\lambda_k'-\rho-2}g^{(j)}(u;\rho+1,P-E_{kj})du\\ =&\,g^{(k)}(t;\rho,P)\,.\end{aligned}$$

Remark 2.1. Property (2.4) may be used to analytically continue (with respect to ρ) the functions $g(t;\rho,P)$ to arbitrary values of ρ, hence we observe that $g(t;\rho,P)$, for fixed t in the cut plane and arbitrary index matrix P, is an *entire function of* ρ. By differentiation (with respect to t) of (2.1), (2.2) one finds that Proposition 1 holds for arbitrary values of ρ; in fact, if $\rho-\lambda_1'$ is a negative integer, we may strengthen (2.1), (2.2) for those values of q for which the exponent of $(t-\lambda_1)$ is negative to read

$$g(t;\rho,P)=\operatorname{reg}(t-\lambda_1) \quad \text{if} \quad |P|=2q+1\,, \qquad 0\leqq q\leqq\lambda_1'-\rho-2\,,$$

resp.

$$g(t;\rho,P)=\operatorname{reg}(t-\lambda_1) \quad \text{if} \quad |P|=2q\,, \qquad 0\leqq q\leqq\lambda_1'-\rho-1\,.$$

Using (2.4) and Proposition 1, one can also see that for arbitrary ρ, (1.12) remains valid provided $|P|$ is sufficiently large to ensure the existence of the integrals. This will be of use later.

3. Estimates of the coefficient functions

In order to establish the convergence of (1.10), we are going to give estimates upon the growth of $g(t;\rho,P)$ with respect to $|P|$. To do so, we denote by $M(K,\delta)$, for arbitrarily fixed positive reals K and δ, the set of all t in the cut plane, for which

$$|t-\lambda_1|\leqq K, \qquad |t-\lambda_j|\geqq\delta, \quad 2\leqq j\leqq n.$$

Proposition 3. *Under our basic assumptions, let ρ be arbitrarily given, and $\beta=\mathrm{Re}(\rho-\lambda_1')$. Then for arbitrarily fixed positive real constants K and δ, there exists a positive constant K_1 (independent of P and t) such that for every index matrix P and every $t\in M(K,\delta)$ (with nonnegative integer q)*

$$|g^{(k)}(t;\rho,P)|\leqq K_1^{2q+2}|t-\lambda_1|^{\beta+q+1}/\Gamma(\beta+q+2), \quad 1\leqq k\leqq n \tag{3.1}$$

if $|P|=2q+1$; respectively

$$|g^{(k)}(t;\rho,P)|\leqq K_1^{2q+1}|t-\lambda_1|^{\beta+q+1}/\Gamma(\beta+q+2), \quad 2\leqq k\leqq n \tag{3.2}$$

and

$$|g^{(1)}(t;\rho,P)|\leqq K_1^{2q+1}|t-\lambda_1|^{\beta+q}/\Gamma(\beta+q+1) \tag{3.3}$$

if $|P|=?q$.

Proof. We proceed by induction with respect to q: For finitely many q, the estimates are certainly correct, due to Proposition 1 resp. Remark 2.1, if we only take K_1 sufficiently large. Therefore, as an induction assumption we suppose that (3.2), (3.3) hold for arbitrary index matrices with "modulus" $2q$, for some fixed $q\geqq\max\{0,-\beta\}$. Then for arbitrarily fixed P with $|P|=2q+1$, the integrals in (1.12) exist, hence according to Remark 2.1, the coefficient $g(t;\rho,P)$ is given by (1.12). Without loss in generality, let K be sufficiently large and δ be sufficiently small, such that $M(K,\delta)$ is simply connected. In order to estimate (1.12), we fix a path of integration from λ_1 to t, which lies in $M(K,\delta)$ and may be parametrized as

$$u\colon[0,L_t]\to M(K,\delta), \quad u(0)=\lambda_1, \quad u(L_t)=t,$$

and the parameter s may be the arc length parameter. Since for t close to λ_1, we may always integrate along a line segment, it is clear that we may always pick a path for which

$$L_t\leqq K_2|t-\lambda_1| \quad \text{for every} \quad t\in M(M,\delta), \tag{3.4}$$

for some positive constant K_2 independent of t and q. Therefore (with suitable $c>0$, independent of t and q)

$$\begin{aligned}|g^{(1)}(t;\rho,P)|&\leqq c|t-\lambda_1|^{\beta}\sum_{j=2}^{n}\int_0^{L_t}|u(s)-\lambda_1|^{-\beta-1}|g^{(j)}(u(s);\rho,P-E_{kj})|ds\\&\leqq|t-\lambda_1|^{\beta}\sum_{j=2}^{n}\frac{cK_1^{2q+1}}{\Gamma(q+2+\beta)}\int_0^{L_t}|u(s)-\lambda_1|^{q}ds.\end{aligned}$$

Observing $|u(s)-\lambda_1|\leqq s$ and (3.4), we obtain

$$|g^{(1)}(t;\rho,P)|\leqq(n-1)\frac{K_1^{2q+1}cK_2}{(q+1)\Gamma(q+2+\beta)}|t-\lambda_1|^{q+1+\beta}\,,$$

hence (3.1) (for $k=1$) follows, if we take

$$K_1\geqq(n-1)cK_2\geqq\frac{n-1}{q+1}cK_2\,.$$

For $k=2,\cdots,n$, we have

$$|(t-\lambda_k)^{\rho-\lambda_k'}(u-\lambda_k)^{\lambda_k'-\rho-1}|\leqq K_8 \quad \text{for every} \quad u,t\in M(K,\delta)\,,$$

with a suitable constant $K_8>0$ (independent of u, t), hence

$$|g^{(k)}(t;\rho,P)|\leqq K_8K_1^{2q+1}\int_0^{L_t}\left\{\frac{|u(s)-\lambda_1|^{q+\beta}}{\Gamma(q+1+\beta)}+(n-2)\frac{|u(s)-\lambda_1|^{q+1+\beta}}{\Gamma(q+2+\beta)}\right\}ds$$
$$\leqq K_8K_1^{2q+1}K_2\left\{1+\frac{n-2}{q+2+\beta}|t-\lambda_1|\right\}\frac{|t-\lambda_1|^{q+1+\beta}}{\Gamma(q+2+\beta)}\,,$$

and (3.1) (for $k=2,\cdots,n$) follows if

$$K_1\geqq K_8K_2\left\{1+\frac{n-2}{2}K\right\}\geqq K_8K_2\left\{1+\frac{n-2}{q+2+\beta}|t-\lambda_1|\right\}\,.$$

If we now fix P with $|P|=2q+2$, then by quite the same arguments one can prove (3.2), (3.3) (with $q+1$ in place of q). This completes the proof.

Remark 3.1. It is easily seen from the foregoing proof that the estimates (3.1), (3.2), (3.3) remain valid for t in arbitrary compact sets not having $\lambda_2,\cdots,\lambda_n$ as boundary points and having the property that every t can be connected to λ_1 by a path for which the quotient of the length of the path divided by $|t-\lambda_1|$ remains bounded. Therefore, if we continue $g(t;\rho,P)$ across the cut $\arg(t-\lambda_k)=\eta$ (for arbitrarily fixed k, $1\leqq k\leqq n$) in the positive sense, say, then the estimates (with somewhat enlarged K_1) hold for the analytic continuation of $g(t;\rho,P)$ for every t with

$$|t-\lambda_k|=\delta\,,\qquad \eta<\arg(t-\lambda_k)<\eta+2\pi\,.$$

This will be of use later.

As a consequence of Proposition 3, we obtain

Theorem 1. *In addition to our basic assumptions, let* ρ *satisfy* (1.2). *Then the series*

$$\sum_P\{(t-\lambda_1)^{\lambda_1'-\rho}g(t;\rho,P)\}A^P$$

converges absolutely and uniformly for $t\in M(K,\delta)$ (*with arbitrarily fixed* K, $\delta>0$)

and

$$|a_{jk}| \leqq c\,, \quad j \neq k\,, \quad 1 \leqq j, k \leqq n\,,$$

(*with arbitrarily given* $c>0$). *Furthermore,*

$$y(t;\rho, A)=\sum_P g(t;\rho, P)A^P\,. \tag{3.5}$$

Proof. To establish the convergence of (3.5), let

$$\|A\|_\infty = \max_{j\neq k} |a_{jk}|\;;$$

then

$$|A^P| \leqq \|A\|_\infty{}^p \leqq c^p\,, \quad p=|P|\,.$$

For fixed non-negative integer p, the number of index sets P satisfying $|P|=p$ is not larger than the number of index sets satisfying $0\leqq p_{jk}\leqq p$, $1\leqq j$, $k\leqq n$, $j\neq k$, and the latter is equal to

$$(p+1)^{n(n-1)}\,.$$

Hence, according to Proposition 3,

$$\sum_P |t-\lambda_1|^{-\beta}|g(t;\rho, P)|\,|A^P| \leqq \sum_{p=0}^{\infty} c^p \sum_{|P|=p} |g(t;\rho, P)|\,|t-\lambda_1|^{-\beta}$$
$$\leqq \sum_{p=0}^{\infty} c^p(p+1)^{n(n-1)}c_1{}^p/\Gamma(1+p/2)\,,$$

for some sufficiently large positive constant c_1, depending upon K and δ, but independent of t and p, as long as $t\in M(K,\delta)$. This shows the convergence as stated. If we for the moment make (3.5) the definition of $y(t;\rho, A)$, then it follows that $y(t;\rho, A)$ is analytic with respect to t in the cut t-plane, and for $\operatorname{Re}(\rho-\lambda_1')>-1$ we obtain from (1.11), (1.12) that $y(t;\rho, A)$ satisfies (1.4) (if $y^{(k)}(t)$ is the k^{th} component of $y(t;\rho, A)$, $1\leqq k\leqq n$). This in turn shows that $y(t;\rho, A)$ is a solution of (1.1), and according to Proposition 1 it is of the form

$$y(t;\rho, A)=(t-\lambda_1)^{\rho-\lambda_1'}\{\delta_1/\Gamma(1+\rho-\lambda_1')+\operatorname{reg}_0(t-\lambda_1)\}$$

i.e. is the unique solution of (1.2) satisfying (1.3) for $k=1$. Using (2.4) resp. Remark 2.1, it is then easy to see that the restriction $\operatorname{Re}(\rho-\lambda_1')>-1$ may be omitted.

4. The nature of the singularities of the coefficient functions

While the behavior of $g(t;\rho, P)$ at $t=\lambda_1$ has been explained in Proposition 1, we are now going to investigate the nature of its singularities at the points

$\lambda_2, \cdots, \lambda_n$.

Proposition 4. *Additionally to our basic assumptions, let* ρ *satisfy* (1.2). *Then for arbitrarily fixed* μ, $2 \leqq \mu \leqq n$, *and every index matrix* P, *there exist unique vectors*

$$a_\mu(t;\rho,P) \quad \textit{and} \quad b_\mu(t;\rho,P)$$

being analytic in the cut plane, such that (with scalar $d_\mu(\rho,P)$*)*

$$a_\mu(t;\rho,P)=d_\mu(\rho,P)\delta_\mu+\mathrm{reg}_0\,(t-\lambda_\mu)\,, \tag{4.1}$$

$$b_\mu(t;\rho,P)=\mathrm{reg}\,(t-\lambda_\mu)\,, \tag{4.2}$$

$$g(t;\rho,P)=a_\mu(t;\rho,P)(t-\lambda_\mu)^{\rho-\lambda'_\mu}+b_\mu(t;\rho,P)\,. \tag{4.3}$$

Proof. Suppose that the existence of $a_\mu(t;\rho,P)$ and $b_\mu(t;\rho,P)$ with (4.1), (4.2), (4.3) had been shown. Then, if $\tilde{g}(t;\rho,P)$ denotes the analytic continuation of $g(t;\rho,P)$ across the μ^{th} cut in the positive sense, we obtain

$$\tilde{g}(t;\rho,P)-g(t;\rho,P)=(e^{2\pi i(\rho-\lambda_\mu')}-1)a_\mu(t;\rho,P)(t-\lambda_\mu)^{\rho-\lambda_\mu'}\,. \tag{4.4}$$

Since $e^{2\pi i(\rho-\lambda_\mu')}\neq 1$, according to (1.2), we may solve (4.4) for $a_\mu(t;\rho,P)$, which shows the uniqueness of $a_\mu(t;\rho,P)$, and $b_\mu(t;\rho,P)$.

For an existence proof, we may assume that additionally $\mathrm{Re}\,(\rho-\lambda_1')>-1$ (for the generalization to arbitrary ρ with (1.2) use (2.4)), and for these ρ we proceed by induction with respect to $|P|=p$.

For $P=0$, the Proposition holds for $a_\mu(t;\rho,0)=0$, $b_\mu(t;\rho,0)=g(t;\rho,0)$, hence we may assume it to hold for every index matrix of "modulus" p (with $p\geqq 0$ fixed). If P is arbitrarily given with $|P|=p+1$, then according to (1.12), for arbitrarily fixed k, $1\leqq k\leqq n$,

$$g^{(k)}(t;\rho,P)=-(t-\lambda_k)^{\rho-\lambda_k'}\sum_{j\neq k}\left\{c_j(t_0)+\int_{t_0}^{t}(u-\lambda_k)^{\lambda_k'-\rho-1}g^{(j)}(u;\rho,P-E_{kj})du\right\}\,,$$

where t_0 may be taken close to λ_μ (in the cut plane), and

$$c_j(t_0)=\int_{\lambda_1}^{t_0}(u-\lambda_k)^{\lambda_k'-\rho-1}g^{(j)}(u;\rho,P-E_{kj})du\,.$$

If $a_\mu^{(j)}(u;\rho,P-E_{kj})$, resp. $b_\mu^{(j)}(u;\rho,P-E_{kj})$ are the components of $a_\mu(u;\rho,P-E_{kj})$, resp. $b_\mu(u;\rho,P-E_{kj})$, then expanding them as power series in $u-\lambda_\mu$ and integrating termwise yields for $k\neq\mu$ and $j\neq k$, $1\leqq j\leqq n$

$$\int_{t_0}^{t}(u-\lambda_k)^{\lambda_k'-\rho-1}b_\mu^{(j)}(u;\rho,P-E_{kj})du=\mathrm{reg}\,(t-\lambda_\mu)\,,$$

$$\int_{t_0}^{t}(u-\lambda_k)^{\lambda_k'-\rho-1}(u-\lambda_\mu)^{\rho-\lambda_\mu'}a_\mu^{(j)}(u;\rho,P-E_{kj})du=(t-\lambda_\mu)^{\rho-\lambda_\mu'+1}\operatorname{reg}(t-\lambda_\mu)+\text{constant},$$

hence

$$g^{(k)}(t;\rho,P)=(t-\lambda_\mu)^{\rho-\lambda_\mu'+1}\operatorname{reg}(t-\lambda_\mu)+\operatorname{reg}(t-\lambda_\mu)(k\neq\mu).$$

Similarly, for $k=\mu$ and $j\neq\mu$, $1\leqq j\leqq n$

$$\int_{t_0}^{t}(u-\lambda_\mu)^{\lambda_\mu'-\rho-1}b_\mu^{(j)}(u;\rho,P-E_{kj})du=(t-\lambda_\mu)^{\lambda_\mu'-\rho}\operatorname{reg}(t-\lambda_\mu)+\text{constant},$$

and (compare (4.1))

$$\int_{t_0}^{t}(u-\lambda_\mu)^{\lambda_\mu'-\rho-1}a_\mu^{(j)}(u;\rho,P-E_{kj})(u-\lambda_\mu)^{\rho-\lambda_\mu'}du=\operatorname{reg}(t-\lambda_\mu),$$

hence

$$g^{(\mu)}(t;\rho,P)=(t-\lambda_\mu)^{\rho-\lambda_\mu'}\operatorname{reg}(t-\lambda_\mu)+\operatorname{reg}(t-\lambda_\mu).$$

Altogether, this shows (4.1), (4.2), and (4.3) for every P with $|P|=p+1$.

As a main result, we now show

Theorem 2. *Under our basic assumptions, let* ρ *additionally satisfy* (1.2). *Then for every* μ, $2\leqq\mu\leqq n$, *and every sufficiently small* $\delta>0$, *the series*

$$\sum_P a_\mu(t;\rho,P)A^P=e_\mu(t;\rho,A) \tag{4.5}$$

and

$$\sum_P b_\mu(t;\rho,P)A^P=f_\mu(t;\rho,A) \tag{4.6}$$

both are absolutely and uniformly convergent for

$$|t-\lambda_\mu|\leqq\delta, \qquad \|A\|_\infty\leqq c$$

(*with arbitrary* $c\geqq0$), *and*

$$e_\mu(t;\rho,A)=c_\mu(\rho,A)\delta_\mu+\operatorname{reg}_0(t-\lambda_\mu), \tag{4.7}$$

$$f_\mu(t;\rho,A)=\operatorname{reg}(t-\lambda_\mu), \tag{4.8}$$

where the scalar constant $c_\mu(\rho,A)$ *is given by the power series*

$$c_\mu(\rho,A)=\sum_P d_\mu(\rho,P)A^P \tag{4.9}$$

(*converging absolutely and uniformly for* $\|A\|_\infty\leqq c$).

Proof. For every t in the cut plane with $|t-\lambda_\mu|=\delta$ we have, according to (4.4), with $c>0$ sufficiently large (independent of t)

$$|a_\mu(t;\rho,P)|\leqq c\delta^{\mathrm{Re}(\lambda_\mu'-\rho)}\{|\tilde{g}(t;\rho,P)|+|g(t;\rho,P)|\},$$

and from Proposition 3 and Remark 3.1 we obtain for t as above (with $p=|P|$)

$$|g(t;\rho,P)|\leqq\frac{K^p}{\Gamma(1+p/2)},$$

$$|\tilde{g}(t;\rho,P)|\leqq\frac{K^p}{\Gamma(1+p/2)},$$

with sufficiently large constant $K>0$, independent of P, for every index matrix P, Hence (if $K\geqq c\delta^{\mathrm{Re}(\lambda_\mu'-\rho)}$)

$$|a_\mu(t;\rho,P)|\leqq\frac{2K^{p+1}}{\Gamma(1+p/2)}$$

for $|t-\lambda_\mu|=\delta$, and, due to the Maximum Modulus Theorem, the same estimate holds for $|t-\lambda_\mu|\leqq\delta$. This implies the uniform and absolute convergence of (4.5) (compare the proof of Theorem 1), and therefore (4.7) follows from (4.1). If we take $t=\lambda_\mu$, then (4.9) follows from (4.5), (4.7), and (4.1). Finally, to establish the convergence of (4.6) (as stated in the Theorem), we solve (4.3) for $b_\mu(t;\rho,P)$ and estimate $b_\mu(t;\rho,P)$, first for $|t-\lambda_\mu|=\delta$, and then apply again the Maximum Modulus Theorem. Due to the uniform convergence, (4.8) is then obvious.

Remark 4.1. Due to (4.3), we obtain, using (4.5), (4.6), and (3.5) (under the assumptions of Theorem 2)

$$y(t;\rho,A)=e_\mu(t;\rho,A)(t-\lambda_\mu)^{\rho-\lambda_\mu'}+f_\mu(t;\rho,A) \tag{4.10}$$

for $|t-\lambda_\mu|\leqq\delta$, and analogous to (4.4), one can express $e_\mu(t;\rho,A)$ in terms of $y(t;\rho,A)$ and its analytic continuation across the μ^{th} cut (which exists, according to (4.10)). Therefore $e_\mu(t;\rho,A)$ is analytic for t in the cut plane, and so is $f_\mu(t;\rho,A)$, and (4.10) then holds for every t in the cut plane.

5. Power series expansions for the Birkhoff invariants

Additionally to our basic assumptions, let again ρ satisfy (1.2). If $y_1(t),\cdots,y_n(t)$ are the vector solutions of (1.1) satisfying (1.3), then it has been shown in [1], [2] (compare also the Introduction) that there exist unique constants $c_{\mu j}$ with

$$y_j(t)=y_\mu(t)c_{\mu j}+\mathrm{reg}\,(t-\lambda_\mu)\quad(1\leqq j,\ \mu\leqq n)\,. \tag{5.1}$$

For $j=1$, we find (since $y_1(t)=y(t;\rho,A)$):

Proposition 5. *In addition to our basic assumptions, let* ρ *satisfy* (1.2). *Then for every* μ, $2\leqq\mu\leqq n$,

(5.2) $$y_\mu(t)c_{\mu 1}=(t-\lambda_\mu)^{\rho-\lambda_\mu'}e_\mu(t;\rho,A)$$
$$=(t-\lambda_\mu)^{\rho-\lambda_\mu'}\sum_P a_\mu(t;\rho,P)A^P\,,$$

(5.3) $$c_{\mu 1}/\Gamma(1+\rho-\lambda_\mu')=c_\mu(\rho,A)=\sum_P d_\mu(\rho,P)A^P\,.$$

Proof. From (5.1), resp. (4.10), we conclude that both $y_\mu(t)c_{\mu 1}$ and $(t-\lambda_1)^{\rho-\lambda_\mu'}\times e_\mu(t;\rho,A)$ are equal to

$$\{\tilde{y}(t;\rho,A)-y(t;\rho,A)\}(e^{2\pi i(\rho-\lambda_\mu')}-1)^{-1}\,,$$

if $\tilde{y}(t;\rho,A)$ denotes the analytic continuation of $y(t;\rho,A)$ (in the positive sense) across the μ^{th} cut. Hence (5.2) follows, and multiplying both sides of (5.2) by $(t-\lambda_1)^{\lambda_\mu'-\rho}$, taking the limit $t\to\lambda_\mu$ and observing (1.3), (4.7), and (4.9), we obtain (5.3).

Remark 5.1. Proposition 5 shows how the constants $c_{\mu 1}(2\leqq\mu\leqq n)$ can be written as convergent power series in A (i.e. in the variables a_{jk}, $j\neq k$, $1\leqq j$, $k\leqq n$). Similar series can easily be derived for $c_{\mu\nu}$ $(\nu\neq\mu)$ (compare the discussion in Section 1), and by definition $c_{\mu\mu}=1$, $1\leqq\mu\leqq n$. Since the quantities

$$\lambda_1,\cdots,\lambda_n\,,\qquad \lambda_1',\cdots,\lambda_n'\,,\qquad c_{\mu\nu}\,,\quad 1\leqq\mu,\nu\leqq n$$

form a *complete system of Birkhoff invariants* for (0.1) (c.f. [1]), we therefore may think of calculating these invariants by means of the power series derived here. It is obvious that in principle the coefficients in (5.3) only depend upon $\lambda_1,\cdots,\lambda_n$, $\lambda_1',\cdots,\lambda_n'$, however to calculate them, there is, so far, no other way than first calculating the coefficient functions via the recursion formulas (1.11), (1.12), and then "evaluating" them at the points $\lambda_2,\cdots,\lambda_n$. In a second part of this paper, we will try and find more direct means to calculate the the coefficients; for example, one may do better by considering only some elements of A as variables, in order to find simpler series.

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