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ON VARIOUS TYPES OF BARRELLEDNESS OF A TOPOLOGICAL ALGEBRA

By

JOHN ARAHOVITIS

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ABSTRACT. This note aims at giving characterizations for certain classes of topological algebras, namely for the m-barrelled and countably m-barrelled algebras, by means of appropriate dual subsets and in analogy with the situation one encounters in the case of barrelled topological vector spaces. It also gives a characterization for spectrally barrelled algebras, using a structural property of these algebras and in analogy with the case of countably barrelled spaces. Furthermore, we examine the role of the above classes of topological algebras within the frame of certain problems appearing in the general theory of topological algebras.

Introduction

Spectrally barrelled algebras were defined and studied by [17], [18]. They have been proved to be, for many purposes in the whole theory, a convenient class of topological algebras (cf. A. Mallios: *"On a convenient category of topological algebras"* I, II, Prakt. Akad. Athēnōn, **50** (1975), 454–477; **51** (1976), 245–263), respecting the most important properties shared by their genuine subclasses of m-barrelled and barrelled algebras. The purpose of this note is, on one hand, to point out some further similarities in the behavior of m-barrelled and spectrally barrelled algebras (cf. Proposition 2.3, Corollary 2.3) and on the other, to confirm a new their importance, by showing that in many cases (e.g. open mapping-closed graph theorems, coincidence of $\mathfrak{M}(E)$ and $\mathfrak{M}(\hat{E})$) they can adequately replace the classical topological algebras (cf. Propositions 2.4, 2.6 and Corollary 2.1). The above is essentially the content of Section 2, while Section 1 points towards the direction of the Silov's point of view, i.e. the characterization of the structure ("geometry") of an algebra by means of topological properties of its spectrum. This is the aim of Theorem 1.3, which leads to a lacking geometric definition, since spectrally barrelled algebras are defined "spectrally", by isolating a crucial spectral property of m-barrelled algebras, while Theorem 1.1, concerning m-barrelled algebras, is antipodal to the above, since these algebras are defined geometrically and thus it settles a question raised in [18]. Furthermore, a characterisation of countable m-barrelledness, introduced by [10], is given in Theorem

1.2 which is the analogon of ([8], Theorem 1, p. 131) for countable barrelledness. This characterization of countable m-barrelledness permits to answer affirmatively a conjecture of [10], p. 186 (cf. Proposition 1.1). Also, as a consequence of the above, we obtain a class of countably m-barrelled algebras, for which, the normed algebras appearing in their Arens-Michael decomposition, are complete (cf. Corollary 2.6, 2.7, 2.8).

Definitions and Notations

A *topological algebra* E is an algebra (=associative, complex linear algebra) and a Hausdorff topological vector space, such that ring multiplication is separately continuous.

A topological algebra is called *locally convex, metrizable barrelled, σ-complete, Fréchet, Pták, nuclear,* if the underlying topological vector space is such, respectively.

Accordingly to Arens, a locally convex algebra E is *locally m-convex* (locally multiplicatively convex) if it has a local basis at 0 consisting of closed, circled and *m-convex* (=idempotent and convex) subsets of E.

A topological algebra is called *m-barrelled* if every *m-barrel* (=absorbing, balanced, convex indempotent and closed subset) is a 0-neighborhood.

A topological algebra E is called *m-infrabarrelled* if it is a locally convex algebra such that every *m-bornivorous* (=a subset of E that absorbs every idempotent and bounded (=m-bounded) subset of it), m-barrel is a 0-neighborhood.

Accordingly to [10], a topological algebra is called *countably m-barrelled* if it is locally convex and every m-barrel which is the countable intersection of closed, circled and convex 0-neighborhoods, is itself a 0-neighborhood.

The *spectrum* of a topological algebra E, is the set of all non-zero, continuous, scalar-valued morphisms (=characters) of E, denoted (if it is not empty) by $\mathfrak{M}(E)$ and carrying the relative topology from the weak dual E_s' of E.

Now, a topological algebra is *spectrally barrelled* [18], if every (weakly) bounded subset of $\mathfrak{M}(E)$ is equicontinuous.

The *completion* $\hat{E}$ of a topological algebra E with (jointly) continuous ring multiplication, is the completion of the underlying topological vector space and $\hat{E}$ is a (complete) topological algebra with continuous multiplication. Accordingly to [21], a locally convex algebra E with continuous multiplication is called *B-complete* (resp. *B_r-complete*) if every continuous (resp. continuous one-to-one) and almost open (algebra) morphism from E onto any topological algebra, is open.

The *Gel'fand map* of a topological algebra E, is the map:

$$E \to C_c(\mathfrak{M}(E)) : x \mapsto \hat{x} : \hat{x} : \mathfrak{M}(E) \to \boldsymbol{C} : f \mapsto \hat{x}(f) := f(x) ,$$

where c denotes the topology of compact convergence on the algebra of all continuous, complex-valued mappings on $\mathfrak{M}(E)$.

The topological algebra E is called *full* if the Gel'fand map is surjective.

The spectrum $\mathfrak{M}(E)$ of a topological algebra E, is called *locally equicontinuous,* if every element in $\mathfrak{M}(E)$ has an equicontinuous neighborhood.

Given a completely regular space X, we call X a *Nachbin-Shirota* space if any closed and *relatively precompact* subset A of X (i.e., such that every element of $C(X)$ is bounded on A), is compact (cf. [20], [23]). Since X is equal within a homeomorphism to the spectrum of $C_c(X)$ (cf. e.g. [16], p. 478, 4.), we can say that X is a Nachbin-Shirota space if, and only if, the (weakly) bounded and the (weakly) relatively compact subsets of the spectrum of $C_c(X)$ coincide (cf. [17], p. 104, 2. or [24], p. 272, 3.).

Finally we recall that, accordingly to Arens, a topological space is *hemicompact,* if there exists a countable family of compact subsets, such that every compact subset is contained in some member of the family.

1. Characterizations

We begin with the following:

Theorem 1.1. *A locally convex algebra E is m-barrelled if, and only if, the bounded subsets of its weak dual, whose polars are idempotent subsets of E, are equicontinuous.*

Proof. Let B be an m-barrel in E. Then its polar B^0 is weakly bounded and its bipolar $B^{00}=B$ is idempotent. Thus B^0 is equicontinuous and $B^{00}=B$ is therefore a 0-neighborhood in E. It follows that E is m-barrelled, by the very definition of an m-barrelled algebra.

Conversely, let E be m-barrelled and B a bounded subset of its weak dual E_s' such that B^0 be idempotent. Then, B^0 is also balanced, absorbing, convex and $\sigma(E, E')$-closed and hence closed in the initial topology of E, that is, B^0 is an m-barrel in E and hence a 0-neighborhood. Thus, $B \subseteqq B^{00}$ is equicontinuous and the proof is completed.

Concerning countably m-barrelled algebras we have, on the other hand, the following characterization:

Theorem 1.2. *E is countably m-barrelled if, and only if, every bounded subset B of E_s' which is the countable union of equicontinuous subsets and with* *idempotent polar B^0, is itself equicontinuous.*

Proof. Suppose E is countably m-barrelled and let B be a bounded subset of E_s' such that:

i) $B=\bigcup_1^\infty H_n$, where $H_n\subseteqq E_s'$ are equicontinuous for every $n=1, 2, \cdots$.

ii) B^0 is idempotent.

We show that B is equicontinuous. In fact, B^0 is also balanced, absorbing, convex $\sigma(E, E')$-closed and hence closed in the initial topology of E, that is B^0 is an m-barrel and

$$B^0=\bigcap_1^\infty H_n^0,$$

where H_n^0 is balanced, convex $\sigma(E, E')$-closed and hence closed in the initial topology of E, for every $n=1, 2, \cdots$. Besides, H_n^0 is a 0-neighborhood, since H_n is equicontinuous for every $n=1, 2, \cdots$. Therefore, B^0 is a 0-neighborhood, by hypothesis on E and thus $B\subseteqq B^{00}$ is equicontinuous.

Conversely, suppose that now B is an m-barrel in E with $B=\bigcap_1^\infty V_n$, where V_n are closed, balanced and convex 0-neighborhoods, for every $n=1, 2, \cdots$. We show that B is a 0-neighborhood. B^0 is bounded. Besides, since

$$\bigcup_1^\infty V_n^0\subseteqq B^0$$

(cf. the proof of [8], Theorem 1, p. 131), it follows that $\bigcup_1^\infty V_n^0$ is a bounded subset of E_s', where each V_n^0 is equicontinuous. The relation:

$$\left(\bigcup_1^\infty V_n^0\right)^0=\bigcap_1^\infty V_n^{00}=\bigcap_1^\infty V_n=B,$$

shows that $\left(\bigcup_1^\infty V_n^0\right)^0$ is idempotent and the hypothesis, that $\bigcup_1^\infty V_n^0$ is equicontinuous. Therefore B, being the polar of the equicontinuous $\bigcup_1^\infty V_n^0$, is a 0-neighborhood and the proof is completed.

Proposition 1.1. *A metrizable countably m-barrelled algebra E is m-barrelled.*

Proof. Let B be an m-barrel in E. Then, B^0 is bounded in E_s'. Since E is metrizable, $E'=\bigcup_1^\infty V_n^0$, where $(V_n)_{1\leq n\leq\infty}$ is a local basis at 0 in E. Now, $B^0=\bigcup_1^\infty B_n$, where $B_n\equiv B^0\cap V_n^0\subseteqq V_n^0$ is equicontinuous. Besides, $B^{00}=B$ is idempotent in E and thus, since E is countably m-barrelled by hypothesis, the above Theorem 1.2 applies, so that B^0 is equicontinuous. A fortiori then, $B=B^{00}$ is a 0-neighborhood, being the polar of the equicontinuous B^0 and the proof is completed.

Now, for spectrally barrelled algebras we have:

Theorem 1.3. *A locally convex algebra E is spectrally barrelled if, and only if, every m-barrel of E which is the polar of a bounded subset of its spectrum, is a* 0*-neighborhood.*

Proof. Let E be spectrally barrelled and B a bounded subset of $\mathfrak{M}(E)$. Its polar B^0 is a balanced, convex, closed, absorbing and idempotent subset of E, i.e. an m-barrel of it. Since, by definition of E, B is also equicontinuous, it follows that B^0 is a 0-neighborhood (cf. e.g. [5], Proposition 4, p. 220).

Conversely, let B be a bounded subset of $\mathfrak{M}(E)$. To show that E is spectrally barrelled, we have to show that B is also equicontinuous. Consider B^0. Then, B^0 is an m-barrel and therefore a 0-neighborhood. Thus, $B \subseteqq B^{00}$ is equicontinuous and the proof is completed.

By the above, we obtain a uniform way in defining barrelled, m-barrelled, countably m-barrelled and spectrally barrelled algebras and in characterizing these algebras with the help of dual subsets.

To give, now, a characterization of spectral barrelledness in the particular case E is a unital metrizable algebra, we need the following:

It is well known (cf. [17], p. 100) that if $\mathfrak{M}(E)$ is the spectrum of a topological algebra E and $(U_\alpha)_{\alpha \in I}$ is a local base of E, then

$$\mathfrak{M}(E) = \bigcup_{\alpha \in I} \mathfrak{M}_\alpha(E)\,, \tag{1.1}$$

where $\mathfrak{M}_\alpha(E) \equiv \mathfrak{M}(E) \cap U_\alpha{}^0$, $U_\alpha{}^0$ the polar of U_α in the weak dual E_s' of E. Now, if E is unital, $\mathfrak{M}(E)$ is closed in E_s' (cf. [19], Lemma 6.2(b), p. 25) and clearly so is also $\mathfrak{M}_\alpha(E)$ for every $\alpha \in I$. Since $\mathfrak{M}_\alpha(E) \subseteqq U_\alpha$, it follows that $\mathfrak{M}_\alpha(E)$ is equicontinuous and therefore, by the Alaoglu-Bourbaki theorem, it is compact.

Lemma 1.1. *The following assertions are equivalent:*

i) *Every compact subset of $\mathfrak{M}(E)$ is equicontinuous.*

ii) *For every compact subset K of $\mathfrak{M}(E)$, $\exists\, \alpha \in I$: $K \subseteqq \mathfrak{M}_\alpha(E)$.*

Proof. It is clear from the preceding, that ii)$\Rightarrow$i). i)$\Rightarrow$ii). Since K is an equicontinuous subset of $\mathfrak{M}(E)$ (cf. [15], Theorem 3.1), it follows that K^0 is a 0-neighborhood and thus there is a U_α such that $U_\alpha \subseteqq K^0$. Consequently $K \subseteqq K^{00} \subseteqq U_\alpha{}^0$ and therefore $K \subseteqq \mathfrak{M}_\alpha(E)$. This completes the proof.

Lemma 1.2. *Let E be a unital metrizable algebra. Then, the spectrum of E is a Nachbin-Shirota space and the algebra $C_c(\mathfrak{M}(E))$ is barrelled.*

Proof. Since E is metrizable, it has a countable local base $(U_n)_{1\leq n\leq\infty}$. Thus, since $\mathfrak{M}(E)=\bigcup_1^\infty \mathfrak{M}_n(E)$ (cf. (1.1)), it is σ-compact and being completely regular, it is a regular Lindelöf space (cf. [9], Y(b), p. 172). So it is paracompact. But a paracompact space is a Nachbin-Shirota space as it follows from ([20], p. 472), so that $C_c(\mathfrak{M}(E))$ is barrelled by the Nachbin-Shirota theorem (cf. [20], Theorem 1, or [23], Theorem 1). This completes the proof.

Theorem 1.4. *Let E be a full unital metrizable algebra. Then, the following assertions are equivalent:*

i) *E is spectrally barrelled.*

ii) *The Gel'fand map of E is continuous.*

iii) *$\mathfrak{M}(E)$ is hemicompact with respect to $(\mathfrak{M}_n(E))_{1\leq n\leq\infty}$.*

Proof. ii)$\Leftrightarrow$iii). The assertion follows from the above Lemma 1.1 and ([15], Theorem 3.1, p. 305).

i)$\Leftrightarrow$ii). The Gel'fand map of every spectrally barrelled algebra is continuous (cf. [17], Lemma 2.1, p. 105). Conversely, let B be a bounded subset of $\mathfrak{M}(E)$. Then $\bar{B}$ is bounded too. Since E is full, the relation

$$\bigcup_{f\in\bar{B}} f(x)=\bigcup_{f\in\bar{B}}\hat{x}(f)=\bigcup_{f\in\bar{B}} F_f(\hat{x})$$

shows that the image of $\bar{B}$ under the evaluation map

$$\omega:\mathfrak{M}(E)\rightarrow\mathfrak{M}(C_c(\mathfrak{M}(E))): f\mapsto F_f: F_f(\phi)=\phi(f)\ ,$$

is a bounded subset of the spectrum of $C_c(\mathfrak{M}(E))$. By applying Lemma 1.2, we obtain that $C_c(\mathfrak{M}(E))$ is barrelled so that $\omega(\bar{B})$ is compact and thus so is $\bar{B}=\omega^{-1}(\omega(\bar{B}))$, since ω is a homeomorphism. By hypothesis the Gel'fand map is continuous, so that $\bar{B}$ is equicontinuous and a fortiori $B\subseteqq\bar{B}$ too. This completes the proof.

Note that if E is unital metrizable, then ii)$\Rightarrow C_c(\mathfrak{M}(E))$ metrizable (cf. [24], Theorem A).

2. Applications

Given a locally m-convex algebra, [13], p. 173 raised the question whether or not $\mathfrak{M}(E)$ and $\mathfrak{M}(\hat{E})$ coincide in general as topological spaces, where $\hat{E}$ denotes the completion of E. This is answered in the negative for the general case (cf. [6], p. 90; [3], p. 210), while local equicontinuity of $\mathfrak{M}(E)$ has been proved (cf. [14], p. 103) to be a sufficient condition in order that the relation

$$\mathfrak{M}(E)=\mathfrak{M}(\hat{E}) \tag{2.1}$$

to hold true within a homeomorphism.

The following proposition gives an equivalent condition to local equicontinuity of $\mathfrak{M}(E)$, for a certain class of topological algebras. That is we have:

Proposition 2.1. *Let E be a unital metrizable algebra. Then, the following assertions are equivalent:*

i) *$\mathfrak{M}(E)$ is locally equicontinuous.*

ii) *$\mathfrak{M}(E)$ is locally compact.*

Proof. Since always i)⇒ii) (cf. [17], Theorem 3.2), the proof will be completed if we show that ii)⇒i). In fact since $\mathfrak{M}(E)=\bigcup_1^\infty \mathfrak{M}_n(E)$ (cf. the proof of Lemma 1.2), it is σ-compact and being by hypothesis locally compact, it is hemicompact (cf. e.g. [4], Ex. 3, p. 241). Let $f\in\mathfrak{M}(E)$. Then, by local compactness, there is a compact neighborhood V of f and, by hemicompactness, there is an n such that the equicontinuous $\mathfrak{M}_n(E)$ contains V. Then, every $f\in\mathfrak{M}(E)$ has an equicontinuous neighborhood.

Now, we proceed to derive another topological property of $\mathfrak{M}(E)$ which, for a certain class of topological algebras including spectrally barrelled algebras (hence m-barrelled algebras), ensures the validity of (2.1) within a homeomorphism. That is, we have the following:

Proposition 2.2. *Let E be a locally convex algebra, with continuous multiplication, continuous Gel'fand map and first countable spectrum $\mathfrak{M}(E)$. Then, $\mathfrak{M}(E)$ is homeomorphic to $\mathfrak{M}(\hat{E})$.*

Proof. It is well-known that the relation $\mathfrak{M}(E)=\mathfrak{M}(\hat{E})$ holds true within a continuous bijection of the first space onto the second and we proceed to show that, under the above hypothesis, the map

$$\mathfrak{M}(E)\to\mathfrak{M}(\hat{E}):\ f\mapsto\tilde{f}$$

is continuous too, where $\tilde{f}$ denotes the extension of f on $\hat{E}$. Let $f_n\to f$ be a weakly convergent sequence of $\mathfrak{M}(E)$. We must show that $\tilde{f}_n\xrightarrow{s}\tilde{f}$. The set H formed by the f_n and the limit f is a compact subset of $\mathfrak{M}(E)$. Therefore it is equicontinuous by ([15], Theorem 3.1). Then, the set $\tilde{H}$ formed by the extensions of the elements of H, is also equicontinuous (cf. [2], § 2, No. 2, Proposition 4). Now, the sequence $\tilde{f}_n\in\tilde{H}$ converges to $\tilde{f}\in\tilde{H}$, if we equip $\tilde{H}$ with the topology of simple convergence on the finite subsets of E, since $\tilde{f}_n$ coincide with f_n on E. But since E is dense in $\hat{E}$, the sequence $\tilde{f}_n$ will converge to $\tilde{f}$ also if we consider on $\tilde{H}$ the topology of simple convergence on the finite subsets of $\hat{E}$, since $\tilde{H}$ is equicontinuous and the above two topologies coincide on $\tilde{H}$ (cf. e.g. [2], § 2, No. 4, Theorem 1) and this completes the proof.

Corollary 2.1. *Let E be a spectrally barrelled locally m-convex algebra, with first countable spectrum $\mathfrak{M}(E)$. Then, $\mathfrak{M}(E)$ is homeomorphic to $\mathfrak{M}(\hat{E})$.*

Proof. The Gel'fand map of E is ccntinuous (cf. [17], Lemma 2.1), as well as ring multiplication of E (cf. [19], p. 10).

Concerning the completion $\hat{E}$ of a spectrally barrelled algebra E, we know (cf. [18], p. 160, Remark) that it is an algebra of the same kind, when multiplication is jointly continuous in E. The same is true for an m-barrelled locally convex algebra with continuous multiplication, as the following proposition shows:

Proposition 2.3. *The completion $\hat{E}$ of an m-barrelled locally convex algebra E with continuous multiplication, is a (complete) algebra of the same kind.*

Proof. Since E has continuous multiplication, $\hat{E}$ is a locally convex algebra (with continuous multiplication). Let B be a bounded subset of $(\hat{E})_s'$, such that B^0 is an idempotent subset of $\hat{E}$. Then ${}^t j(B)$ is a bounded subset of E_s', where ${}^t j$ is the transpose of the canonical imbedding $j: E \to \hat{E}$, and $({}^t j(B))^0 = j^{-1}(B^0)$ is idempotent in E, since the inverse image of an idempotent subset, under a morphism, is an idempotent subset (cf. [19], Lemma 1.4 a), p. 9). Thus, by Theorem 1.1, ${}^t j(B)$ is an equicontinuous subset of E', since E is, by hypothesis, m-barrelled. But ${}^t j(B) = \{\tilde{f} \circ j \mid \tilde{f} \in B\}$ and therefore the extensions of the elements of this set, i.e. the elements of B, form an equicontinuous subset of $(\hat{E})'$ (cf. [2], § 2, No. 2, Proposition 4). By Theorem 1.1, $\hat{E}$ is m-barrelled and the proof is completed.

Corollary 2.2. *The completion of an m-barrelled locally m-convex algebra is a (complete) algebra of the same kind.*

Proof. By ([19], p. 10) multiplication is jointly continuous.

Corollary 2.3. *Let X be a completely regular (Hausdorff) k-space which is also a Nachbin-Shirota space and E a unital complete nuclear algebra with continuous multiplication. Then, if E is moreover m-barrelled (resp. spectrally barrelled), the algebra $C_c(X, E)$ is a complete m-barrelled (resp. spectrally barrelled) algebra with continuous multiplication.*

Proof. Since X is as in the above hypothesis, the algebra $C_c(X)$ is a barrelled locally m-convex algebra (cf. [20], Theorem 1, or [23], Theorem 1 and [17], p.

104, 2.). Since furthermore, E is nuclear by hypothesis, it follows that

$$C_c(X) \underset{\varepsilon}{\otimes} E = C_c(X) \underset{\pi}{\otimes} E$$

(ε and π denote the biprojective and projective tensorial topology, respectively) (cf. [12], p. 23) and that this projective tensor product has also continuous ring multiplication. Now, by ([16], Lemma 4.1) we have

$$C_c(X, E) = C_c(X) \underset{\pi}{\hat{\otimes}} E$$

within a topological and algebraic isomorphism and thus, by ([1] Lemma 3.1) (resp. by [18], Lemma 4.3 and Remark p. 160), $C_c(X, E)$ is an m-barrelled (resp. spectrally barrelled) complete locally convex algebra with continuous multiplication, which finishes the proof.

We turn, now, to a discussion concerning the form the open-mapping and closed graph theorems take, when morphisms of topological algebras are considered. Thus, we have:

Proposition 2.4. *Let E be a locally m-convex algebra and F an m-barrelled algebra. Then, every surjective morphism $f\colon E \to F$ is almost open.*

Proof. Let U be a balanced, closed, m-convex 0-neighborhood in E. Then, $f(U)$ is an m-barrel in F. In fact, $f(U)$ is balanced and m-convex; it is also absorbing, since f is surjective and U is absorbing. Therefore, $\overline{f(U)}$ is a closed, balanced, absorbing and m-convex subset of F, i.e. an m-barrel and hence a 0-neighborhood, by hypothesis on F. This finishes the proof.

Corollary 2.4. *Let E be a locally m-convex algebra and F a σ-complete. m-infrabarrelled (resp. a metrizable countably m-barrelled) algebra. Then, every surjective morphism $f\colon E \to F$ is almost open.*

Proof. It follows by ([1], Lemma 1.1) (resp. by Proposition 1.1) that σ-completeness (resp. metrizability) of F, implies m-barrelledness of F, F being m-infrabarrelled (resp. countably m-barrelled) by hypothesis.

Corollary 2.5. *Let E be a B-complete (resp. B_r-complete) locally m-convex algebra and F an m-barrelled locally convex algebra. Then, every continuous surjective (resp. bijective) morphism $f\colon E \to F$ is a strict morphism (resp. an isomorphism).*

Proof. By the above Proposition 2.4, f is almost open and thus, by definition of a B-complete (resp. B_r-complete) algebra, it is a strict morphism (resp. an isomorphism).

We comment that since every Fréchet algebra is a B-complete algebra (cf. [21], p. 200, (2)), the above corollary applies when E is a Fréchet locally m-convex algebra.

Proposition 2.5. *Let* $(E, (q_\alpha)_{\alpha\in I})$ *be a countably m-barrelled locally m-convex algebra. Then* $E/q_\alpha^{-1}(0)$ *is m-barrelled, where* q_α *is the gauge of an m-convex, balanced* 0*-neighborhood* V_α.

Proof. We shall prove that $E_\alpha \equiv E/q_\alpha^{-1}(0)$ is countably m-barrelled. Then, the m-barrelledness of E_α will follow from Proposition 1.1. Let $T=\bigcap_1^\infty V_n$ be an $m-N$-barrel in E_α. Since the map $\phi_\alpha\colon E\to E_\alpha$ is a continuous morphism,

$$\phi_\alpha^{-1}(T)=\phi_\alpha^{-1}\left(\bigcap_1^\infty V_n\right)=\bigcap_1^\infty \phi_\alpha^{-1}(V_n)$$

is an $m-N$-barrel in E (cf. [15], proof of Proposition 2.1, p. 303) and hence a 0-neighborhood in the countably m-barrelled algebra E. Therefore $(\phi_\alpha^{-1}(T))^0$ is a weakly compact subset of E'. Consider now the relation:

$$ {}^t\phi_\alpha^{-1}((\phi_\alpha^{-1}(T))^0)=(\phi_\alpha(\phi_\alpha^{-1}(T)))^0 \tag{2.2}$$

(cf. [7], Proposition 2 (a), p. 255). Since ϕ_α is onto, the second member of (2.2) is equal to T^0. The transpose ${}^t\phi_\alpha$ of ϕ_α is an injective strict morphism (cf. [7], Corollary, p. 264). Thus, the first member of (2.2) is compact and so is T^0. Furthermore T^0 is balanced and convex. Thus, T^0 is strongly bounded in E_α' (cf. [22], (5.1), p. 141). But since E_α is metrizable, E_α' has a fundamental sequence of strongly bounded subsets (cf. [11], (6), p. 394), which are the polars U_n^0 of a base $U_1\supseteqq U_2\supseteqq\cdots$ of a sequence of 0-neighborhoods. Thus $T^0\subseteqq U_\varepsilon^0$, from which we obtain $U_\varepsilon^{00}\subseteqq T^{00}$, i.e. $U_\varepsilon\subseteqq T$. Therefore T is a 0-neighborhood in E_α and E_α is countably m-barrelled. This completes the proof.

Corollary 2.6. *Let E be a Pták countably m-barrelled locally m-convex algebra. Then,* E_α *is a Banach algebra.*

Proof. We only have to prove that E_α is complete. The map ϕ_α is a continuous morphism onto E_α and also an almost open map (cf. Proposition 2.4), since E_α is m-barrelled by the above Proposition 2.5. Thus, E_α is a Pták algebra too (cf. [22], Corollary 2, p. 164) and hence complete (cf. [7], Proposition 3 (b), p. 299).

Corollary 2.7. *Let E be a Fréchet locally m-convex algebra. Then,* E_α *is a Banach algebra.*

Proof. E is Pták and barrelled (cf. [7], Proposition 3 (a), p. 299 and Corollary p. 214) and thus countably m-barrelled.

Corollary 2.8. *Let E be a Pták, countably m-barrelled locally m-convex algebra. Then, for the Arens-Michael decomposition of E, we have*: $E=\varprojlim E_\alpha$ (*cf.* [19], Theorem 5.1, p. 20).

Now, we have the following:

Proposition 2.6. *Let E be an m-barrelled algebra and F a locally m-convex algebra. Then, every morphism $f: E \to F$ is almost continuous.*

Proof. Let V be a balanced m-convex 0-neighborhood in F. Then, $f^{-1}(V)$ is a balanced, m-convex set and $\overline{f^{-1}(V)}$ a balanced, m-convex, closed set in E. $f^{-1}(V)$ is also absorbing, since V is. Thus, $\overline{f^{-1}(V)}$ is an m-barrel and hence a 0-neighborhood, by hypothesis on E. This completes the proof.

Corollary 2.9. *Let E be a σ-complete, m-infrabarrelled* (*resp. a metrizable, countably m-barrelled*) *algebra and F a locally m-convex algebra. Then, every morphism $f: E \to F$ is almost continuous.*

Proof. cf. the proof of Corollary 2.4.

Corollary 2.10. *Let E be an m-barrelled locally convex algebra, F a B_r-complete locally m-convex algebra and $f: E \to F$ a morphism. If the graph G of f is closed in the product space $E \times F$ and $\overline{f(E)}=F$, then, f is continuous.*

Proof. The above Proposition 2.6 and ([21], Theorem 5.1, p. 207).

Again, the conclusion of the above corollary holds true if F is a Fréchet algebra. Note also that the above corollary is valid without the density assumption, if F is a Pták locally m-convex algebra. In fact, in this case the proof follows from Proposition 2.6 and ([7], Proposition 8, p. 302).

Corollary 2.11. *Let E be an m-barrelled locally convex algebra, F a B_r-complete locally m-convex algebra with topology τ_0 and $f: E \to F$ a morphism. Let τ be a* (*Hausdorff*) *topology on F which is coarser than τ_0 and suppose that $\overline{f(E)}=F_{\tau_0}$. If f is continuous for τ on F, then, it is also continuous for the topology τ_0 on F.*

Proof. The graph of f is closed in $E \times F$, if F is equipped with τ and thus a fortiori if F is equipped with τ_0. The assertion follows from Corollary 2.10.

If F is a Pták locally m-convex algebra, the above corollary is valid without the density assumption.

Concerning the continuity of a morphism between two topological algebras, we also have the following, in the special case the morphism is the Gel'fand map:

Corollary 2.12. *Let E be a unital metrizable algebra and suppose that its spectrum $\mathfrak{M}(E)$ is locally compact. Then, the Gel'fand map of E is continuous and $C_c(\mathfrak{M}(E))$ is Fréchet.*

Proof. The assertion on the Gel'fand map follows from ([17], Theorem 3.2, p. 107) and Proposition 2.1. Concerning $C_c(\mathfrak{M}(E))$ ([24], Theorem 2, p. 267) applies, since a σ-compact, locally compact space is a hemicompact k-space (cf. [19], p. 25).

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Athens University
Institute of Mathematics
57, Solonos Street, Athens 10679
Greece