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AUTOMORPHISMS AND COVARIANT IRREDUCIBLE REPRESENTATIONS

By

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1. Introduction.

Let A be a separable simple C^*-algebra and α a continuous action of the circle group $\boldsymbol{T}$, the group $\boldsymbol{R}$ of real numbers, or a countable discrete abelian group on A by automorphisms. If α is not inner, it has been shown in [6] that the action of α on the spectrum $\hat{A}$ of A is not trivial. In this note, conversely, we want to study the problem of whether it has fixed points on $\hat{A}$.

In Section 2 we solve this problem affirmatively when the group is $\boldsymbol{T}$, i.e., we show that the system has an α-invariant pure state. The same method can be applied to cyclic groups of prime order but not those of non-prime order. More precisely, in Section 3, we show that for a periodic automorphism α of the C^*-algebra A, (A, α) has a covariant irreducible representation if and only if $\gamma(\alpha)=1$, where $\gamma(\alpha)$ is an outer invariant of α, being defined by Connes [2] as follows: When $p_0(\alpha)$ is the outer period of α and the $p_0(\alpha)$-th power of α is $Ad\, u$ with u a unitary multiplier of A, $\gamma(\alpha)$ is the complex number defined by $\alpha(u)=\gamma(\alpha)u$.

In Section 4 we study the case that the group is $\boldsymbol{R}$, and show the existence of a fixed point of $\hat{A}$ under α when the crossed product of A by α is not simple. And we give an example where α fixes no points of $\hat{A}$ and another example where α fixes points of $\hat{A}$, in case the crossed product is simple. The obstruction, in our example, for α having fixed points on $\hat{A}$ is of the same kind as $\gamma(\alpha)$ not being 1 in Section 3.

In Section 5 we restrict ourselves to the separable nuclear C^*-algebras which admit unique tracial states. Using Ocneanu's result [8], we show that, for an action α of a countable discrete abelian group on the C^*-algebra with trace τ, if α induces a free action on $\pi_\tau(A)''$, then (A, α) has covariant irreducible representations. In particular we show that for any automorphism α of a UHF algebra there exists a positive integer n such that (A, α^n) has covariant irreducible representations.

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2. Periodic one-parameter automorphism groups.

2.1. Theorem. *Let A be a separable C^*-algebra and α a continuous action of $\boldsymbol{T}$ on A. Suppose that α fixes each (closed two-sided) ideal of A. Then there exists an α-invariant pure state of A.*

Let B be the crossed product $A\times_\alpha \boldsymbol{T}$ of A by α and β the action $\hat{\alpha}$ of $\boldsymbol{Z}=\hat{\boldsymbol{T}}$ on B dual to α.

2.2. Lemma. *Each β_n with $n\neq 0$ is freely acting.*

Proof. First of all note [5, Lemma 3.2] that the strong Connes spectrum $\tilde{\boldsymbol{T}}(\beta)$ of β is $\boldsymbol{T}$, because each ideal of A is fixed under α. This implies, by the definition of $\tilde{\boldsymbol{T}}$, that for each non-zero β-invariant hereditary C^*-subalgebra D of B and for each non-empty open subset Ω of $\boldsymbol{T}$, the closed linear span of x^*yz with $x, y, z\in D$, $Sp_\beta(x)\subset\Omega$, $Sp_\beta(z)\subset\Omega$, is the whole D.

Suppose that β_n with $n>0$ is not freely acting (cf. [7], [9]). Then by definition there exists a non-zero β_n-invariant ideal I of B such that the Borchers spectrum of β_n restricted to I is trivial. It follows from [10] that for any $\varepsilon>0$ there exists a non-zero β_n-invariant hereditary C^*-subalgebra D of I such that $Sp(\beta_n|D)\subset\Omega_\varepsilon\equiv\{\exp i\theta : |\theta|<\varepsilon\}$. Choose ε such that there exists a nonempty open subset Ω of T with

$$\bigcup_{k=1}^{n}\bar{\Omega}^k\cap\Omega_\varepsilon=\phi,$$

where $\bar{\Omega}$ is the closure of Ω, and let D_1 be the β-invariant hereditary C^*-subalgebra generated by D, i.e., D_1 is the closed linear span of $\beta_k(D)B\beta_l(D)$ with $0\leqq k,\ l\leqq n-1$. Then denoting by $R(\Omega)$ the set of $x\in D_1$ with $Sp_{\beta_n}(x)\subset\Omega$, and by (e_λ) an approximate identity of D, for any λ we can find, $k_0, k_1, \cdots, k_n$ in $\{0, 1, \cdots, n-1\}$ with $k_0=0$, $\lambda_i\geqq\lambda$, and $x_i\in R(\Omega)$ such that

$$\beta_{k_0}(e_{\lambda_0})x_1\beta_{k_1}(e_{\lambda_1})\cdots x_n\beta_{k_n}(e_{\lambda_n})\neq 0\,.$$

Hence there are at least two i and j with $k_i=k_j$. Since we may assume that $Sp_{\beta_n}(e_\lambda)$ shrinks to $\{1\}$, we can conclude that $DR(\Omega_1)^kD\neq(0)$ with $k=|i-j|$ for any open set Ω_1 with $\Omega_1\supset\bar{\Omega}$. Thus $Sp(\beta_n|D)\cap\bar{\Omega}^k\neq\phi$, which is a contradiction.

2.3. Lemma. *Let $\{\sigma_n : n=1, 2, \cdots\}$ be a countable set of freely acting automorphisms of B. Then there exists a pure state ϕ of B such that $\|\phi-\phi\circ\sigma_n\|=2$ for $n=1, 2, \cdots$.*

Proof. See the proof of 2.1 in [6] as well as [7, 2.1] and [9, 6.6].

Note that we have not used the separability assumption so far. But we need it in the following.

Proof of 2.1. Let $\{u_k: k=1, 2, \cdots\}$ be a countable dense subset of the unitary group of the C^*-algebra $\tilde{B}$ obtained by adjoining an identity to B. By applying the above lemma to $\{Ad\, u_k \circ \beta_n: k, n=1, 2, \cdots\}$ we obtain a pure state ϕ of B such that ϕ is disjoint from $\phi\circ\beta_n$ for each non-zero n.

Let f be a state of $B\times_\beta \boldsymbol{Z}$ such that the restriction of f to B is ϕ. Let v be the canonical unitary multiplier of $B\times_\beta \boldsymbol{Z}$ which implements β_1 on B, and let e be a positive element of B with $\|e\|=1$ such that for a state ψ of B, $\psi(e)=1$ if and only if $\psi=\phi$. Then for any unitary u of $\tilde{B}$ and $n\neq 0$,

$$|f(uv^n)|=|f(uv^n e^k)|=|f(u\beta_n(e^k)v^n)|\leqq|\phi(u\beta_n(e^{2k})u^*)|^{1/2}$$

which converges to zero as k tends to infinity. Hence $f(xv^n)=0$, $x\in B$, $n\neq 0$, which implies that f is $\hat{\beta}$-invariant. Thus f is a pure state since the $\hat{\beta}$-invariant extension of ϕ is unique.

Now we use the duality for C^*-crossed products due to Takai (cf. [10, 7.9.3]). $B\times_\beta \boldsymbol{Z}$ with $\hat{\beta}$ is covariantly isomorphic to $A\otimes K$ with $\alpha\otimes Ad\,\tilde{\lambda}$, where K is the compact operators $K(L^2(\boldsymbol{T}))$ on $L^2(\boldsymbol{T})$, and $\tilde{\lambda}$ is the right regular representation of $\boldsymbol{T}$ on $L^2(\boldsymbol{T})$. Thus we may regard f as $\alpha\otimes Ad\,\tilde{\lambda}$-invariant pure state of $A\otimes K$. Let p be a minimal projection of K such that $f|A\otimes p\neq 0$, and $Ad\,\tilde{\lambda}(p)=p$, and define a state ψ_p of $A\otimes K$ by

$$\psi_p(x)=f((1\otimes p)x(1\otimes p))/f(1\otimes p)\,.$$

Then ψ_p is an $\alpha\otimes Ad\,\tilde{\lambda}$-invariant pure state of $A\otimes K$. Since $(1\otimes p)A\otimes K(1\otimes p)$ is naturally isomorphic to A, we may regard the restriction of ψ_p to $(1\otimes p)A\otimes K(1\otimes p)$ as a state of A, which is apparently an α-invariant pure state.

2.4. Remark. One has a similar result to the above theorem for any finite cyclic group G of prime order instead of $\boldsymbol{T}$. Because in this case any non-zero $n\in\hat{G}$ is a generator and so the counterpart to Lemma 2.2 holds trivially.

3. Finite cyclic group actions.

3.1. Theorem. *Let A be a separable simple C^*-algebra and α a periodic automorphism of A. Then $\gamma(\alpha)=1$ (cf. Sect. 1) if and only if (A, α) has a covariant irreducible representation.*

Proof. Suppose that (A, α) has a covariant irreducible representation, say, (π, U). Let p_0 be the outer period of α, i.e., the smallest positive integer n such that α^n is inner. Let v be a unitary multiplier of A with $\alpha^{p_0}=Ad\,v$. Then, since π is irreducible, $U_{p_0}=\lambda\pi(v)$ with $\lambda\in\boldsymbol{C}$, $|\lambda|=1$. Hence $\alpha(v)=v$, i.e., $\gamma(\alpha)=1$.

Suppose that $\gamma(\alpha)=1$ and let p_0 and v be as above. Let e be a minimal projection of the finite-dimensional algebra generated by v. Then eAe is an α-invariant hereditary C^*-subalgebra of A and $\alpha|eAe$ is periodic with period p_0.

It suffices to show that there exists an α-invariant pure state of eAe.

Now we assume that p_0 is the period p of α. We only have to show that $\hat{\alpha}_k$, with $k \in \{1, \cdots, p-1\}$ is outer on the (simple) crossed product $A \times_\alpha \boldsymbol{Z}_p$, where $\boldsymbol{Z}_p = \boldsymbol{Z}/p\boldsymbol{Z}$. Suppose that $\hat{\alpha}_k$ is inner, say, $Ad\, v$ with v a unitary multiplier. Then v can be written as

$$\sum_{n=0}^{p-1} v_n \lambda(n)$$

where v_n is a multiplier of A and $\{\lambda(n)\}$ is the canonical unitary multipliers of $A \times_\alpha \boldsymbol{Z}_p$. Since v commutes with elements of A, one can conclude that α_n is inner if v_n is non-zero. This implies that v is a scalar, i.e., $k=0$.

3.2. Remark. For any $p=2, 3, \cdots$ and $\gamma \in \boldsymbol{C}$ with $\gamma^p=1$, there exists a periodic automorphism α of the UHF algebra of the type p^∞ such that the outer period of α is p and $\gamma(\alpha)=\gamma$. (See [2].)

3.3. Remark. For the periodic automorphism α the least positive integer k with $\gamma(\alpha)^k=1$ is equal to the least number of the dimensions of $\pi(A)'$ with π covariant representations of (A, α).

4. One-parameter automorphism groups.

4.1. Theorem. *Let A be a separable C^*-algebra and α a strongly continuous one-parameter automorphism group of A. Suppose tht α fixes each ideal of A and that the strong Connes spectrum $\tilde{\boldsymbol{R}}(\alpha)$ of α is not $\boldsymbol{R}$. Then (A, α) has a covariant irreducible representation.*

Proof. Let π be an irreducible representation of $A \times_\alpha \boldsymbol{R}$ such that $P = \ker \pi$ is not $\hat{\alpha}$-invariant. Then the intersection of $\hat{\alpha}_p(P)$, $p \in \boldsymbol{R}$, is equal to $J \times_\alpha \boldsymbol{R}$ for some ideal J of A. Denoting $\bar{\pi}$ the normal extension of π to a representation of the multiplier algebra of $A \times_\alpha \boldsymbol{R}$, J is given by $\ker \bar{\pi} | A$. For any ideal I of A, the support projection of $\bar{\pi}(I)$ is zero or the identity, since it should be in the center of $\pi(A \times_\alpha R)''$. Thus J is a prime ideal. Dividing A by J we may assume that A is prime.

Define an action γ of $\boldsymbol{R}$ on $\bar{\pi}(A)'$ by $\gamma_t = Ad\, \bar{\pi}(\lambda(t))$. If $\bar{\pi}|A$ is not irreducible, the spectrum $Sp(\gamma)$ of γ is a non-trivial closed subgroup of $\boldsymbol{R}$ since γ is ergodic. Since P is invariant under $\hat{\alpha}_p$ for $p \in Sp(\gamma)$ and P is not $\hat{\alpha}$-invariant, $Sp(\gamma)$ should be of the form $p_0\boldsymbol{Z}$ with $p_0>0$, which implies that

$$P_{p_0} = \cap \{\hat{\alpha}_q(P) : 0 \leqq q \leqq p_0\}$$

is zero. Since P_ε is non-zero for sufficiently small $\varepsilon>0$, and since

$$\bigcap_{n=0}^{k} \hat{\alpha}_{n\varepsilon}(P_\varepsilon) = (0)$$

for k with $k\varepsilon \geqq p_0$, it follows that the crossed product is not prime, i.e., $\boldsymbol{R}(\alpha)\neq \boldsymbol{R}$.

If $\boldsymbol{R}(\alpha)=(0)$, then (since A is prime) for any $n=1, 2, \cdots$, there exists a non-zero α-invariant hereditary C^*-subalgebra D of A such that $Sp(\alpha|D)\cap[-n, n] \subset[-1, 1]$ ([10]). Thus P is not invariant under any $\hat{\alpha}_p$ with $p\neq 0$, which is a contradiction. Hence if either $\boldsymbol{R}(\alpha)=\boldsymbol{R}$ or $\boldsymbol{R}(\alpha)=(0)$, then $\bar{\pi}|A$ is irreducible (and covariant), which completes the proof in these cases.

The remaining case, i.e., the case $\boldsymbol{R}(\alpha)=p_0\boldsymbol{Z}$ with $p_0>0$, can be reduced to that treated in 2.1. Because, in this case there exists a non-zero α-invariant hereditary C^*-subalgebra B of A such that

$$Sp(\alpha|B)\subset p_0\boldsymbol{Z}+(-p_0/4, p_0/4).$$

Then perturbing $\alpha|B$ by the bounded derivation $-\mathrm{Log}(\alpha_s|B)$ with $s=2\pi/p_0$ gives a periodic one-parameter automorphism group β of B with period s. Then 2.1 gives a β-invariant pure state ϕ of B, and π_ϕ is a covariant irreducible representation of $(B, \alpha|B)$. It easily follows that the unique extension of π_ϕ to an irreducible representation of A is covariant.

Incidentally we remark that for $(A, \boldsymbol{R}, \alpha)$ with A simple, if $\tilde{\boldsymbol{R}}(\alpha)\neq\boldsymbol{R}$ and $\tilde{\boldsymbol{R}}(\alpha)$ contains a non-zero subgroup of $\boldsymbol{R}$, then $\tilde{\boldsymbol{R}}(\alpha)=\boldsymbol{R}(\alpha)$. (First one shows that $\boldsymbol{R}(\alpha)\neq\boldsymbol{R}$ as in the second paragraph of the above proof. If $\boldsymbol{R}(\alpha)=p\boldsymbol{Z}$ we may assume that α is periodic with period $s=2\pi/p$. Then regarding α as an action of $\boldsymbol{R}/s\boldsymbol{Z}$, one obtains that $\tilde{\boldsymbol{R}}(\alpha)=\boldsymbol{R}(\alpha)$ by a similar reasoning to that for $\boldsymbol{R}(\alpha)\neq\boldsymbol{R}$.)

4.2. Remark. There are examples of dynamical systems $(A, \boldsymbol{R}, \alpha)$ which have covariant irreducible representations even if A is separable and simple and $\tilde{\boldsymbol{R}}(\alpha)=\boldsymbol{R}$. A may be chosen to be a Cuntz algebra. See Sections 4 and 5 in [5].

4.3. Remark. Let A_θ be the irrational rotation algebra with $\theta\in[0, 1]\backslash\boldsymbol{Q}$, i.e., A_θ is generated by two unitaries u and v satisfying $uvu^*v^*=\exp 2\pi i\theta$. Let α be the one-parameter automorphism group of A defined by $\alpha_t(u)=e^{i2\pi t}u$, $\alpha_t(v)=e^{i2\pi\theta t}v$. Then α is ergodic in a strong sense, i.e., there are no non-trivial α-invariant hereditary C^*-subalgebras of A. Hence $\tilde{\boldsymbol{R}}(\alpha)=Sp(\alpha)=\boldsymbol{R}$. Since α_1 is inner, it easily follows that (A_θ, α) does not have a covariant irreducible representation, nor does $(A_\theta, \alpha_{1/n})$ with $n\geqq 2$.

5. Countable discrete abelian group actions.

Let A be a separable nuclear C^*-algebra with unique tracial state τ. Let G be a countable discrete abelian group and α a homomorphism of G into the automorphism group of A. Since τ is α-invariant, one can define a unitary representation W of G on the GNS representation space H_τ associated with τ by

$$W_g\pi_\tau(x)\Omega_\tau=\pi_\tau\circ\alpha_g(x)\Omega_\tau, \quad x\in A.$$

Thus one can define an action $\bar{\alpha}$ of G on $\pi_\tau(A)''$ by $\bar{\alpha}_g = Ad\, W_g$, $g \in G$. Note that $\pi_\tau(A)''$ is a factor since τ is supposed to be a unique tracial state.

5.1. Proposition. *In the above situation further suppose $\bar{\alpha}$ is free, i.e., for each non-zero $g \in G$, $\bar{\alpha}_g$ is an outer automorphism of $\pi_\tau(A)''$. Then (A, G, α) has covariant irreducible representations.*

Proof. Let B be the crossed product of A by α and let β be the action of $\Gamma = \hat{G}$ on B dual to α. Let ϕ be the β-invariant extension of τ to a (tracial) state of B. Then $N \equiv \pi_\phi(A)''$ is isomorphic to $\pi_\tau(A)''$ and is the hyperfinite type II_1 factor ([1], [3]) (since it is not finite-dimensional due to the assumption on $\bar{\alpha}$). In passing we remark that $M \equiv \pi_\phi(B)''$ is the W^*-crossed product of $\pi_\tau(A)''$ by $\bar{\alpha}$ and is a factor ([10], 8.9.4, 8.11.5).

Define a unitary representation V of Γ on H_ϕ by

$$V_t Q \Omega_\phi = \bar{\beta}_t(Q) \Omega_\phi, \quad Q \in M.$$

Let u be the canonical unitary group of multipliers of B which implements α on A and let $U_g = \pi_\phi(u_g)$ for $g \in G$. Denoting by $\bar{\alpha}$ also the inner action of G on M defined by $Ad\, U_g$, $g \in G$, define a unitary representation W of G on H_ϕ by

$$W_g Q \Omega_\phi = \bar{\alpha}_g(Q) \Omega_\phi, \quad Q \in M.$$

Then W_g and U_h commute with each other for any $g, h \in G$, and $\{W_g U_g^* : g \in G\}$ is a unitary group in the commutant M' of M, since W and U implement the same action $\bar{\alpha}$ of G on M.

Any two free actions of G on the hyperfinite type II_1 factor with separable predual are outer conjugate with each other [8]. We note

5.2. Lemma. *There exists a free action γ of G on N such that the fixed point algebra N^γ of N under γ contains a maximal abelian subalgebra of N.*

We shall give the proof of this later. Now we may assume that there exists a unitary cocycle X for $\bar{\alpha}$, i.e., $X_g \in N$, $g \in G$ with

$$X_{g+h} = X_g \bar{\alpha}_g(X_h),$$

such that N^γ for the action γ of G defined by $Ad\, X_g \circ \bar{\alpha}_g$, $g \in G$, contains a maximal abelian subalgebra of N, say C.

Let J be the canonical involution defined for (M, Ω_ϕ), i.e., J is the conjugate-linear operator satisfying $J Q \Omega_\phi = Q^* \Omega_\phi$, $Q \in M$. Then $JMJ = M'$ and $JW_g = W_g J$, $g \in G$. Set $Y_g = J X_g J W_g U_g^*$ for $g \in G$. Then $Y_g \in M'$, and

$$\begin{aligned} Y_g Y_h &= J X_g W_g X_h W_h J U_g^* U_h^* = J X_g \alpha_g(X_h) W_{g+h} J U_{g+h}^* \\ &= Y_{g+h} \end{aligned}$$

and for $Q \in N$,

$$Y_g JQJ Y_g^* = JX_g W_g Q W_g^* X_g^* J$$
$$= J\gamma_g(Q) J.$$

Thus for $Q \in N$, JQJ commute with Y_g, $g \in G$ if and only if $Q \in N^\gamma$.

Let C_1 be the von Neumann algebra generated by JCJ and Y_g, $g \in G$. Then since $C \subset N^\gamma$, and Y is a representation of G, C_1 is abelian. We claim that $M' \cap \{V_t\}' = JNJ$ and that C_1 is a maximal abelian subalgebra of M'. The first part is obvious since M' is generated by JNJ and JU_gJ, $g \in G$. For the second part note that C_1 is invariant under $Ad\, V_t$, $t \in \Gamma$ since $V_t Y_g V_t^* = \overline{\langle g, t \rangle} Y_g$, and $JCJ \subset \{V_t\}'$. For any $Q \in M' \cap C_1'$ invariant under $Ad\, V_t$, $t \in \Gamma$, JQJ belongs to N, and so to C, since C is maximal in N. We next claim that $M' \cap (JCJ)' = C_1$. It is obvious that $M' \cap (JCJ)' \supset C_1$. Since $M' \cap (JCJ)'$ is invariant under the action $Ad\, V$, we may consider only an eigen-operator Q in $M' \cap (JCJ)'$ with eigenvalue $g \in G$. Then QY_g belongs to

$$M' \cap (JCJ)' \cap \{V_t\}' = JNJ \cap (JCJ)' = JCJ,$$

which implies that $Q \in C_1$.

We now use Decomposition theory of states (cf. [11]). Let K_1 be the (compact) space of characters of C_1 with the probability measure coming from the state defined by Ω_ϕ. Then there exists a subset D_1 of measure one of K_1 whose elements correspond to pure states of B ([11], 3.1.16, 3.4.2). Here the state f_χ corresponding to $\chi \in K_1$ is given by

$$f_\chi(x) = \chi(e\pi_\phi(x)e), \quad x \in B,$$

where e is the projection onto $[C_1\Omega_\phi]$ and the isomorphism of eC_1 with C_1 is used. Further there exists a subset D_2 of measure one of K_1 such that if $\chi \in D_2$, then f_χ is not zero on some $e\pi_\phi(Au_{g_1})e\pi_\phi(Au_{g_2})e \cdots e\pi_\phi(Au_{g_n})e$ with $g_1 + g_2 + \cdots + g_n = g$ for each $g \in G$, because the weak closure of the algebra generated by $e\pi_\phi(B)e$ is eC_1. Note that for $\chi \in D_2$, the set of $g \in G$ with $f_\chi | Au_g \neq 0$ generates G.

Similarly let K be the space of characters of JCJ with the canonical probability measure. Then there exists a subset D_3 of measure one of K such that each element of D_3 corresponds to a state of B whose restriction to A is pure, since JCJ is maximal abelian in $\pi_\phi(A)'$ when restricted to $[\pi_\phi(A)\Omega_\phi]$. Further there exists a subset D_4 of measure one of K such that each element of D_4 corresponds to a state f of B with the property that $\pi_f(B)'$ is commutative, since $\pi_\phi(B)' \cap (JCJ)'$ is commutative (cf. [11], 3.4, 3.2.12). Let F be the (continuous) map of K_1 onto K obtained by restriction. Then, since F maps the measure on K_1 to the oneon K, $F^{-1}(D_j)$, $j=3, 4$ have measure one.

Let $\chi \in D_1 \cap D_2 \cap F^{-1}(D_3) \cap F^{-1}(D_4)$ and let $f = f_{F(\chi)}$. Then, since JCJ is the fixed point algebra of C_1 under $Ad\,V$,

$$f=\int_{\Gamma} f_\chi\circ\beta_t dt$$

where dt is the normalized Haar measure on Γ. Since $f|A$ is pure, f is an extreme β-invariant state of B.

Let V be the unitary representation of Γ on the GNS representation space H_f defined by

$$V_t\pi_f(x)\Omega_f=\pi_f\circ\beta_t(x)\Omega_f\,,\quad x\in B.$$

Then we claim that every spectrum appears for the action $Ad\,V$ of Γ on $\pi_f(B)'$. Let $g\in G$ be such that there exists $x\in A$ with $f_\chi(xu_g)\neq 0$. For Ω being a small open neighbourhood of $1\in\Gamma$ with

$$\int_{\Omega}\langle g,\,t\rangle dt\neq 0\,,$$

we have a positive operator T in $\pi_f(B)'$ such that

$$\langle\pi_f(xu_g)\Omega_f,\,V_sTV_s^*\Omega_f\rangle=\overline{\langle g,\,s\rangle}\int_{\Omega} f_\chi\circ\beta_t(xu_g)dt\,.$$

Hence the Arverson spectrum of T with respect to $Ad\,V$ contains $-g\in G$. Since $\pi_f(B)'$ is commutative and so, is the center of $\pi_f(B)''$, and $Ad\,V$ acts ergodically on $\pi_f(B)'$, we have a commutative family $\{W_g : g\in G\}$ of unitaries in $\pi_f(B)'$ such that

$$V_tW_gV_t^*=\langle g,\,t\rangle W_g\,,\quad t\in\Gamma.$$

By arranging phase factors for $\{W_g\}$ we can assume that $\{W_g\}$ forms a unitary representation of G.

The rest of the proof procedes as that of 2.1. We have now the irreducible representation $\rho=\pi_f\times V$ of the crossed product $B\times_\beta\Gamma$ on H_f where $Ad\,W^*$ induces the dual action of G on $B\times_\beta\Gamma$. Then regarding this representation as that of $A\otimes K(l^2(G))$ by the duality, $Ad\,W^*$ induces the action $\alpha\otimes Ad\,\tilde{\lambda}$. Hence $\{\rho(1\otimes\tilde{\lambda}(g)^*)W_g^*\}$ is a unitary representation of G and induces the action $\alpha\otimes 1$. Therefore by cutting down H_f by $\rho(1\otimes p)$ with p a one-dimensional projection of $K(l^2(G))$, we obtain a covariant irreducible representation of $(A,\,G,\,\alpha)$.

Proof of 5.2. We shall embed G into the direct product $\boldsymbol{T}^\infty$ of countably infinitely many copies of the circle group $\boldsymbol{T}$.

We write G as $\{g_1,\,g_2,\,\cdots\}$ since G is countable. Let G_1 be the subgroup of G generated by g_1. We define an injective homomorphism φ_1 of G_1 into $\boldsymbol{T}$ by $\varphi_1(g_1)=t_1$ where t_1 is irrational if G_1 is infinite and $t_1=1/n$ if G_1 is of order n. Now suppose that we have defined an injection φ_k of the group G_k generated by $\{g_1,\,\cdots,\,g_k\}$ into T^k for $k\leqq n$ such that $\varphi_k|G_{k-1}$ is φ_{k-1} composed with the natural embedding $\boldsymbol{T}^{k-1}\to\boldsymbol{T}^k$. Let $H=\{k\in\boldsymbol{Z} : kg_{n+1}\in G_n\}$, which is a subgroup of $\boldsymbol{Z}$. If $H=(0)$, then define φ_{n+1} by $\varphi_{n+1}(g_{n+1})=\{0, t_{n+1}\}\in\boldsymbol{T}^n\times\boldsymbol{T}$ (and $\varphi_{n+1}(h)=\{\varphi_n(h), 0\}\in\boldsymbol{T}^n\times\boldsymbol{T}$) where t_{n+1} is irrational. If $H=k\boldsymbol{Z}$ with $k>0$, define φ_{n+1} by $\varphi_{n+1}(g_{n+1})=\{t, t_{n+1}\}\in\boldsymbol{T}_n\times\boldsymbol{T}$ where $kt=\varphi_n(kg_{n+1})$ and $t_{n+1}=1/k$. If $H=\boldsymbol{Z}$, set $\varphi_{n+1}=\varphi_n$ (composed with the embedding of $\boldsymbol{T}^n$ into $\boldsymbol{T}^{n+1}$). It is easy to show that φ_{n+1} satisfies the required properties. Then, having $\{\varphi_n\}$, define a map φ of G into $\boldsymbol{T}^\infty$ by $\varphi(h)=\{\varphi_n(h), 0\}\in\boldsymbol{T}^n\times\boldsymbol{T}^\infty\cong\boldsymbol{T}^\infty$, for $h\in G_n$, which gives the desired embedding.

Define an action of $\boldsymbol{T}$ on the UHF algebra of (2^∞) type by the infinite tensor product of

$$Ad\begin{pmatrix}1 & 0\\ 0 & \exp 2\pi it\end{pmatrix}, \quad t\in\boldsymbol{T}$$

and define an action of $\boldsymbol{T}^\infty$ on $A\cong A\otimes A\otimes\cdots$ by the tensor product of infinitely many copies of the action of $\boldsymbol{T}$ just obtained. Since there exists a shift automorphism of A which commutes the action of $\boldsymbol{T}$, we also have a shift automorphism which commutes the action of $\boldsymbol{T}^\infty$. By using this shift, we can easily conclude that the action of G on $\pi_\tau(A)''$ is free, where τ is the unique tracial state of A. It is clear that the fixed point algebra of $\pi_\tau(A)''$ under G contains a maximal abelian subalgebra of $\pi_\tau(A)''$.

5.3. Remark. Since each χ of a dense subset of the non-atomic measure space K produces a state of B whose restriction to A is pure in the way that the states so obtained are mutually orthogonal, and since A is separable, one has uncountably many equivalence classes of covariant irreducible representations of (A, G, α).

5.4. Proposition. *Let A be a separable C^*-algebra and α an automorphism of A. Let f be an α-invariant state of A and let $\bar{\alpha}$ be the automorphism of $\pi_f(A)''$ induced by α. If $\bar{\alpha}$ is inner, then (A, α) has a convariant irreducible representation.*

Proof. Define a unitary U on H_f by $U\pi_f(x)\Omega_f=\pi_f\circ\alpha(x)\Omega_f$, $x\in A$. By the assumption there is a unitary V in $\pi_f(A)''$ such that $V^*U\equiv W\in\pi_f(A)'$. Let C be a maximal abelian subalgebra of $\pi_f(A)'$ containing W. Then, since $C\subset\pi_f(A)'\cap U'$, one can use Decomposition theory of states to conclude the existence of an α-invariant pure state of A.

5.5. Corollary. *Let A be a UHF algebra and α an automorphism of A. Then the action of α on the spectrum $\hat{A}$ of A has uncountably many finite orbits.*

Incidentally we remark that there are at least countably infinitely many outer conjugacy classes of automorphisms of a UHF algebra A which are not outer periodic. To show this we just point out that whether or not an automorphism of A induces a weakly inner automorphism of $\pi_\tau(A)''$ is an outer invariant, where

τ is the unique tracial state (cf. [4]).

5.6. Example. Let γ be the action of $\boldsymbol{T}^2$ on the irrational rotation algebra A_θ (cf. 4.3) defined by

$$\gamma_{(t,s)}(u)=e^{2\pi it}u\,,\quad \gamma_{(t,s)}(v)=e^{2\pi is}v\,.$$

Then of course there are no $\boldsymbol{T}^2$-covariant pure states of A, since γ is ergodic. But for a countable subgroup G of T^2 such that $\gamma|G$ is free, i.e., γ_g is outer for any non-trivial $g\in G$, there are G-covariant pure states of A. To show this, since A is nuclear and has a unique tracial state, say, τ, it suffices to prove that if γ_g is outer with $g\in\boldsymbol{T}^2$, then γ_g induces an outer automorphism of $\pi_\tau(A_\theta)''$. But this follows by easy computations.

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