YOKOHAMA MATHEMATICAL
JOURNAL VOL. 31, 1983

NOTES ON THE ALMOST SURE ABSOLUTE CONVERGENCE OF THE FOURIER SERIES OF A STOCHASTIC PROCESS

By

TATSUO KAWATA

(Received March 1, 1983)

1. Introduction.

Throughout this paper we suppose that $X(t, \omega)$ is a measurable 2π-periodic stochastic process of $L^r(T\times\Omega)$ for some $r\geqq 1$, where $T=[-\pi, \pi]$ and $(\Omega, \mathfrak{F}, P)$ is the probability space:

(1.1) $$|||X(t, \omega)|||_r\equiv\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}E|X(t, \omega)|^r dt\right)^{1/r}<\infty,$$

and

(1.2) $$\|X(t+2\pi, \omega)-X(t, \omega)\|_1=0,$$

for every $t\in R^1$. We generally write $\|X(\omega)\|_r=(E|X(\omega)|^r)^{1/r}$. We also write

$$D_h X(t, \omega)=\frac{1}{h}[X(t+h, \omega)-X(t, \omega)].$$

If there exists an $X_M'(t, \omega)\in L^r(T\times\Omega)$ such that

(1.3) $$|||D_h X(t, \omega)-X_M'(t, \omega)|||_r\to 0,$$

as $h\to 0$, then $X(t, \omega)$ is said to have the mean derivative $X_M'(t, \omega)$ in $L^r(T\times\Omega)$.

If $X(t, \omega)$ has the mean derivative $X_M'(t, \omega)$ in $L^r(T\times\Omega)$ and furthermore $X_M'(t, \omega)$ has the mean derivative $X_M''(t, \omega)$ in $L^r(T\times\Omega)$, then we say that $X(t, \omega)$ has the second mean derivative $X_M''(t, \omega)$ in $L^r(T\times\Omega)$. In a similar way we successively define the k-th mean derivative $X_M^{(k)}(t, \omega)$ in $L^r(T\times\Omega)$.

Let the Fourier series of $X(t, \omega)\in L^1(T\times\Omega)$ be

(1.4) $$X(t, \omega)\sim\sum_{n=-\infty}^{\infty}C_n(\omega)e^{int},$$

where

(1.5) $$C_n(\omega)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(t, \omega)e^{-int}dt.$$

The author recently has studied the almost sure convergence of

$$\sum_{n=-\infty}^{\infty} |n|^{k+\alpha}|C_n(\omega)|, \quad (0 \leqq \alpha < 1), \tag{1.6}$$

k being a nonnegative integer, when $X(t, \omega)$ has the k-th mean derivative in $L^r(T\times\Omega)$, $1<r\leqq 2$. In this paper we aim at giving some sufficient condition for the almost sure convergence of (1.6) when $X(t, \omega)$ has the k-th mean derivative in $L^1(T\times\Omega)$ and not necessarily in $L^r(T\times\Omega)$, $r>1$.

2. Variation of a stochastic process.

Let $X(t, \omega)$ be of $L^1(T\times\Omega)$. Suppose moreover for each t, $X(t, \omega)\in L^r(\Omega)$ for some $r\geqq 1$. If

$$\sup_D \sum_{j=1}^{n} \|X(t_j, \omega)-X(t_{j-1}, \omega)\|_r = V_r(X) < \infty, \tag{2.1}$$

where sup is taken over all the divisions D of T, $D: -\pi\leqq t_0<t_1<\cdots<t_n\leqq\pi$, then $X(t, \omega)$ is said to be of bounded variation in $L^r(\Omega)$. This was defined in [3]. $V_r(X)$ is called the total variation of X in $L^r(\Omega)$.

Lemma 1. *Suppose that* $X(t, \omega)\in L^1(T\times\Omega)$ *and has the mean derivative* $X'_M(t, \omega)$ *in* $L^1(T\times\Omega)$. *Suppose also that for each* t, $X(t, \omega)$ *and* $X'_M(t, \omega)$ *belong to* $L^r(\Omega)$ *for some* $r\geqq 1$ *with* $\|X'_M(t, \omega)\|_r\in L^1(T)$ *and* $X(t, \omega)$ *is continuous in* $L^r(\Omega)$ *as a function of* t:

$$\|X(t+h, \omega)-X(t, \omega)\|_r \to 0 \tag{2.2}$$

as $h\to 0$, *then* $X(t, \omega)$ *is of bounded variation in* $L^r(\Omega)$ *and*

$$V_r(X) \leqq \int_{-\pi}^{\pi} \|X'_M(t, \omega)\|_r dt. \tag{2.3}$$

Proof. From [3] (Lemma 2), for each fixed h,

$$X(t+h, \omega)-X(t, \omega)=\int_t^{t+h} X'_M(u, \omega)du$$

for almost all (t, ω) in $T\times\Omega$. Hence for each h,

$$\|X(t+h, \omega)-X(t, \omega)\|_r = \left\|\int_t^{t+h} X'_M(u, \omega)du\right\|_r \tag{2.4}$$

holds for almost all $t\in T$. Namely there is a subset $H(h)$ of T with $mH(h)=2\pi$, generally depending on h, such that, for $u\in H(h)$, (2.4) with $t=u$ holds good. m is the Lebesgue measure.

Let $t_1<t_2$ be numbers in T and fix them. Take $h=t_2-t_1$ and write $H=H(h)$. For any $\varepsilon>0$, choose δ in such a way that

$$\int_S \|X'_M(t, \omega)\|_r dt < \varepsilon, \tag{2.5}$$

for $mS<\delta$, and for $|v|<\delta$

$$\|X(t+v,\ \omega)-X(t,\ \omega)\|_r<\varepsilon,\quad t\in T. \tag{2.6}$$

Choose $u_1\in H$ so that

$$|t_1-u_1|<\delta. \tag{2.7}$$

This implies

$$|t_2-(u_1+h)|<\delta. \tag{2.8}$$

Using (2.4) we then have

$$\begin{aligned}\|X(t_2,\ \omega)-X(t_1,\ \omega)\|_r&\leqq\|X(t_1,\ \omega)-X(u_1,\ \omega)\|_r\\&\quad+\|X(u_1,\ \omega)-X(u_1+h,\ \omega)\|_r+\|X(u_1+h,\ \omega)-X(t_2,\ \omega)\|_r\\&\leqq2\varepsilon+\|X(u_1,\ \omega)-X(u_1+h,\ \omega)\|_r\\&=2\varepsilon+\int_{u_1}^{u_1+h}\|X_M'(t,\ \omega)\|_r dt\end{aligned}$$

which is, because of (2.5), (2.7) and (2.8)

$$\leqq2\varepsilon+2\varepsilon+\int_{t_1}^{t_2}\|X_M'(t,\ \omega)\|_r dt.$$

This gives us

$$\|X(t_2,\ \omega)-X(t_1,\ \omega)\|_r\leqq\int_{t_1}^{t_2}\|X_M'(t,\ \omega)\|_r dt. \tag{2.9}$$

Now for any division $-\pi\leqq t_0\leqq t_1\leqq\cdots\leqq t_n\leqq\pi$, we have, from (2.9)

$$\sum_{j=1}^{n}\|X(t_j,\ \omega)-X(t_{j-1},\ \omega)\|_r\leqq\int_{-\pi}^{\pi}\|X_M'(t,\ \omega)\|_r dt$$

which shows (2.3).

We shall prove the following lemma just for completeness, although we do not use it in this paper.

Lemma 2. *Let $r\geqq1$. Suppose $X(t,\omega)$ is of $L^r(T\times\Omega)$ and is continuous in $L^r(\Omega)$ as a function of t. If $X(t,\omega)$ has the mean derivative in $L^r(T\times\Omega)$, then $X(t,\omega)$ is of bounded variation in $L^r(\Omega)$ and*

$$V_r(X)=\int_{-\pi}^{\pi}\|X_M'(t,\ \omega)\|_r dt. \tag{2.10}$$

Proof. In view of Lemma 1 it is sufficient to show the reverse inequality of (2.3). Let $h=2\pi/n$, n being any positive integer.

$$\int_{-\pi}^{\pi}\|X_M'(t,\ \omega)\|_r dt\leqq\int_{-\pi}^{\pi}\|D_hX(t,\ \omega)-X_M'(t,\ \omega)\|_r dt+\int_{-\pi}^{\pi}\|D_hX(t,\ \omega)\|_r dt.$$

The first of the right hand side converges to zero as $n\to\infty$. The second is

$$\int_{-\pi}^{\pi}\left\|\frac{1}{h}[X(t+h,\omega)-X(t,\omega)\right\|_r dt$$

$$=\frac{n}{2\pi}\sum_{j=1}^{n}\int_{-\pi+2\pi(j-1)/n}^{-\pi+2\pi j/n}\left\|X\left(t+\frac{2\pi}{n},\omega\right)-X(t,\omega)\right\|_r dt$$

$$=\frac{n}{2\pi}\sum_{j=1}^{n}\int_{-\pi}^{-\pi+2\pi/n}\left\|X\left(t+\frac{2\pi j}{n},\omega\right)-X\left(t+\frac{2\pi(j-1)}{n},\omega\right)\right\|_r dt$$

$$\leqq\frac{n}{2\pi}\int_{-\pi}^{-\pi+2\pi/n}V_r(X)dt=V_r(X).$$

We remark that the condition that $X(t,\omega)$ has the mean derivative in $L^r(T\times\Omega)$ can be replaced by the slightly more general condition that

$$\int_{-\pi}^{\pi}\|D_hX(t,\omega)-X_M'(t,\omega)\|_r dt\to 0,\quad h\to 0. \tag{2.11}$$

3. Almost sure absolute convergence of the Fourier series of a stochastic process.

Let $X(t,\omega)$ be a 2π-periodic stochastic process of $L^1(T\times\Omega)$ and have the mean derivative $X_M'(t,\omega)$ in $L^1(T\times\Omega)$. Let the Fourier series of $X(t,\omega)$ be given by (1.4). Furthermore we impose the following conditions:

(i) $X(t,\omega)\in L^r(\Omega)$ for each t and as a function of t, $X(t,\omega)$ is continuous in $L^r(\Omega)$ on T, for some $r>1$.

(ii) $X_M'(t,\omega)\in L^r(\Omega)$ for each t and as a function of t, $\|X_M'(t,\omega)\|_r\in L^1(T)$.

Because of (i), $X(t,\omega)$ actually belongs to $L^r(T\times\Omega)$. However we are supposing that $X_M'(t,\omega)$ exists merely in $L^1(T\times\Omega)$. As a matter of fact, we do not assume that $\|X_M'(t,\omega)\|_r\in L^r(T)$. This is the point in what follows. If this were assumed, Theorem 1 below we are going to show is just a special case of [3] (Theorem 2).

For $X(t,\omega)\in L^r(T\times\Omega)$, $1<r\leqq 2$, we write

$$M_r^*(X,\delta)=\sup_{|h|\leqq\delta}\int_{-\pi}^{\pi}E|X(t+h,\omega)-X(t,\omega)|^r dt. \tag{3.1}$$

We here mention the following theorem which is a special case of [2] (Theorem 3.1).

Theorem A. *Let $X(t,\omega)\in L^r(T\times\Omega)$, $1<r\leqq 2$. If, for $0\leqq\alpha<1$,*

$$\sum_{n=1}^{\infty}n^{-(1-1/r)+\alpha}M_r^*(X,1/n)<\infty, \tag{3.2}$$

then

$$\sum_{n=-\infty}^{\infty} |n|^{\alpha}|C_n(\omega)| < \infty, \tag{3.3}$$

almost surely.

We shall prove the following theorem.

Theorem 1. *Let* $2 \geqq r > 1$. *Suppose* $X(t, \omega) \in L^r(T \times \Omega)$ *and has the mean derivative* $X'_M(t, \omega)$ *in* $L^1(T \times \Omega)$. *Suppose* (i) *and* (ii) *above. If*

$$\int_{-\pi}^{\pi} \|X'_M(t, \omega)\|_r \log^+ \|X'_M(t, \omega)\|_r dt < \infty, \tag{3.4}$$

then (3.3) *with* $\alpha=0$ *holds almost surely.*

This theorem is thought of as an analogue of the well known theorem of Zygmund [1] (Theorem 2, p. 162), [2] (Theorem 3.9, p. 242) that the Fourier series of $f(t)$ which is absolutely continuous and satisfies

$$\int_{-\pi}^{\pi} |f'(t)| \log^+ |f'(t)| dt < \infty, \tag{3.5}$$

is absolutely convergent. The proof of Theorem 1 is carried out based on the arguments used by Wik [4] who made a detailed study on the above Zygmund theorem.

In proving Theorem 1, we note that because of Lemma 1, $X(t, \omega)$ is of bounded variation in $L^r(\Omega)$. We use the following lemma employed by Wik [4].

Lemma 3. *Let* $\phi(t) \in L^1(T)$. *Write* $V_n = V_n(\phi) = \int_{|\phi|>n} |\phi(t)| dt$. *Then for any* $a>1$,

$$\int_{-\pi}^{\pi} |\phi(t)| \log^+ |\phi(t)| dt < \infty \tag{3.6}$$

is equivalent to

$$\sum_{q=1}^{\infty} V_{a^q} < \infty. \tag{3.7}$$

Proof of Theorem 1. From (2.4)

$$M_r^*(X, 1/n) = \sup_{|h| \leq 1/n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} E \left| \int_t^{t+h} X'_M(s, \omega) ds \right|^r dt \right)^{1/r}$$

which is, by Minkowski inequality,

$$\leqq \sup_{|h| \leq 1/n} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\int_t^{t+h} \|X'_M(s, \omega)\|_r ds \right)^r \right]^{1/r}.$$

Writing

$$Y(t) = \int_{-\pi}^{t} \|X'_M(s, \omega)\|_r ds,$$

we have

$$M_r^*(X, 1/n) \leqq \sup_{|h| \leq 1/n} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |Y(t+h) - Y(t)|^r dt \right\}^{1/r} \tag{3.8}$$

$$\leqq \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} [Y(t+1/n) - Y(t)]^r dt \right\}^{1/r}$$

$$+ \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} [Y(t) - Y(t-1/n)]^r dt \right\}^{1/r}$$

$$= I_1 + I_2,$$

say. Now let us write

$$E_n = \{t \in T,\ \|X_M'(t, \omega)\|_r \geqq n\}$$

and let N be any positive integer. We write

$$Y(t+1/n) - Y(t) = \int_{E_N \cap (t, t+1/n)} \|X_M'(s, \omega)\|_r ds$$

$$+ \int_{E_N^c \cap (t, t+1/n)} \|X_M'(s, \omega)\|_r ds$$

$$= I_{1,1} + I_{1,2},$$

say. Then

$$I_{1,1} \leqq \int_{E_N} \|X_M'(s, \omega)\|_r ds = W_N \tag{3.9}$$

say, and defining $Z_n(s) = \|X_M'(s, \omega)\|$, for $s \in E_n$ and $=0$, for $s \in E_n^c$, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} I_{1,1} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_t^{t+1/n} Z_N(s) ds dt \tag{3.10}$$

$$\leqq \frac{1}{2n\pi} \int_{-\pi}^{\pi+1/n} Z_N(s) ds = \frac{1}{2n\pi} \int_{E_N} \|X_M'(s, \omega)\|_r ds$$

$$= \frac{1}{2n\pi} W_N.$$

Hence using (3.9) and (3.10), we have

$$\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} I_{1,1}^r dt \right)^{1/r} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} I_{1,1}^{r-1} \cdot I_{1,1} dt \right)^{1/r} \tag{3.11}$$

$$\leqq W_N^{1-1/r} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} I_{1,1} dt \right)^{1/r} \leqq \frac{1}{(2n\pi)^{1/r}} W_N.$$

Now

$$I_{1,2} \leqq \int_{E_N^c \cap (t, t+1/n)} \|X_M'(s, \omega)\|_r ds \leqq \frac{N}{n}.$$

Hence as in (3.11) and (3.10),

(3.12)
$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} I_{1,2}^{r}dt\right)^{1/r} \leqq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} I_{1,2}^{r-1}\cdot I_{1,2}dt\right)^{1/r}$$
$$\leqq \left(\frac{N}{n}\right)^{1-1/r}\frac{C}{(2n\pi)^{1/r}} = \frac{C}{(2\pi)^{1/r}}\cdot\frac{N^{1-1/r}}{n}, \quad C=\int_{-\pi}^{\pi}\|X'_M(t,\omega)\|_r dt\,.$$

Therefore from (3.11) and (3.12)

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} I_1^r dt\right)^{1/r} \leqq \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} I_{1,1}^r dt\right)^{1/r} + \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} I_{1,2}^r dt\right)^{1/r}$$
$$\leqq \left(\frac{1}{2\pi}\right)^{1/r}\left(\frac{W_N}{n^{1/r}} + \frac{CN^{1-1/r}}{n}\right).$$

The same thing is true for $\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} I_2^r dt\right)^{1/r}$ and we have

(3.13)
$$M_r^*(X, 1/n) \leqq C_r\left(\frac{W_N}{n^{1/r}} + \frac{N^{1-1/r}}{n}\right),$$

where C_r is a constant depending only on r.

Now take $n=2^q$, q being a positive integer and $N=a^q$, $1<a<2$. By monotoneness of $n^{1/r-1}$ and $M_r(1/n)$, (3.3) with $\alpha=0$ in Theorem B is equivalent to the convergence of $\sum_{q=1}^{\infty} 2^{q/r}M_r(X, 2^{-q})$. From (3.13) the last series converges when

$$\sum_{q=1}^{\infty} W_{a^q} + \sum_{q=1}^{\infty}\left(\frac{a}{2}\right)^{q-q/r}$$

converges. The second series obviously converges and the first series also converges by Lemma 3. Hence in view of Theorem B the proof of Theorem 1 is complete.

4. More results.

Let $X(t, \omega)$ be a 2π-periodic stochastic process of $L^1(T\times\Omega)$. Let its Fourier series be given by (1.4). Suppose $X(t, \omega)$ has the mean derivative $X'_M(t, \omega)$ in $L^1(T\times\Omega)$. Then the Fourier series of $X'_M(t, \omega)$ is given by

(4.1)
$$\sum_{n=-\infty}^{\infty}(-in)C_n(\omega)e^{int}.$$

Actually the Fourier coefficient $C'_n(\omega)$ of $X'_M(t, \omega)$ is $(-in)C_n(\omega)$ for all n, almost surely. Because

$$\left\|C'_n(\omega) - \frac{1}{2\pi}\int_{-\pi}^{\pi} D_h X(t,\omega)e^{-int}dt\right\|_1 \leqq \|\|X'_M(t,\omega) - D_h X(t,\omega)\|\|_1 \to 0,$$

as $h\to 0$ and

$$\left\|\frac{1}{2\pi}\int_{-\pi}^{\pi} D_h X(t,\omega)e^{-int}dt + \frac{in}{2\pi}\int_{-\pi}^{\pi} X(t,\omega)e^{-int}dt\right\|_1$$

$$=\left|\frac{1}{h}(e^{-inh}-1+in)\right|\cdot\|C_n(\omega)\|\to 0$$

as $h\to 0$, and from these we see that $\|C_n'(\omega)-(-in)C_n(\omega)\|=0$. Hence $C_n'(\omega)=-inC_n(\omega)$ almost surely for each n, from which this holds for all n, almost surely.

More generally if $X(t,\omega)$ has the k-th mean derivative $X_M^{(k)}(t,\omega)$ in $L^1(T\times\Omega)$, then the Fourier series of $X_M^{(k)}(t,\omega)$ is given by

$$\sum_{n=-\infty}^{\infty}(-in)^k C_n(\omega)e^{int}. \tag{4.2}$$

From Theorem 1 with $X_M^{(k)}(t,\omega)$ in place of $X(t,\omega)$, we have the following theorem.

Theorem 2. *Suppose a 2π-periodic stochastic process $X(t,\omega)\in L^1(T\times\Omega)$ has the $k+1$-st mean derivative in $L^1(T\times\Omega)$. Assume that there is an $\hat{X}_M^{(k)}(t,\omega)$ such that $\hat{X}_M^{(k)}(t,\omega)\in L^r(\Omega)$ for some $1<r\leqq 2$, for each t and is continuous in $L^r(\Omega)$ on T, and $\hat{X}_M^{(k)}(t,\omega)=X_M^{(k)}(t,\omega)$ in $L^1(T\times\Omega)$. Moreover suppose $X_M^{(k+1)}(t,\omega)\in L^r(\Omega)$ for each t and $\|X_M^{(k+1)}(t,\omega)\|_r\in L^1(T)$. If*

$$\int_{-\pi}^{\pi}\|X_M^{(k+1)}(t,\omega)\|_r\log^+\|X_M^{(k+1)}(t,\omega)\|_r dt<\infty, \tag{4.3}$$

then

$$\sum_{n=-\infty}^{\infty}|n|^k|C_n(\omega)|<\infty, \tag{4.4}$$

almost surely.

We also have the following theorem.

Theorem 3. *Let $1<r\leqq 2$, $1/r+1/r'=1$. We assume all the conditions in Theorem 2 with*

$$\int_{-\pi}^{\pi}\|X_M^{(k+1)}(t,\omega)\|_r^{1+\alpha}dt<\infty \tag{4.5}$$

for some $0<\alpha<r-1$ in place of (4.3), then for every β such that $0\leqq\beta<\alpha/(1+\alpha r')$,

$$\sum_{n=-\infty}^{\infty}|n|^{k+\beta}|C_n(\omega)|<\infty, \tag{4.6}$$

almost surely.

If $\alpha=r-1$ and hence $X_M^{(k+1)}(t,\omega)\in L^r(T\times\Omega)$, and $X(t,\omega)$ has the $k+1-st$ derivative in $L^r(T\times\Omega)$, then for $\beta<1/r'$, (4.6) holds almost surely [3] (Corollary 1). Our theorem for $\alpha<r-1$ is not covered by this corollary.

We also note that the condition for β implies $\beta<1/r'$. For the case $\beta=1/r'$ the theorem is not necessarily true. See the remark given for [3] (Corollary 1).

Proof of Theorem 3. From (3.13) with $X_M^{(k)}(t, \omega)$ in place of $X(t, \omega)$, we see that

$$n^{\beta} M_r(X_M^{(k)}, 1/n) \leqq C_r\Big(\frac{W_N}{n^{1/r-\beta}} + \frac{N^{1-1/r}}{n^{1-\beta}}\Big), \tag{4.7}$$

where

$$W_N = \int_{E_N} \|X_M^{(k+1)}(t, \omega)\|_r dt,$$

and

$$E_N = \{t \in T,\ \|X_M^{(k+1)}(t, \omega)\|_r \geqq N\}.$$

Choose a so that $2^{1-r'\beta} \geqq a > 2^{\beta/\alpha}$. This is possible. Take $n=2^q$, $N=a^q$ in (4.7), $q=1, 2, \cdots$. The convergence of

$$\sum_{n=1}^{\infty} n^{\beta-1/r'} M_r^*(X_M^{(k)}, 1/n) \tag{4.8}$$

is equivalent to the convergence of

$$\sum_{q=1}^{\infty} 2^{q(\beta+1/r)} M_r^*(X_M^{(k)}, 2^{-q}). \tag{4.9}$$

From (4.7), this converges if

$$\sum_{q=1}^{\infty} V_{a^q} 2^{\beta q} + \sum_{q=1}^{\infty} a^{q/r'} / 2^{(1/r'-\beta)q} < \infty. \tag{4.10}$$

Since $a^{1/r'} < 2^{1/r'-\beta}$, the second series of (4.9) converges.

Now

$$2^{\beta q} V_{a^q} = 2^{\beta q} \int_{E_N} \|X_M^{(k+1)}(t, \omega)\|_r dt$$

$$\leqq 2^{\beta q} a^{-\alpha q} \int \|X_M^{(k+1)}(t, \omega)\|_r^{1+\alpha} dt$$

$$\leqq C(2^{\beta}/a^{\alpha})^q,$$

where C is a constant independent of q.

Since $2^{\beta}/a^{\alpha} < 1$, the first series of (4.10) is convergent. Thus (4.9) converges and the proof is complete.

5. Sample properties of a periodic stochastic process.

In [2] and [3] it was shown that the almost sure absolute convergence of the Fourier series of a stochastic process immediately yields some sample properties of the process. By the same argument used there, we readily have the following theorem.

Theorem 4. (i) *Let $X(t, \omega) \in L^1(T \times \Omega)$ be stochastically continuous. If all the conditions in Theorem* 1 *are satisfied, then there exists a modification $X_0(t, \omega)$*

of $X(t, \omega)$ which is almost surely continuous.

(ii) *If all the conditions in Theorem* 2 *are satisfied, then there exists a modification $X_0(t, \omega)$ of $X(t, \omega)$ which has almost surely the k-th contindous derivative.*

(iii) *If all the conditions in Theorem* 3 *are satisfied, then there exists a modification $X_0(t, \omega)$ of $X(t, \omega)$ which has almost surely the k-th derivative belonging to Lipschitz class Λ_β of order β.*

References

[1] N. Bary: *A treatise on trigonometric series,* Engl. transl. Pergamon, Oxford, 1964.
[2] T. Kawata: *Absolute convergence of Fourier series of periodic stochastic processes and its applications* (to appear).
[3] T. Kawata: *The mean derivatives and the absolute convergence of the Fourier series of a stochastic process* (to appear).
[4] I. Wik: *Criteria for absolute convergence of Fourier series of functions of bounded variation,* Trans. Amer. Math. Soc. **163** 1-24 (1972).
[5] A. Zygmund: *Trigonometric series,* Camb. Univ. Press, New York, 1959.

Added in proof: [2] appeared in Tohoku Math. J. **35** 459-474 (1983) and [3] in Keio Sci. Tech. Rep. **36** 11-24 (1983).

Department of Mathematics
Keio University
Hiyoshi, Kohoku
Yokohama 223, Japan