YOKOHAMA MATHEMATICAL
JOURNAL VOL. 31, 1983

AN ITERATED LOGARITHM RESULT FOR PARTIAL SUMS OF A STATIONARY LINEAR PROCESS

By

RYÔZÔ YOKOYAMA

(Received February 7, 1983)

1. Introduction and results

In time series analysis, an important class of second-order stationary models is that generated by a linear process of the form

(1.1) $$z(n)=\sum_{j=-\infty}^{\infty} a(n-j)\varepsilon(j), \quad \sum_{j=-\infty}^{\infty} a^2(j)<\infty,$$

where the $\varepsilon(n)$ are independent random variables with $E\varepsilon(n)=0$ and $E\varepsilon^2(n)=\sigma^2$ for each n. A process of the kind (1.1) arises, for example, from a mixed autoregressive and moving average process ([3], Chapter I). Lai and Wei [7] have recently obtained iterated logarithm results for partial sums $\sum_{j=1}^{n} z(j)$ of linear processes under very general situations. One of their results is

Theorem A [7]. *Let $\{\varepsilon(n), -\infty<n<\infty\}$ be a sequence of independent random variables such that $E\varepsilon(n)=0$ for all n,*

(1.2) $$E\varepsilon^2(n)=\sigma^2 \qquad \textit{for all } n,$$

and

(1.3) $$\sup_n E|\varepsilon(n)|^r<\infty \quad \textit{for some } r>2,$$

and let $\{z(n)\}$ be a linear process defined by (1.1). Let $S_n=\sum_{j=1}^{n} z(j)$ and let $g(n)=ES_n^2$. Suppose that

(1.4) $$\liminf_{n\to\infty} g(Kn)/g(n)>K^{2/r} \quad \textit{for some integer } K\geqq 2,$$

and

(1.5) $$^{\forall}\gamma>0,\ ^{\exists}\delta<1 \quad \textit{such that } \limsup_{n\to\infty}\{\max_{\delta n\leq i\leq n} g(i)/g(n)\}<1+\gamma.$$

Then

(1.6) $$\limsup_{n\to\infty} |S_n| / \{2g(n) \log\log g(n)\}^{1/2} \leqq 1 \quad \text{a.s.}$$

In Theorem A, the case $a(j)=0$, $j<0$, is of special interest. In this case the process $\{z(n)\}$ is a one-sided moving average. The one-sided case is of fundamental importance in prediction theory. In fact, if $\{y(n)\}$ is a second-order stationary, purely nondeterministic, process (cf. [3], Chapter III), then it may be represented in the form

$$y(n)=\sum_{j=-\infty}^{n} a(n-j)\varepsilon(j), \quad \sum_{j=0}^{\infty} a^2(j)<\infty, \quad a(0)=1;$$

$$E\varepsilon(n)=0 \text{ for all } n, \quad E(\varepsilon(m)\varepsilon(n))=0 \text{ if } m\neq n$$

with the $\varepsilon(n)$ as the linear prediction errors, each having variance $\sigma^2>0$.

In this paper we consider a process of the form

(1.7) $$x(n)=\sum_{j=-\infty}^{n} a(n-j)\varepsilon(j), \quad \sum_{j=0}^{\infty} a^2(j)<\infty,$$

where the $\{\varepsilon(n), \mathcal{F}_n, -\infty<n<\infty\}$ are martingale differences, i.e.,

(1.8) $$E(\varepsilon(n)|\mathcal{F}_{n-1})=0 \quad \text{a.s. for each } n,$$

$\mathcal{F}_n$ being the σ-field generated by $\varepsilon(m)$, $m\leqq n$, and $E\varepsilon^2(n)=\sigma^2$ for each n. This model is important in prediction theory, the martingale condition (1.8) corresponding to the condition that the best linear predictor is the best predictor (both in the least squares sense; see Hannan and Heyde [4]).

It is our object here to give the following generalization of Theorem A (in the one-sided case).

Theorem 1. *Let* $\{\varepsilon(n), \mathcal{F}_n, -\infty<n<\infty\}$ *be a martingale difference sequence such that*

(1.9) $$E(\varepsilon^2(n)|\mathcal{F}_{n-1})=\sigma^2 \quad \text{a.s. } \textit{for all } n,$$

and

(1.10) $E(|\varepsilon(n)|^r|\mathcal{F}_{n-1})\leqq A$ a.s. *for all* n, *some* $r>2$ *and some constant* $A<\infty$,

and let $\{x(n)\}$ *be a linear process defined by* (1.7). *Let* $S_n=\sum_{j=1}^{n} x(j)$ *and let* $g(n)=ES_n^2$. *Suppose that* g *satisfies conditions* (1.4) *and* (1.5). *Then* (1.6) *holds.*

In the field of applications, however, it will be often difficult to check conditions (1.9) and (1.10), while conditions (1.2) and (1.3) are satisfied in many applications. For this reason, replacing the assumption (1.10) of Theorem 1 by the weaker assumption (1.3) (and replacing (1.9) by (1.2)), we give the following iterated logarithm and stability results.

Theorem 2. *Let* $\beta>1$. *In the notation of Theorem* 1, *suppose that* (1.3) *and* (1.9) *hold, and that* g *satisfies* (1.5) *and*

$$\liminf_{n\to\infty} g(Kn)/g(n)\geqq K^{2\beta/r(\beta-1)} \quad \textit{for some integer } K\geqq 2. \tag{1.11}$$

Then

$$\limsup_{n\to\infty} |S_n|/\{2\beta g(n)\log\log g(n)\}^{1/2}\leqq 1 \quad \text{a.s.} \tag{1.12}$$

Let us introduce the following notation. We denote by Ψ_c the set of functions $\psi(n)$ defined on the positive integers such that

(i) $\psi(n)$ is positive and nondecreasing, and

(ii) $\sum_{n=1}^{\infty} 1/n\psi(n)<\infty$.

Examples of $\psi(n)$ are $(\log n)^{1+\delta}$, $\log n(\log\log n)^{1+\delta}$, etc., where $\delta>0$.

Theorem 3. *In the notation of Theorem* 1, *suppose* (1.2), (1.3), (1.4) *and* (1.5) *hold. Then for any* $\psi\in\Psi_c$,

$$\lim_{n\to\infty} S_n/\{g(n)\psi^{2/r}(n)\}^{1/2}=0 \quad \text{a.s.} \tag{1.13}$$

Remarks and Examples. (I) Let $\{X_n, \mathcal{F}_n, -\infty<n\leqq N\}$ be a martingale difference sequence with $\sum_{n=-\infty}^{N} EX_n^2<\infty$, and let $U_n=\sum_{j=n}^{N} X_j$. Since $|U_k-U_m|\leqq |U_n-U_m|+|U_n-U_k|$ for $n\leqq k\leqq m$, by Kolmogorov's inequality for submartingales,

$$\begin{aligned} &P[\max_{n\leq k<m} |U_k-U_m|>\varepsilon] \\ &\leqq P[|U_n-U_m|>\varepsilon/2]+P[\max_{n<k\leq m} |U_n-U_k|>\varepsilon/2] \\ &\leqq (8/\varepsilon^2)E(U_n-U_m)^2=(8/\varepsilon^2)\sum_{j=n}^{m-1} EX_j^2 \end{aligned}$$

for each $\varepsilon>0$. Thus U_n converges a.s. as $n\to-\infty$ (cf. [2], p. 238). From this point of view, the series in (1.7) converges a.s.

(II) As pointed out by Lai and Stout [5] (see also Lai and Wei [7]), the conditions (1.4) (or (1.11)) and (1.5) cover a wide range of correlation structures for the sequences $\{x(n)\}$. A simple example is

(i) Ler $\gamma(n)=E(x(1)x(n+1))$. Suppose that $\gamma(0)>0$ and $\gamma(n)\geqq 0$ for all $n\geqq 1$. Then $g(n)$ is increasing and hence (1.5) holds. Trivially, $g(2n)\geqq 2g(n)$ and hence (1.4) holds and if $\beta\geqq r/(r-2)$, then (1.11) also holds. Another examples considered in [5] and [7] are

(ii) Suppose that $1<\alpha<2$ and

$$\gamma(n)\sim n^{\alpha-2}L(n),$$

where $L(n)$ is a positive slowly varying function. Then

$$g(n)=n\gamma(0)+2\sum_{j=1}^{n-1}(n-j)\gamma(j)\sim 2\{\alpha(\alpha-1)\}^{-1}n^{\alpha}L(n),$$

and hence (1.4) holds, and if $\beta\geqq\alpha r/(\alpha r-2)$, then (1.11) holds.

(iii) Suppose that $0<\alpha<1$ and

$$\gamma(n)\sim -n^{\alpha-2}L(n);$$

$$\gamma(0)+2\sum_{n=1}^{\infty}\gamma(n)=0.$$

Then

$$g(n)\sim 2\{\alpha(1-\alpha)\}^{-1}n^{\alpha}L(n).$$

Hence, if $r>2/\alpha$, then (1.4) holds, and if in addition $\beta\geqq\alpha r/(\alpha r-2)$, then (1.11) holds.

(iv) Suppose that

$$\sum_{j=0}^{\infty}|a(j)|<\infty,\quad \sum_{j=0}^{\infty}a(j)\neq 0 \quad\text{and}\quad \sigma^2(=E\varepsilon^2(0))>0.$$

Then

$$g(n)\sim n\sigma^2\Big(\sum_{j=0}^{\infty}a(j)\Big)^2,$$

and hence (1.4) holds, and if $\beta\geqq r/(r-2)$, then (1.11) holds. Obviously all the g in (ii)-(iv) satisfy (1.5). For the particular case,

$$a(0)=1,\quad a(j)=0 \text{ if } j\geqq 1,\quad \sigma^2>0,$$

S_n reduces to the partial sum $\sum_{j=1}^{n}\varepsilon(j)$ of martingale differences and $g(n)=\sigma^2 n$, which is a special case of (i) (and (iv)) and has been extensively studied in the literature. In the case (iii), since $0<\alpha<1$ and $|\gamma(n)|\sim n^{\alpha-2}L(n)$, $\sum_{n=1}^{\infty}|\gamma(n)|<\infty$. Hence $\rho^2=\gamma(0)+2\sum_{n=1}^{\infty}\gamma(n)$ is a finite nonnegative number. The case (iii) therefore deals with the case $\rho^2=0$. Let $f(\lambda)$ be the spectral density of the process $\{x(n)\}$. We note that if $\sigma^2>0$, then

$$\rho^2=0 \Leftrightarrow \sum_{j=0}^{\infty}a(j)=0 \Leftrightarrow f(0)=0,$$

since

$$\rho^2=\sigma^2\Big(\sum_{j=0}^{\infty}a(j)\Big)^2 \quad\text{and}\quad f(\lambda)=\sigma^2(2\pi)^{-1}\Big|\sum_{j=0}^{\infty}a(j)e^{i\lambda j}\Big|^2.$$

2. Proofs

The proofs of Theorems 1 and 2 are based on the following lemma due to Lai and Wei [7].

Lemma 1 [7]. *Let $\{Y_n\}$ be a sequence of random variables and $\{B_n\}$ be a sequence of positive constants such that $\lim_{n\to\infty} B_n=\infty$. Suppose that there exist $\theta>1$ aud $\tau(\theta)>0$ such that as $n\to\infty$*

$$P[|Y_n|\geqq\tau(\theta)(B_n \log\log B_n)^{1/2}]=O(\exp(-\theta \log\log B_n)).$$

Suppose further that there exist $q>0$ and $f:\{1, 2, \cdots\}\to(0, \infty)$ such that

$$E|Y_n-Y_m|^q\leqq f(n-m) \quad \text{for } n>m\geqq m_0,$$

$$f(n)=O(B_n^{q/2}) \qquad \text{as} \quad n\to\infty,$$

$$\liminf_{n\to\infty} f(Kn)/f(n)\geqq K^\lambda \quad \text{for some } \lambda>\theta/(\theta-1) \text{ and integer } K\geqq 2$$

and

$$\forall\gamma>0,\ \exists\delta<1 \text{ such that } \limsup_{n\to\infty}\{\max_{\delta n\leq i\leq n} f(i)/f(n)\}<1+\gamma.$$

Then

$$\limsup_{n\to\infty} |Y_n|/(B_n \log\log B_n)^{1/2}\leqq\tau(\theta) \quad \text{a.s.}$$

We first prove the following lemma.

Lemma 2. *Let $\{\varepsilon(n), \mathcal{F}_n, -\infty<n<\infty\}$ be a martingale difference sequence such that $E(\varepsilon^2(n)|\mathcal{F}_{n-1})\leqq\sigma^2$ a.s. for all n, and (1.3) holds. Let $\{b_{nj}: n\geqq 1, -\infty<j\leqq n\}$ be a double array of constants such that*

(2.1) $$\sum_{j=-\infty}^{n} b_{nj}^2<\infty \qquad \text{for all } n$$

(2.2) $$A_n=\sum_{j=-\infty}^{n} b_{nj}^2 \longrightarrow \infty \qquad \text{as } n\to\infty;$$

and

(2.3) $$\sup_j b_{nj}^2=o(A_n(\log A_n)^{-\rho}) \quad \text{for all } \rho>0.$$

Let $S_n=\sum_{j=-\infty}^{n} b_{nj}\varepsilon(j)$. Then for all $\zeta>1$ and $\theta>0$, as $n\to\infty$

(2.4) $$P[|S_n|>\zeta\sigma(2\theta A_n \log\log A_n)^{1/2}]=O(\exp(-\theta \log\log A_n)).$$

Proof. We first note that for all $\theta>0$,

(2.5) $$\begin{aligned} &P[|b_{nj}\varepsilon(j)|>A_n^{1/2}(\log\log A_n)^{-1} \text{ for some } j] \\ &\leqq \sum_{j=-\infty}^{n} A_n^{-r/2}(\log\log A_n)^r|b_{nj}|^r E|\varepsilon(j)|^r \\ &\leqq (\sup_j E|\varepsilon(j)|^r)\{(\sup_j b_{nj}^2)/A_n\}^{(r-2)/2}(\log\log A_n)^r \\ &=o(\exp(-\theta \log\log A_n)), \quad \text{by (2.3)}. \end{aligned}$$

Let

$$\varepsilon_n(j)=\varepsilon(j)I[|b_{nj}\varepsilon(j)|\leqq A_n^{1/2}(\log\log A_n)^{-1}],$$

and

$$\tilde{\varepsilon}_n(j)=\varepsilon_n(j)-E(\varepsilon_n(j)|\mathcal{F}_{j-1})$$

for $j\leqq n$ and $n\geqq 1$. Then for each $n\geqq 1$, $\{\tilde{\varepsilon}_n(j), \mathcal{F}_j, -\infty<j\leqq n\}$ is a martingale difference sequence with

(2.6) $$E(\tilde{\varepsilon}_n^2(j)|\mathcal{F}_{j-1})\leqq\sigma^2 \quad \text{a.s. for all } j,$$

and

(2.7) $$|b_{nj}\tilde{\varepsilon}_n(j)|\leqq 2A_n^{1/2}(\log\log A_n)^{-1} \quad \text{a.s. for all } j.$$

Let $\zeta>\zeta'>1$ and put

(2.8) $$c_n=2A_n^{1/2}(\log\log A_n)^{-1}, \quad \lambda_n=\zeta'\sigma^{-1}A_n^{-1/2}(2\theta\log\log A_n)^{1/2}.$$

Then, by virtue of (2.2), there exists n_0 such that $\lambda_n c_n\leqq 1$ for all $n\geqq n_0$. Let $\tilde{S}_n=\sum_{j=-\infty}^{n} b_{nj}\tilde{\varepsilon}_n(j)$, and define nonnegative random variables T_n by

(2.9) $$T_n=\exp(\lambda_n\tilde{S}_n)\exp\left[-(\lambda_n^2/2)(1+\lambda_n c_n/2)\sum_{j=-\infty}^{n} b_{nj}^2 E(\tilde{\varepsilon}_n^2(j)|\mathcal{F}_{j-1})\right].$$

In view of a remark made in Section 1, (2.1) and (2.6), both the series in (2.9) converge a.s., and therefore the T_n are well-defined.

We now show that for each $n\geqq n_0$

(2.10) $$ET_n\leqq 1.$$

Combining this with a simple relation $ET_n\geqq\alpha P[T_n>\alpha]$ implies

(2.11) $$P[T_n>\alpha]\leqq\alpha^{-1} \quad \text{for all } n\geqq n_0 \text{ and } \alpha>0.$$

In order to prove (2.10), we let

$$T_{k,m}^{(n)}=\exp\left(\lambda_n\sum_{j=m}^{k} b_{nj}\tilde{\varepsilon}_n(j)\right)\exp\left[-(\lambda_n^2/2)(1+\lambda_n c_n/2)\sum_{j=m}^{k} b_{nj}^2 E(\tilde{\varepsilon}_n^2(j)|\mathcal{F}_{j-1})\right]$$

for $m\leqq k\leqq n$ and $n\geqq n_0$, and $T_{m-1,m}^{(n)}=1$ a.s. Then, in view of (2.7) and (2.8), $\{T_{k,m}^{(n)}, \mathcal{F}_k, m-1\leqq k\leqq n\}$ forms a nonnegative supermartingale for each $m\leqq n$ and $n\geqq n_0$ (cf. [8], p. 299). Thus

$$ET_{n,m}^{(n)}\leqq ET_{n-1,m}^{(n)}\leqq\cdots\leqq ET_{m,m}^{(n)}\leqq 1$$

On the other hand,

$$T_n=\lim_{m\to-\infty} T_{n,m}^{(n)} \quad \text{a.s.}$$

Hence by the Fatou lemma

$$ET_n\leqq\liminf_{m\to-\infty} ET_{n,m}^{(n)}\leqq 1,$$

which proves (2.10).

By (2.6) and (2.11), we obtain that

$$(2.12)\qquad P[\tilde{S}_n>\zeta'\sigma(2\theta A_n \log\log A_n)^{1/2}]$$
$$=P[\exp(\lambda_n\tilde{S}_n)>\exp\{\lambda_n\zeta'\sigma(2\theta A_n \log\log A_n)^{1/2}\}]$$
$$=P\Big[T_n>\exp\Big\{\lambda_n\zeta'\sigma(2\theta A_n \log\log A_n)^{1/2}$$
$$-(\lambda_n^2/2)(1+\lambda_n c_n/2)\sum_{j=-\infty}^{n} b_{nj}^2 E(\tilde{\varepsilon}_n^2(j)\,|\,\mathcal{F}_{j-1})\Big\}\Big]$$
$$\leqq\exp\{-\lambda_n\zeta'\sigma(2\theta A_n \log\log A_n)^{1/2}+(\lambda_n^2/2)(1+\lambda_n c_n/2)\sigma^2 A_n\}$$
$$\leqq\exp(-\theta\log\log A_n)$$

for all n sufficiently large. By a similar argument as above, we have for all large n,

$$(2.13)\qquad P[-\tilde{S}_n>\zeta'\sigma(2\theta A_n \log\log A_n)^{1/2}]\leqq\exp(-\theta\log\log A_n)\,.$$

Combining (2.12) and (2.13), we get

$$(2.14)\qquad P[|\tilde{S}_n|>\zeta'\sigma(2\theta A_n \log\log A_n)^{1/2}]\leqq 2\exp(-\theta\log\log A_n)$$

for all n large enough.

Since $E(\varepsilon(j)\,|\,\mathcal{F}_{j-1})=0$ a.s., we have for all $\tau>0$ and $\theta>0$,

$$(2.15)\qquad P\Big[\Big|\sum_{j=-\infty}^{n} b_{nj}E(\varepsilon_n(j)\,|\,\mathcal{F}_{j-1})\Big|>\tau(A_n\log\log A_n)^{1/2}\Big]$$
$$\leqq\tau^{-1}(A_n\log\log A_n)^{-1/2}E\Big|\sum_{j=-\infty}^{n} b_{nj}E(\varepsilon_n(j)\,|\,\mathcal{F}_{j-1})\Big|$$
$$\leqq\tau^{-1}(A_n\log\log A_n)^{-1/2}$$
$$\times\sum_{j=-\infty}^{n} E\{|b_{nj}\varepsilon(j)|I[|b_{nj}\varepsilon(j)|>A_n^{1/2}(\log\log A_n)^{-1}]\}$$
$$\leqq\tau^{-1}(\sup_j E|\varepsilon(j)|^r)\{(\sup_j b_{nj}^2)/A_n\}^{(r-2)/2}(\log\log A_n)^{r-3/2}$$
$$=o(\exp(-\theta\log\log A_n))\,,\quad\text{by (2.3)}.$$

From (2.5), (2.14) and (2.15), (2.4) follows as desired.

Proof of Theorem 1. Let for $j\leqq n$ and $n\geqq 1$,

$$(2.16)\qquad b_{nj}=\sum_{i=1}^{n} a(i-j)\,,\quad\text{where } a(i-j)=0 \text{ if } i<j\,.$$

Then we note that

$$S_n=\sum_{j=1}^{n} x(j)=\sum_{j=-\infty}^{n} b_{nj}\varepsilon(j).$$

Therefore

$$g(n)=ES_n^2=\sigma^2 A_n, \quad \text{where } A_n=\sum_{j=-\infty}^{n} b_{nj}^2.$$

Note that $\lim_{n\to\infty} g(n)=\infty$ under the conditions (1.4) and (1.5) (cf. [5]). Therefore $\sigma^2>0$, $\lim_{n\to\infty} A_n=\infty$ and (2.2) holds. Note also that (2.3) holds for the double array $\{b_{nj}\}$ defined by (2.16) (see [7], p. 327).

Let $0<\delta<1$. In view of (1.10), we can choose $B>0$ such that

$$E(\varepsilon^2(j)I[|\varepsilon(j)|>B]|\mathcal{F}_{j-1})\leqq\delta^2\sigma^2 \quad \text{a.s. for all } j. \tag{2.17}$$

Let

$$e(j)=\varepsilon(j)I[|\varepsilon(j)|\leqq B]-E(\varepsilon(j)I[|\varepsilon(j)|\leqq B]|\mathcal{F}_{j-1}).$$

Then $\{e(j), \mathcal{F}_j, -\infty<j<\infty\}$ is a martingale difference sequence with

$$E(e^2(j)|\mathcal{F}_{j-1})\leqq E(\varepsilon^2(j)|\mathcal{F}_{j-1})=\sigma^2 \quad \text{a.s. for all } j.$$

Applying Lemma 2 with $\theta=1+\delta$ and $\zeta=(1+\delta)^{1/2}$, we obtain that

$$P\Big[\Big|\sum_{j=-\infty}^{n} b_{nj}e(j)\Big|>(1+\delta)\sigma(2A_n\log\log A_n)^{1/2}\Big] =O(\exp\{-(1+\delta)\log\log A_n\}). \tag{2.18}$$

Now we note that for $n>m\geqq 0$

$$\sum_{j=-\infty}^{n} b_{nj}e(j)-\sum_{j=-\infty}^{m} b_{mj}e(j)=\sum_{j=-\infty}^{n} b_{n-m,j-m}e(j).$$

Since $\sup_j|e(j)|\leqq 2B$ a.s., by Burkholder's inequality [1], we obtain that for every $p>1$ and $n>m\geqq 0$

$$E\Big|\sum_{j=-\infty}^{n} b_{nj}e(j)-\sum_{j=-\infty}^{m} b_{mj}e(j)\Big|^p\leqq C_p(2B)^p\Big(\sum_{j=-\infty}^{n-m} b_{n-m,j}^2\Big)^{p/2} =C_p(2B)^p A_{n-m}^{p/2}, \tag{2.19}$$

where C_p is a positive constant depending only on p. Take $s>2/r$ such that $\liminf_{n\to\infty} g(Kn)/g(n)\geqq K^s$, where K is as given in (1.4), and choose p large enough such that $ps/2>(1+\delta)/\delta$. Then in view of (2.18) and (2.19), we can apply Lemma 1 (with $q=p$ and $f(n)=C_p(2B)^p A_n^{p/2}$) and obtain that

$$\limsup_{n\to\infty}\Big|\sum_{j=-\infty}^{n} b_{nj}e(j)\Big|\Big/(A_n\log\log A_n)^{1/2}\leqq(1+\delta)2^{1/2}\sigma \quad \text{a.s.} \tag{2.20}$$

Let $d(j)=\varepsilon(j)-e(j)$. Then $\{d(j), \mathcal{F}_j, -\infty<j<\infty\}$ is a martingale difference sequence with

$$E(d^2(j)\,|\,\mathcal{F}_{j-1})\leqq E(\varepsilon^2(j)I[\,|\varepsilon(j)|>B]\,|\,\mathcal{F}_{j-1})\leqq\delta^2\sigma^2 \quad \text{a.s. for all } j,$$

by (2.17). Hence by Lemma 2,

$$P\Big[\Big|\sum_{j=-\infty}^{n} b_{nj}d(j)\Big|>(1+\delta)\delta\sigma(2\theta A_n \log\log A_n)^{1/2}\Big] = O(\exp(-\theta \log\log A_n)) \tag{2.21}$$

for all $\theta>0$. Moreover, by Burkholder's inequality again,

$$\begin{aligned} E\Big|\sum_{j=-\infty}^{n} b_{nj}d(j)-\sum_{j=-\infty}^{m} b_{mj}d(j)\Big|^r &= E\Big|\sum_{j=-\infty}^{n} b_{n-m,j-m}d(j)\Big|^r \\ &\leqq 2^r AC_r\Big(\sum_{j=-\infty}^{n-m} b_{n-m,j}^2\Big)^{r/2} \\ &= 2^r AC_r A_{n-m}^{r/2} \end{aligned} \tag{2.22}$$

for $n>m\geqq 0$. Choose θ large enough so that

$$\theta>1, \quad \theta/(\theta-1)<rs/2. \tag{2.23}$$

Then in view of (2.21), (2.22) and (2.23), we can again apply Lemma 1 (with $q=r$ and $f(n)=2^r AC_r A_n^{r/2}$) and obtain that

$$\limsup_{n\to\infty}\Big|\sum_{j=-\infty}^{n} b_{nj}d(j)\Big|\Big/(A_n\log\log A_n)^{1/2}\leqq(2\theta)^{1/2}(1+\delta)\delta\sigma \quad \text{a.s.} \tag{2.24}$$

Since δ is arbitrary, the upper half (1.6) of the law of the iterated logarithm follows from (2.20) and (2.24).

Proof of Theorem 2. Let $\delta>0$. By Burkholder's inequality, we obtain that for $n>m\geqq 0$

$$\begin{aligned} E|S_n-S_m|^r &= E\Big|\sum_{j=-\infty}^{n} b_{n-m,j-m}\varepsilon(j)\Big|^r \\ &\leqq C_r(\sup_j E|\varepsilon(j)|^r)\Big(\sum_{j=-\infty}^{n-m} b_{n-m,j}^2\Big)^{r/2} \\ &= C_r(\sup_j E|\varepsilon(j)|^r)A_{n-m}^{r/2}. \end{aligned} \tag{2.25}$$

Hence applying Lemma 2 with $\theta=\beta+\delta$ and $\zeta=1+\delta$, and further applying Lemma 1 with $q=r$ and $f(n)=C_r(\sup_j E|\varepsilon(j)|^r)A_n^{r/2}$, we have

$$\limsup_{n\to\infty}|S_n|/(A_n\log\log A_n)^{1/2}\leqq(1+\delta)(\beta+\delta)^{1/2}2^{1/2}\sigma \quad \text{a.s.} \tag{2.26}$$

Since δ is arbitrary, the desired conclusion (1.12) follows from (2.26).

Proof of Theorem 3. The desired conclusion (1.13) immediately follows from (2.25) and Theorems 5 and 7 of Lai and Stout [6].

References

[1] D.L. Burkholder: *Martingale transforms,* Ann. Math. Statist. **37** (1966), 1494–1504.
[2] W. Feller: *An Introduction to Probability Theory and Its Applications,* Vol. II. 2nd ed. Wiley, New York 1971.
[3] E.J. Hannan: *Multiple Time Series,* Wiley, New York 1970.
[4] E.J. Hannan and C.C. Heyde: *On limit theorems for quadratic functions on discrete time series,* Ann. Math. Statist. **43** (1972), 2058-2066.
[5] T.L. Lai and W. Stout: *The law of the iterated logarithm and upper-lower class tests for partial sums of stationary Gaussian sequences,* Ann. Probability **6** (1978), 731-750.
[6] T.L. Lai and W. Stout: *Limit theorems for sums of dependent random variables,* Z. Wahrscheinlichkeitstheorie verw. Gebiete **51** (1980), 1-14.
[7] T.L. Lai and C.Z. Wei: *A law of the iterated logarithm for double arrays of independent random variables with applications to regression and time series models,* Ann. Probability **10** (1982), 320-335.
[8] W.F. Stout: *Almost Sure Convergence,* Academic, New York 1974.

Department of Mathematics
Osaka Kyoiku University
Tennoji, Osaka 543, Japan