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ISOTROPIC IMMERSIONS WITH PARALLEL SECOND FUNDAMENTAL FORM II

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0. Introduction

Recently, Sakamoto [6] has classified isotropic submanifolds M with parallel second fundamental form in the Euclidean sphere S^m. He stated that M is locally isometric to compact symmetric spaces of rank one and the immersion is locally congruent to the second of first standard immersion according as M is a sphere or not.

Motivated by his work, we have already characterized the second standard minimal immersion of a sphere into S^m in terms of isotropic immersions. Namely, we obtained the following (for details, see [3]).

Theorem. *Let M be an n-dimensional real space form of constant curvature c, and $\tilde{M}$ be an $\left(n+\frac{1}{2}n(n+1)-1\right)$-dimensional real space form of constant curvature $\tilde{c}$. If $c<\tilde{c}$ and M is an isotropic submanifold of $\tilde{M}$, then M is immersed as a Veronese manifold into $\tilde{M}$.*

The purpose of this paper is to characterize in terms of isotropic immersions the first standard minimal immersions of other compact symmetric spaces of rank one into a sphere. We get the following.

Theorem 1. *Let M be a real $2n$-dimensional complex space form and $\tilde{M}^{2n+p}$ be a $(2n+p)$-dimensional real space form of constant curvature $\tilde{c}>0$. If $p\leqq n^2-1$ and M is an isotropic submanifold of $\tilde{M}$, then $p=n^2-1$, M is locally isometric to a complex projective space and the immersion is locally congruent to the first standard minimal immersion.*

Theorem 2. *Let M be a real $4n$-dimensional quaternionic space from and $\tilde{M}^{4n+p}$ be a $(4n+p)$-dimensional real space form of constant curvature $\tilde{c}>0$. If $p\leqq(n-1)(2n+1)$ and M is an isotropic submanifold of $\tilde{M}$, then $p=(n-1)(2n+1)$, M is locally isometric to a quaternion projective space and the immersion is locally congruent to the first standard minimal immersion.*

Theorem 3. *Let M be an open connected submanifold of either the Cayley* *plane or its noncompact dual and* $\tilde{M}^{16+p}$ *be a* $(16+p)$*-dimensional real space form of constant curvature* $\tilde{c}>0$. *If* $p\leqq 9$ *and* M *is an isotropic submanifold of* $\tilde{M}$, *then* $p=9$, M *is not an open connected submanifold of the noncompact dual of the Cayley plane and the immersion is locally congruent to the first standard minimal immersion.*

Remark. Due to Sakamoto [6], we have only to show that the second fundamental form of the immersion is parallel in order to prove Theorems 1, 2 and 3.

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1. Preliminies

Let M be an n-dimensional submanifold of a Riemannian manifold $\tilde{M}^{n+p}$ with $\langle\,,\rangle$. Let ∇ and $\tilde{\nabla}$ be the Riemannian connections of M and $\tilde{M}$, respectively. Then the second fundamental form σ of the immersion is given by $\sigma(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y$, where X and Y are tangent vector fields on M. We call $\mathcal{S}=(1/n)(tr\ \sigma)$the *mean curvature vector* of M in $\tilde{M}$. The *mean curvature* H of M is the length of $\mathcal{S}$. If $\mathcal{S}$ is identically zero, the submanifold M is said to be *minimal*. The submanifold M is *totally umbilic* provided that $\sigma(X, Y)=\langle X, Y\rangle\mathcal{S}$ for any tangent vector field X, Y on M. In particular, if σ vanishes identically, M is said to be a *totally geodesic* submanifold of $\tilde{M}$. For a vector field ξ normal to M, we write $\tilde{\nabla}_X\xi=-A_\xi X+D_X\xi$, where $-A_\xi X$ (resp. $D_X\xi$) denotes tangential (resp. the normal) component of $\tilde{\nabla}_X\xi$. A normal vector field ξ is said to be *parallel* if $D_X\xi=0$ for any vector field X tangent to M. We define the covariant differentiation ∇' of the second fundamental form σ with respect to the connection in (tangent bundle)+(normal bundle) as follows: $(\nabla'_X\sigma)(Y, Z)=D_X(\sigma(Y, Z))-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z)$. The second fundamental form σ is said to be *parallel* if $(\nabla'_X\sigma)(Y, Z)=0$ for all tangent vector fields X, Y and Z on M. Let $\xi_1, \cdots, \xi_p$ be an orthonormal basis of the normal bundle $T^\perp(M)$ and A_α be the second fundamental form with respect to ξ_α: $\langle A_\alpha X, Y\rangle=\langle\sigma(X, Y), \xi_\alpha\rangle$. $\|\sigma\|$ is the length of the second fundamental form σ of the immersion so that $\|\sigma\|^2=\sum_{\alpha=1}^{p} tr\ A_\alpha^2$. A λ*-isotropic* immersion is an isometric immersion such that all its normal curvature vectors have the same length λ at each point. Namely, the length λ of the normal curvature vector is a function on the submanifold. In particular, if the function λ is constant, then the immersion is said to be constant λ-isotropic. Here and in the sequel, we study isotropic submanifolds in case that the ambient manifold $\tilde{M}$ is a real space form of curvature $\tilde{c}$. A Riemannian manifold of constant curvature is called a *real space form*. For later use, we write down Gauss and Coddazi equations:

$$\text{(1.1)} \qquad \langle\sigma(X, Y), \sigma(Z, W)\rangle-\langle\sigma(Z, Y), \sigma(X, W)\rangle$$
$$=\langle R(Z, X)Y, W\rangle-\tilde{c}(\langle X, Y\rangle\langle Z, W\rangle-\langle Z, Y\rangle\langle X, W\rangle),$$

$$\text{(1.2)} \qquad (\nabla'_X\sigma)(Y, Z)=(\nabla'_Y\sigma)(X, Z),$$

where R denotes the curvature tensor of M.

Now we write the curvature tensors of symmetric spaces of rank one except real space forms. A Kaehler manifold M of constant holomorphic sectional curvature is called a *complex space form*. The curvature tensor R of a complex space form M with complex structure J of constant holomorphic sectional curvature $4c$ is given by

$$\text{(1.3)} \qquad R(X, Y)Z=c\{\langle Y, Z\rangle X-\langle X, Z\rangle Y+\langle JY, Z\rangle JX$$
$$-\langle JX, Z\rangle JY+2\langle X, JY\rangle JZ\}$$

for all vector fields X, Y and Z tangent to M. A quaternionic Kaehler manifold of constant Q-sectional curvature is called a *quaternionic space form*. As is well-known (cf. [2]), the curvature tensor R of a quaternionic space form M of constant Q-sectional curvature $4c$ is given by

$$\text{(1.4)} \qquad R(X, Y)Z=c\{\langle Y, Z\rangle X-\langle X, Z\rangle Y+\langle IY, Z\rangle IX-\langle IX, Z\rangle IY$$
$$+\langle JY, Z\rangle JX-\langle JX, Z\rangle JY+\langle KY, Z\rangle KX-\langle KX, Z\rangle KY$$
$$+2\langle X, IY\rangle IZ+2\langle X, JY\rangle JZ+2\langle X, KY\rangle KZ\}$$

for all vector fields X, Y and Z tangent to M, where $\{I, J, K\}$ is a canonical local basis of M.

Let M be either the Cayley plane or its noncompact dual. Here we denote by Cay the Cayley numbers, which is an 8-dimensional non-associative division algebra over the real numbers. It has a multiplicative identity and a positive definite bilinear form $\langle\, ,\rangle$. The tangent space of M may be identified with $V=\mathrm{Cay}\oplus\mathrm{Cay}$, viewed as ordered pairs of Cayley numbers. The vector space V has a positive definite symmetric form $\langle\, ,\rangle$ given by $\langle(a, c), (b, d)\rangle=\langle a, b\rangle+\langle c, d\rangle$ (for details, see [1]). The curvature tensor R of M is given by

$$\text{(1.5)} \qquad \langle R((a, b), (c, d))(e, f), (g, h)\rangle$$
$$=\frac{\alpha}{4}(4\langle c, e\rangle\langle a, g\rangle-4\langle a, e\rangle\langle c, g\rangle+\langle ed, gb\rangle-\langle eb, gd\rangle$$
$$+\langle ad-cb, gf\rangle+\langle cf, ah\rangle-\langle af, ch\rangle-4\langle b, f\rangle\langle d, h\rangle$$
$$+4\langle d, f\rangle\langle b, h\rangle-\langle ad-cb, eh\rangle),$$

where α is a nonzero real number.

2. Lemmas

For orthonormal vectors $X, Y \in T_x(M)$, we denote by $K(X, Y)$ (resp. $\bar{K}(X, Y)$) the sectional curvature of the plane spanned by X and Y for M (resp. for $\bar{M}$) and we put $\Delta_{XY}=K(X, Y)-\bar{K}(X, Y)$. We call Δ the discriminant at $x \in M$. The following lemma is due to O'Neill [5].

Lemma 1. *Let M^n be a λ (>0)-isotropic submanifold in a Riemannian manifold $\bar{M}$. Assume that the discriminant Δ at $x \in M$ is constant. Then the following inequalities hold at x:*

$$-((n+2)/2(n-1))\lambda^2 \leqq \Delta \leqq \lambda^2.$$

Let N_x^1 be the first normal space at x of the above immersion, that is, the vector space spanned by all vectors $\sigma(X, Y)$. Then we have

(1) $\Delta=\lambda^2 \Leftrightarrow M$ *is totally umbilic at* $x \Leftrightarrow \dim N_x^1=1$,

(2) $\Delta=-((n+2)/2(n-1))\lambda^2 \Leftrightarrow M$ *is minimal at* $x \Leftrightarrow \dim N_x^1=n(n+1)/2-1$,

(3) $-((n+2)/2(n-1))\lambda^2<\Delta<\lambda^2 \Leftrightarrow \dim N_x^1=n(n+1)/2$.

Finally we prepare the following lemma, which is indebted to Nakagawa and Itoh [4].

Lemma 2. *Let M^n be a constant λ-isotropic submanifold in a real space form $\tilde{M}^{n+p}$ of curvature $\tilde{c}$. We assume that M is locally symmetric and the first normal space equals the normal space at any point of M. Then the second fundamental form of the immersion is parallel.*

Proof. By assumption, for all vector fields X on M, we have

$$\langle\sigma(X, X), \sigma(X, X)\rangle=\lambda^2\langle X, X\rangle\langle X, X\rangle .$$

This is equivalent to

$$\begin{aligned}(2.1)\qquad &\langle\sigma(X, Y), \sigma(Z, W)\rangle+\langle\sigma(X, Z), \sigma(Y, W)\rangle+\langle\sigma(X, W), \sigma(Y, Z)\rangle\\ &=\lambda^2(\langle X, Y\rangle\langle Z, W\rangle+\langle X, Z\rangle\langle X, W\rangle+\langle X, W\rangle\langle Y, Z\rangle)\end{aligned}$$

for all vector fields X, Y, Z and W tangent to M. On the other hand, exchanging X and Y in (1.1), we get

$$\begin{aligned}(2.2)\qquad &\langle\sigma(Y, X), \sigma(Z, W)\rangle-\langle\sigma(Z, X), \sigma(Y, W)\rangle\\ &=\langle R(Z, Y)X, W\rangle-\tilde{c}(\langle Y, X\rangle\langle Z, W\rangle-\langle Z, X\rangle\langle Y, W\rangle) .\end{aligned}$$

Summing up (1.1), (2.1) and (2.2), we obtain

$$(2.3)\qquad \langle\sigma(X, Y), \sigma(Z, W)\rangle=\frac{1}{3}(\langle R(Z, X)Y, W\rangle+\langle R(Z, Y)X, W\rangle)$$

$$-\frac{\tilde{c}}{3}(2\langle X, Y\rangle\langle Z, W\rangle-\langle Y, Z\rangle\langle X, W\rangle-\langle Z, X\rangle\langle Y, W\rangle)$$

$$+\frac{\lambda^2}{3}(\langle X, Y\rangle\langle Z, W\rangle+\langle X, Z\rangle\langle Y, W\rangle+\langle X, W\rangle\langle Y, Z\rangle).$$

Since λ is constant and M is locally symmetric, differentiating (2.3) with respect to any tangent vector field T on M, we have the following:

$$\langle(\nabla'_T\sigma)(X, Y), \sigma(Z, W)\rangle=-\langle\sigma(X, Y), (\nabla'_T\sigma)(Z, W)\rangle. \tag{2.4}$$

By using (2.4) and the Codazzi equation (1.2) repeatedly, we find

$$\begin{aligned}&\langle(\nabla'_T\sigma)(X, Y), \sigma(Z, W)\rangle=-\langle\sigma(X, Y), (\nabla'_Z\sigma)(T, W)\rangle\\ &=\langle(\nabla'_X\sigma)(Z, Y), \sigma(T, W)\rangle=-\langle\sigma(Z, Y), (\nabla'_W\sigma)(X, T)\rangle\\ &=\langle(\nabla'_Y\sigma)(Z, W), \sigma(X, T)\rangle=-\langle\sigma(Z, W), (\nabla'_T\sigma)(X, Y)\rangle.\end{aligned}$$

So we see that $\langle(\nabla'_T\sigma)(X, Y), \sigma(Z, W)\rangle=0$. This, together with the assumption that the first normal space equals the normal space at any point of M, shows that the second fundamental form of our immersion is parallel. Q.E.D.

3. Proof of Theorem 1

Let M be a real $2n$-dimensional complex space form with complex structure J of constant holomorphic sectional curvature $4c$. We fix an arbitrary point x of M. By assumption, all normal curvature vectors at x have the same length, say, λ. Substituting (1.3) into the right-hand side of (2.3), we have

$$\begin{aligned}\langle\sigma(X, Y), \sigma(Z, W)\rangle&=\frac{\lambda^2+2(c-\tilde{c})}{3}\langle X, Y\rangle\langle Z, W\rangle\\ &+\frac{\lambda^2-(c-\tilde{c})}{3}(\langle X, W\rangle\langle Y, Z\rangle+\langle X, Z\rangle\langle Y, W\rangle)\\ &+c(\langle JX, Z\rangle\langle JY, W\rangle+\langle JY, Z\rangle\langle JX, W\rangle)\end{aligned} \tag{3.1}$$

for all vectors X, Y, Z and W of $T_x(M)$. Now we investigate the first normal space of M by using (3.1). We choose a local field of orthonormal frame $e_1, \cdots, e_n, e_{n+1}=Je_1, \cdots, e_{2n}=Je_n$ around x. Since the curvature tensor R of M is a nice form, see (1.3), we immediately find $\langle R(e_i, e_j)e_k, e_l\rangle=c(\delta_{jk}\delta_{il}-\delta_{ik}\delta_{jl})$, where i, j, k and l run over the range $\{1, 2, \cdots, n\}$. So we may apply Lemma 1 to the linear subspace of $T_x(M)$, which is generated by $\{e_1, \cdots, e_n\}$. Our aim here is to show that the case (2) of Lemma 1 occurs at any point of M. First we consider the case (1) of Lemma 1, that is, $\lambda^2=c-\tilde{c}$. From (2.1), we have

$$2\langle\sigma(e_i, e_j), \sigma(e_i, e_j)\rangle+\langle\sigma(e_i, e_i), \sigma(e_j, e_j)\rangle=(c-\tilde{c})(2\delta_{ij}\delta_{ij}+1), \tag{3.2}$$

where i and j run over the range $\{1, 2, \cdots, 2n\}$. Hence, (3.2) yields

$$2\|\sigma\|^2+4n^2H^2=4n(n+1)(c-\tilde{c}), \tag{3.3}$$

where $\|\sigma\|$ is the length of the second fundamental form σ and H is the mean curvature of M.

On the other hand the Gauss equation (1.1), combined with (1.3), shows

$$-\|\sigma\|^2+4n^2H^2=4n(n+1)c-2n(2n-1)\tilde{c}. \tag{3.4}$$

As an immediate consequence of (3.3) and (3.4), we get $\|\sigma\|^2=-2n\tilde{c}<0$. This is a contradiction. Moreover a long but straightforward calculation, by virtue of (3.1), yields the following orthogonal relations:

$$\langle\sigma(e_i, Je_j), \sigma(e_k, Je_l)\rangle=\frac{\lambda^2-(c-\tilde{c})}{3}\delta_{ik}\delta_{jl} \tag{3.5}$$

$$\text{for } 1\leqq i<j\leqq n \text{ and } 1\leqq k<l\leqq n.$$

$$\langle\sigma(e_i, e_j), \sigma(e_k, Je_l)\rangle=0 \quad \text{for } 1\leqq i\leqq j\leqq n \text{ and } 1\leqq k<l\leqq n. \tag{3.6}$$

Then, in consideration of (3.5) and (3.6), we see that the case (3) of Lemma 1 does not occur at x. In fact, in this case we find that the codimension $p\geqq\frac{n(n+1)}{2}+\frac{n(n-1)}{2}=n^2$, which is a contradiction. Hence the case (2) of Lemma 1 occurs at any point x of M so that λ is constant on M and $p=n^2-1$. Thus, in view of Lemma 2, the second fundamental form of our immersion is parallel.
Q. E. D.

4. Proof of Theorem 2

Let M be a real $4n$-dimensional quaternionic space form with canonical local basis $\{I, J, K\}$ of constant Q-sectional curvature $4c$. By the same calculation as in the section 3, we have

$$\begin{aligned}\langle\sigma(X, Y), \sigma(Z, W)\rangle&=\frac{\lambda^2+2(c-\tilde{c})}{3}\langle X, Y\rangle\langle Z, W\rangle\\&+\frac{\lambda^2-(c-\tilde{c})}{3}(\langle X, W\rangle\langle Y, Z\rangle+\langle X, Z\rangle\langle Y, W\rangle)\\&+c(\langle IX, Z\rangle\langle IY, W\rangle+\langle IY, Z\rangle\langle IX, W\rangle+\langle JX, Z\rangle\langle JY, W\rangle\\&+\langle JY, Z\rangle\langle JX, W\rangle+\langle KX, Z\rangle\langle KY, W\rangle+\langle KY, Z\rangle\langle KX, W\rangle).\end{aligned} \tag{4.1}$$

Fix an arbitrary point x of M. We choose a local field of orthonormal frame $e_1, \cdots, e_n, Ie_1, \cdots, Ie_n, Je_1, \cdots, Je_n, Ke_1, \cdots, Ke_n$ around x. We here remark that we may also apply Lemma 1 to the linear subspace of $T_x(M)$, which is generated by $\{e_1, \cdots, e_n\}$. Similarly we find that the case (1) of Lemma 1 does not occur. Moreover a long but straightforward calculation, with the help of (4.1), shows the following orthogonal relations:

(4.2) $$\langle\sigma(e_i, Ie_j), \sigma(e_k, Ie_l)\rangle=\langle\sigma(e_i, Je_j), \sigma(e_k, Je_l)\rangle$$
$$=\langle\sigma(e_i, Ke_j), \sigma(e_k, Ke_l)\rangle=\frac{\lambda^2-(c-\tilde{c})}{3}\delta_{ik}\delta_{jl}$$
$$\text{for } 1\leqq i<j\leqq n \text{ and } 1\leqq k<l\leqq n.$$

(4.3) $$\langle\sigma(e_i, Ie_j), \sigma(e_k, Je_l)\rangle=\langle\sigma(e_i, Je_j), \sigma(e_k, Ke_l)\rangle$$
$$=\langle\sigma(e_i, Ke_j), \sigma(e_k, Ie_l)\rangle=0 \quad \text{for } 1\leqq i<j\leqq n \text{ and } 1\leqq k<l\leqq n.$$

(4.4) $$\langle\sigma(e_i, e_j), \sigma(e_k, Ie_l)\rangle=\langle\sigma(e_i, e_j), \sigma(e_k, Je_l)\rangle$$
$$=\langle\sigma(e_i, e_j), \sigma(e_k, Ke_l)\rangle=0 \quad \text{for } 1\leqq i\leqq j\leqq n \text{ and } 1\leqq k<l\leqq n.$$

Then, in consideration of (4.2), (4.3) and (4.4), we see that the case (3) of Lemma 1 does not occur at x. In fact, in this case we find that the codimension $p\geqq\frac{n(n+1)}{2}+\frac{3n(n-1)}{2}=2n^2-n$, which is a contradiction. Hence the case (2) of Lemma 1 occurs at any point x of M so that λ is constant on M and $p=(n-1)(2n+1)$. Thus, in view of Lemma 2, the second fundamental form of our immersion is parallel. Q. E. D.

5. Proof of Theorem 3

We immediately find from (1.5) that

$$K((a, 0), (b, 0))=\langle R((a, 0), (b, 0))(b, 0), (a, 0)\rangle=\alpha$$

if $(a, 0)\wedge(b, 0)\neq 0$. Hence Lemma 1 asserts that $\lambda^2=\alpha-\tilde{c}$, since $p\leqq 9$. A direct calculation from (1.5), (2.3) and $\lambda^2=\alpha-\tilde{c}$ gives

(5.1) $$\langle\sigma((a, b), (c, d)), \sigma((e, f), (g, h))\rangle$$
$$=(\alpha-\tilde{c})(\langle g, e\rangle\langle a, c\rangle+\langle h, f\rangle\langle b, d\rangle)-\tilde{c}(\langle e, g\rangle\langle b, d\rangle+\langle f, h\rangle\langle a, c\rangle)$$
$$+\frac{\alpha}{3}(\langle g, e\rangle\langle b, d\rangle+\langle h, f\rangle\langle a, c\rangle+\langle a, g\rangle\langle f, d\rangle+\langle b, h\rangle\langle c, e\rangle$$
$$+\langle c, g\rangle\langle f, b\rangle+\langle d, h\rangle\langle e, a\rangle)$$
$$+\frac{\alpha}{12}(\langle eh, cb\rangle+\langle ah, cf\rangle+\langle gf, ad\rangle+\langle gb, ed\rangle+\langle gf, cb\rangle$$
$$+\langle af, ch\rangle+\langle eh, ad\rangle+\langle eb, gd\rangle-2\langle eb, ch\rangle-2\langle af, gd\rangle$$
$$-2\langle gb, cf\rangle-2\langle ah, ed\rangle).$$

Here, for simplicity, we put $X_i=(e_i, 0)$, $Y_i=(0, e_i)$ for $0\leqq i\leqq 7$, where $e_0=1$, $e_1, \cdots, e_7$ is a basis of Cay. By using (5.1), we find that the nonzero vectors $\{\sigma(X_0, X_0), \sigma(X_0, Y_i)\}_{0\leq i\leq 7}$ are mutually orthogonal so that $p=9$. Thus, in view of Lemma 2, the second fundamental form of our immersion is parallel. Q. E. D.

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