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RATES OF CONVERGENCE IN THE ERDÖS-KAC TYPE INVARIANCE PRINCIPLE FOR WEAKLY DEPENDENT RANDOM VARIABLES

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1. Introduction and results.

Let $\{X_i, i\in \boldsymbol{Z}\}$ be a strictly stationary sequence of random variables. Let $\mathfrak{F}_p^q$ denote the σ-field generated by random variables $\{X_i, i=p, p+1, \cdots, q\}$. Suppose that the sequence $\{X_i\}$ satisfies the condition of the absolute regularity, that is

$$\beta(n)=E\{\sup_{A\in\mathfrak{F}_n^\infty}|P(A|\mathfrak{F}_{-\infty}^0)-P(A)|\}\downarrow 0, \tag{1.1}$$

as $n\to\infty$.

Suppose $EX_1=0$ and $E|X_1|^{2+\delta}<\infty$ for some $\delta>0$. Besides if we assume

$$\sum_{i=1}^\infty(\beta(n))^{\delta/(2+\delta)}<\infty,$$

then the series

$$\sigma^2\equiv EX_1^2+2\sum_{i=1}^\infty E(X_1X_{i+1}) \tag{1.2}$$

converges absolutely. (See e.g. [8].) We suppose moreover that $\sigma>0$.

Define a continuous polygonal line $\{X_n(t), 0\leqq t\leqq 1\}$ by

$$X_n(t)=(\sigma n^{1/2})^{-1}\sum_{i=1}^{[nt]}X_i+(nt-[nt])(\sigma n^{1/2})^{-1}X_{[nt]+1},$$

where $[x]$ denotes the integral part of x. Let $\{B(t), t\geqq 0\}$ be a standard Brownian motion. Moreover let P_n be distribution of $\{X_n(t), 0\leqq t\leqq 1\}$ and W the Wiener measure on the space of continuous functions $C[0, 1]$. As is well known, if the above conditions are satisfied, then the invariance principle of Donsker type holds, that is, as $n\to\infty$

$$P_n\Rightarrow W. \quad \text{(Weak convergence in } C[0, 1].) \tag{1.3}$$

(See e.g. Oodaira and Yoshihara [4], Kato [3].) Furthermore, from Theorem 5.1 in Billingsley [1], (1.3) implies the Erdös-Kac type invariance principle, that is

$$\lim_{n\to\infty} P(\max_{1\leq k\leq n} |S_k| < \sigma n^{1/2}\lambda) = T(\lambda) = P(\sup_{0\leq t\leq 1} |B(t)| < \lambda) \tag{1.4}$$

for all λ, where

$$S_k = \sum_{i=1}^{k} X_i$$

and

$$T(\lambda) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left(-\frac{(2k+1)^2\pi^2}{8\lambda^2}\right). \tag{1.4'}$$

For the convergence rate in (1.4) Yoshihara [8] gave the following result; if $E|X_1|^{4+\varepsilon} < \infty$ for some $\varepsilon > 0$ and

$$\sum_{k=1}^{\infty} k(\beta(k))^{\varepsilon/(4+\varepsilon)} < \infty, \tag{1.5}$$

then

$$\Delta_n \equiv \sup_{\lambda} |P(\max_{1\leq k\leq n} |S_k| < \sigma n^{1/2}\lambda) - T(\lambda)| = O(n^{-1/8}(\log n)^{1/2}). \tag{1.6}$$

The method of the proof in Yoshihara [8] highly depends on the assumption that $E|X_1|^{4+\varepsilon} < \infty$ for some ε. No stronger condition on moments seems to imply better rates in (1.6). In this paper we shall give the better rate in (1.4) under stronger moment condition and furthermore we give the more results under several other moment conditions.

Theorem 1. *Let $\{X_i, i \in \boldsymbol{Z}\}$ be a sequence of strictly stationary random variables satisfying $EX_1 = 0$ and*

$$P(|X_1| \geqq \lambda) \leqq K_1 \exp(-\alpha\lambda^{\mu}) \tag{1.7}$$

for some $\alpha > 0$ and $\mu > 0$, where K_1 is a positive constant. Suppose that $\{X_i\}$ satisfies the condition of the absolute regularity with coefficient $\beta(\cdot)$ such that

$$\beta(n) = o(e^{-\gamma n}) \tag{1.8}$$

for some $\gamma > 0$. Then we have

$$\Delta_n = O(n^{-1/4}(\log n)^{\zeta}), \tag{1.9}$$

where $\zeta = \max(1+\varepsilon, 1/2+1/\mu)$ for any $\varepsilon > 0$.

Theorem 2. *In Theorem 1, replace condition (1.7) by*

$$E|X_1|^r < \infty \quad \text{for some } r > 6. \tag{1.10}$$

Then we have for any $\delta < (r-2)/4(r+1)$

$$\Delta_n = o(n^{-\delta}) \tag{1.11}$$

as $n \to \infty$.

Theorem 3. *In Theorem* 1, *replace concition* (1.7) *by*

(1.12) $$E|X_1|^r<\infty \quad \textit{for some } r \textit{ with } 4<r\leqq 6.$$

Then we have

(1.13) $$\Delta_n=o(n^{-1/7})$$

as $n\to\infty$. *Moreover if*

(1.14) $$E|X_1|^r<\infty \quad \textit{for some } r \textit{ with } 2<r\leqq 4,$$

then we have for any $\delta<(r-2)/2(2r-1)$,

(1.15) $$\Delta_n=o(n^{-\delta})$$

as $n\to\infty$.

2. Preliminaries.

The basic idea of the proofs of Theorems 1-3 is to use the following lemma which is an approximation theorem given by Yoshihara [7].

Lemma 1. *Let* $\{X_i\}$ *be an absolutely regular sequence of random variables such that* $EX_1=0$ *and* $E|X_1|^{2+\varepsilon}<\infty$ *for some* $\varepsilon>0$. *Let* $g(t_1, \cdots, t_k)$ *be any Borel function on* $\boldsymbol{R}^k$ *with* $|g(t_1, \cdots, t_k)|\leqq K_2$, K_2 *being a positive constant. Then there exists a sequence of independent random variables* $\{Y_i\}$ *such that each* Y_i *has the same distribution as that of* X_i *and*

$$|E\{g(X_{i_1}, X_{i_2}, \cdots, X_{i_k})\}-E\{g(Y_{i_1}, Y_{i_2}, \cdots, Y_{i_k})\}|\leqq 2K_2 k\beta(d)$$

for any $1\leqq i_1<i_2<\cdots<i_k$, *where*

$$d=\max_{1\leq j\leq k-1}(i_{j+1}-i_j)\,.$$

The following lemma is also due to Yoshihara [7].

Lemma 2. *Let* $\{X_i\}$ *be a strictly stationary and absolutely regular sequence of random variables with zero mean and* $E|X_1|^s<\infty$ *for some* $s\geqq 3$. *If the assumption* (1.6) *holds and* $\sigma>0$ *in* (1.2), *then*

$$\sup_{t>((s-2)\log n)^{1/2}}\{t^s(\log n)^{-1}P(|S_n|\geqq t(EX_1^2)^{1/2}(\log n)^{1/2}n^{1/2})\}\leqq K_s n^{-(s-2)/2},$$

where K_s *is a positive constant depending only on* s.

We now define several sequences of random variables associated with $\{X_i\}$ in the following way. Let $M=[n^{1/2}]+1$. For $j=1, \cdots, M-1$, define

$$I_j=\{(j-1)[n^{1/2}]+1,\ (j-1)[n^{1/2}]+2,\ \cdots,\ j[n^{1/2}]\}$$

and

$$I_M=\{(M-1)[n^{1/2}]+1,\ (M-1)[n^{1/2}]+2,\ \cdots,\ n\}\ .$$

Let

$$y_j=\sum_{i\in I_j} n^{-1/2}X_i \quad \text{for } j=1,\ \cdots,\ M.$$

Moreover define $U_j=\{j[n^{1/2}]-[\theta \log n]+1,\ j[n^{1/2}]-[\theta \log n]+2,\ \cdots,\ j[n^{1/2}]\}$ and

$$v_j=\sum_{i\in U_j} n^{-1/2}X_i \quad \text{for } j=1,\ \cdots,\ M-1 \text{ and } v_M=0,$$

where θ is some positive constant. Denote

$$\xi_j=y_j-v_j \quad \text{for } j=1,\ \cdots,\ M.$$

Using Theorem 1 in Yokoyama [6] we easily have the following lemma.

Lemma 3. *As* $n\to\infty$,

(i) $Ey_j^2=n^{-1/2}+O(n^{-1}),\quad j=1,\ \cdots,\ M-1,$

(ii) $Ey_M^2=(n-(M-1)[n^{1/2}])/n+O(n^{-1})$

and

(iii) $Ev_j^2=(\theta \log n)/n+O(n^{-1}),\quad j=1,\ \cdots,\ M-1.$

For each positive integer m,

(iv) $E|y_j|^{2m}\leqq K_\beta K_m E|X_1|^{2m}n^{-m/2},\quad j=1,\ \cdots,\ M$

and

(v) $E|v_j|^{2m}\leqq K_\beta K_m E|X_1|^{2m}(\theta \log n)^m n^{-m},\quad j=1,\ \cdots,\ M-1,$

where $K_m=2m!(2m-1)$ *and* K_β *is a positive constant depending only on the coefficient* $\beta(\cdot)$.

Define $M+1$ points $\{a_k\}$ on $[0,1]$ by $a_k=k[n^{1/2}]/n$ for $k=0,1,\cdots,M-1$ and $a_M=1$. Let $\{X_n(t),\ 0\leqq t\leqq 1\}$, $\{\tilde{X}_n(t),\ 0\leqq t\leqq 1\}$ and $\{\hat{X}_n(t),\ 0\leqq t\leqq 1\}$ be continuous polygonal lines defined respectively by

$$X_n(t)=\begin{cases} n^{1/2}tX_1, & \text{for } t\in[0,\ 1/n], \\ n^{-1/2}S_k+(nt-k)n^{-1/2}X_{k+1}, & \text{for } t\in(k/n,\ (k+1)/n],\ k=1,\ \cdots,\ M-1, \end{cases}$$

$$\tilde{X}_n(t)=\begin{cases} \dfrac{t}{a_1}y_1, & \text{for } t\in[0,\ a_1], \\ \displaystyle\sum_{i=1}^{k} y_i+\frac{t-a_k}{a_{k+1}-a_k}y_{k+1}, & \text{for } t\in(a_k,\ a_{k+1}],\ k=1,\ \cdots,\ M-1, \end{cases}$$

and

$$\hat{X}_n(t)=\begin{cases} \dfrac{t}{a_1}y_1, & \text{for } t\in[0,\ a_1], \\ \displaystyle\sum_{i=1}^{k} \xi_i+\frac{t-a_k}{a_{k+1}-a_k}\xi_{k+1}, & \text{for } t\in(a_k,\ a_{k+1}],\ k=1,\ \cdots,\ M-1. \end{cases}$$

3. More lemmas.

We need more lemmas. Let $\varepsilon_n = n^{-1/4}(\log n)^{\zeta}$. Define

$$\xi_j^* = \begin{cases} \xi_j & \text{for } |\xi_j| \leqq \varepsilon_n \\ 0 & \text{for } |\xi_j| > \varepsilon_n, \end{cases}$$

and

$$\eta_j = \xi_j^* - E\xi_j^* \quad \text{for } j=1, \cdots, M.$$

Moreover define continuous polygonal lines $\{X_n^*(t), 0 \leqq t \leqq 1\}$ and $\{X_n^{**}(t), 0 \leqq t \leqq 1\}$ by

$$X_n^*(t) = \begin{cases} \dfrac{t}{a_1}\xi_1^*, & \text{for } t \in [0, a_1], \\ \displaystyle\sum_{i=1}^{k} \xi_i^* + \frac{t-a_k}{a_{k+1}-a_k}\xi_{k+1}^*, & \text{for } t \in (a_k, a_{k+1}],\ k=1, \cdots, M-1, \end{cases}$$

and

$$X_n^{**}(t) = \begin{cases} \dfrac{t}{a_1}\eta_1, & \text{for } t \in [0, a_1], \\ \displaystyle\sum_{i=1}^{k} \eta_i + \frac{t-a_k}{a_{k+1}-a_k}\eta_{k+1}, & \text{for } t \in (a_k, a_{k+1}],\ k=1, \cdots, M-1. \end{cases}$$

The expectations of ξ_j^* are estimated as follows.

Lemma 4. *As $n \to \infty$, for any $u > 1/4$,*

$$|E\xi_j^*| \leqq K_u n^{-u} \quad \text{for } j=1, \cdots, M,$$

where K_u is a positive constant depending only on u.

Proof. Since $E\xi_j = 0$ for each j,

$$\text{(3.1)} \qquad |E\xi_1^*| = \left| \int_{|x|>\varepsilon_n} xP\left(n^{-1/2}\sum_{i=1}^{[n^{1/2}]-[\theta \log n]} X_i \in dx\right) \right|$$
$$\leqq \left| \int_{|x|>(\log n)^{\zeta}} n^{-1/4} xP\left(n^{-1/4}\sum_{i=1}^{[n^{1/2}]-[\theta \log n]} X_i \in dx\right) \right|.$$

On the other hand we have by Lemma 2 that

$$\text{(3.2)} \qquad P\left(\left|n^{-1/4}\sum_{i=1}^{[n^{1/2}]-[\theta \log n]} X_i\right| \geqq x\right) \leqq K_s x^{-s} n^{-(s-2)/4}(\log n)^{1+s/2},$$

for $x > (\log n)^{\zeta}$. Thus, from (3.1) and (3.2), the lemma is proved.

Lemma 5. *As $n \to \infty$,*

$$P(\sup_{0 \leq t \leq 1} |X_n(t) - \hat{X}_n(t)| \geqq \varepsilon_n) = o(\varepsilon_n)$$

Proof. We see from the definitions of $X_n(t)$ and $\hat{X}_n(t)$ that

$$(3.3)\qquad \begin{aligned} & P(\sup_{0\le t\le 1} |X_n(t)-\hat{X}_n(t)| \geqq \varepsilon_n) \\ & \leqq P(\sup_{0\le t\le 1} |X_n(t)-\tilde{X}_n(t)| \geqq \varepsilon_n/2)+P(\sup_{0\le t\le 1} |\tilde{X}_n(t)-\hat{X}_n(t)| \geqq \varepsilon_n/2) \\ & \leqq \sum_{i=1}^{M} P\Big(\sup_{\alpha_{i-1}\le t\le \alpha_i} |X_n(t)-\hat{X}_n(t)| \geqq \varepsilon_n/2\Big)+P\Big(\max_{1\le k\le M} \Big|\sum_{i=1}^{k} v_i\Big| \geqq \varepsilon_n/2\Big) \\ & \leqq MP\Big(\max_{1\le k\le [n^{1/2}]} \Big|\sum_{i=1}^{k} n^{-1/2}X_i\Big| \geqq \varepsilon_n/2\Big)+\sum_{k=1}^{M} P\Big(\Big|\sum_{i=1}^{k} v_i\Big| \geqq \varepsilon_n/2\Big) \\ & \equiv A_1+A_2, \qquad \text{say.} \end{aligned}$$

Recall that $\zeta>1$. Using Lemma 2 with $s=8$, we have that

$$(3.4)\qquad \begin{aligned} P\Big(\Big|\sum_{i=1}^{k} n^{-1/2}X_i\Big| \geqq \varepsilon_n/2\Big) &= P\Big(\Big|\sum_{i=1}^{k} n^{-1/4}X_i\Big| \geqq (\log n)^{\zeta}/2\Big) \\ &= o(n^{-3/2}), \end{aligned}$$

uniformly in k with $1\leqq k\leqq [n^{1/2}]$. Hence

$$(3.5)\qquad A_1 \leqq M\sum_{k=1}^{[n^{1/2}]} P\Big(\Big|\sum_{i=1}^{k} n^{-1/2}X_i\Big| \geqq \varepsilon_n/2\Big)=o(\varepsilon_n).$$

As to A_2 we also apply Lemma 2 to see that

$$(3.6)\qquad \begin{aligned} A_2 & \leqq \sum_{k=1}^{M} P\Big(\Big|\sum_{i=1}^{k} v_i\Big| \geqq \varepsilon_n/2\Big) \\ & \leqq \sum_{k=1}^{M} P\Big(\Big|\sum_{i=1}^{k} v_i\Big| \geqq \theta^{-1/2}(Ev_i^2)^{1/2}(\log n)^{\zeta-1/2}n^{1/4}\Big) \\ & = o(\varepsilon_n). \end{aligned}$$

Thus the lemma is proved from the relations (3.3), (3.5) and (3.6).

Lemma 6. *As* $n\to\infty$,

$$P(\sup_{0\le t\le 1} |\hat{X}_n(t)-X_n^{**}(t)| \geqq \varepsilon_n)=o(\varepsilon_n).$$

Proof. From the definitions of $X_n(t)$ and $X_n^{**}(t)$, we see that

$$(3.7)\qquad \begin{aligned} & P(\sup_{0\le t\le 1} |\hat{X}_n(t)-X_n^{**}(t)| \geqq \varepsilon_n) \\ & \leqq P(\sup_{0\le t\le 1} |\hat{X}_n(t)-X_n^{*}(t)| \geqq \varepsilon_n/2)+P(\sup_{0\le t\le 1} |X_n^{*}(t)-X_n^{**}(t)| \geqq \varepsilon_n/2) \\ & = P\Big(\max_{1\le k\le M} \Big|\sum_{i=1}^{k} (\xi_i-\xi_i^*)\Big| \geqq \varepsilon_n/2\Big)+P\Big(\max_{1\le k\le M} \Big|\sum_{i=1}^{k} E\xi_i^*\Big| \geqq \varepsilon_n/2\Big) \\ & \equiv C_1+C_2, \qquad \text{say.} \end{aligned}$$

Since $\xi_i=\xi_i^*$ if $|\xi_i|\leqq \varepsilon_n$ for each i, we have from (3.4) that

$$C_1 \leqq P(\max_{1\le k\le M} |\xi_k| \geqq \varepsilon_n) \leqq MP(|\xi_1| \geqq \varepsilon_n) = o(\varepsilon_n)\,. \tag{3.8}$$

Moreover, as to C_2, since we have from Lemma 3 that

$$\max_{1\le k\le M}\left|\sum_{i=1}^{k} E\xi_i^*\right| \leqq \sum_{i=1}^{M} |E\xi_i^*| = o(\varepsilon_n)\,,$$

it follows that for sufficiently large n

$$C_2 \equiv 0\,. \tag{3.9}$$

Therefore we obtain the lemma from the relations (3.7)-(3.9).

4. Proof of Theorem 1.

We now turn out to prove Theorem 1. Define the distribution function $F_n^*(\cdot)$ as follows,

$$F_n^*(\lambda) = P\left(\max_{1\le k\le M}\left|\sum_{i=1}^{k}\eta_i\right| \leqq \lambda\right).$$

Furthermore denote $\mathcal{M}_p^q$ the σ-field generated by the random variables $\{\eta_p, \eta_{p+1}, \cdots, \eta_q\}$ for all p and q with $1 \leqq p \leqq q \leqq M$. We have from the assumption (1.1) that

$$\max_{1\le k\le M-1} E\{\sup_{A\in\mathcal{M}_{k+1}^M} |P(A|\mathcal{M}_1^k) - P(A)|\} = O(n^{-\gamma\theta})\,.$$

Therefore, using Lemma 1, we see that there exists a sequence of independent random variables $\{Y_1, \cdots, Y_M\}$ such that Y_i has the same distribution as that of η_i for each i and

$$\left|P\left(\max_{1\le k\le M}\left|\sum_{i=1}^{k}\eta_i\right| \geqq \lambda\right) - P\left(\max_{1\le k\le M}\left|\sum_{i=1}^{k}Y_i\right| \geqq \lambda\right)\right| = O(n^{-\gamma\theta+1/2}) \tag{4.1}$$

for any λ.

On the other hand, applying the Skorokhod embedding theorem (see e.g. [5]) to $\{Y_i\}$, we can construct a Brownian motion $\{B(t), t \geqq 0\}$ and a sequence of independent and positive random variables $\{T_i, 1 \leqq i \leqq M\}$ such that the joint distributions of

$$\left\{B(T_1),\ B(T_1+T_2) - B(T_1),\ \cdots,\ B\left(\sum_{i=1}^{M} T_i\right) - B\left(\sum_{i=1}^{M-1} T_i\right)\right\}$$

are same as those of $\{Y_1, Y_2, \cdots, Y_M\}$ and

$$ET_i = EY_i^2 \quad \text{and} \quad E|T_i|^m \leqq 2m!\, E|Y_i|^{2m},\ i=1, \cdots, M, \tag{4.2}$$

for each positive integer m.

Combining this argument and the equation (4.1), we have that

$$F_n^*(\lambda) = P\left(\max_{1\le k\le M}\left|\sum_{i=1}^{k} Y_i\right| \leqq \lambda\right) + o(\varepsilon_n) \tag{4.3}$$

$$=P\Big(\max_{1\le k\le M}\Big|B\Big(\sum_{i=1}^{k}T_i\Big)\Big|\leqq\lambda\Big)+o(\varepsilon_n).$$

Now since $E\eta_i=0$ and $|\eta_i|\leqq 2\varepsilon_n$ for each i,

$$\max_{1\le k\le M}\ \sup_{0\le t\le T_k}\Big|B\Big(t+\sum_{i=1}^{k-1}T_i\Big)-B\Big(\sum_{i=1}^{k-1}T_i\Big)\Big|\leqq 2\varepsilon_n.$$

Therefore we have from (4.3) that

$$F_n^*(\lambda)\leqq P\Big(\sup_{0\le t\le \sum_{i=1}^{M}T_i}|B(t)|\leqq\lambda+2\varepsilon_n\Big)+o(\varepsilon_n) \tag{4.4}$$

$$\leqq P\Big(\sup_{0\le t\le 1-\varepsilon_n}|B(t)|\leqq\lambda+2\varepsilon_n\Big)+P\Big(\Big|\sum_{i=1}^{M}T_i-1\Big|\geqq\varepsilon_n\Big)+o(\varepsilon_n),$$

and

$$F_n^*(\lambda)\geqq P\Big(\sup_{0\le t\le 1+\varepsilon_n}|B(t)|\leqq\lambda-2\varepsilon_n\Big)-P\Big(\Big|\sum_{i=1}^{M}T_i-1\Big|\geqq\varepsilon_n\Big)-o(\varepsilon_n). \tag{4.5}$$

As to the right hand sides of (4.4) and (4.5), the same method as in the proof of Theorem 1 stated in Sawyer [5] gives us in view of the inequality (4.2) that

$$P\Big(\Big|\sum_{i=1}^{M}T_i-1\Big|\geqq\varepsilon_n\Big)=o(\varepsilon_n). \tag{4.6}$$

On the other hand we see from Lemmas 5 and 6 that

$$F_n(\lambda)=P\Big(\sup_{0\le t\le 1}|X_n(t)|\leqq\lambda\Big) \tag{4.7}$$

$$\leqq P\Big(\sup_{0\le t\le 1}|X_n(t)|\leqq\lambda,\ \sup_{0\le t\le 1}|X_n(t)-X_n^{**}(t)|\leqq 2\varepsilon_n\Big)$$

$$+P\Big(\sup_{0\le t\le 1}|X_n(t)-\hat{X}_n(t)|\geqq\varepsilon_n\Big)+P\Big(\sup_{0\le t\le 1}|\hat{X}_n(t)-X_n^{**}(t)|\geqq\varepsilon_n\Big)$$

$$\leqq F^*(\lambda+2\varepsilon_n)+o(\varepsilon_n)$$

and

$$F_n(\lambda)\geqq F^*(\lambda-2\varepsilon_n)-o(\varepsilon_n). \tag{4.8}$$

Thus we have from the inequalities (4.4)-(4.8) that

$$P\Big(\sup_{0\le t\le 1+\varepsilon_n}|B(t)|\leqq\lambda-4\varepsilon_n\Big)-o(\varepsilon_n)$$

$$\leqq F_n(\lambda)\leqq P\Big(\sup_{0\le t\le 1-\varepsilon_n}|B(t)|\leqq\lambda+4\varepsilon_n\Big)+o(\varepsilon_n).$$

The rest of the proof is the same as that of Theorem 1 in [5] and is omitted.

5. Proof of Theorem 2.

By the truncation argument which was used in the proof of Theorem 2 in [5], we can easily prove the theorem.

6. Proof of Theorem 3.

In the proof of Theorem 1 in [2], replace Lemma 3 in [2] by Lemma 1, then we can obtain Theorem 3.

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