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CONVERGENCE RATES FOR INTEGRAL TYPE FUNCTIONALS OF WEAKLY DEPENDENT RANDOM VARIABLES

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1. Introduction and results.

Let $\{X_i, i\in \boldsymbol{Z}\}$ be a strictly stationary sequence of random variables. Let $\mathcal{F}_p^q$ denote a σ-field generated by random variables $\{X_i, i=p, p+1, \cdots, q\}$. Suppose the sequence $\{X_i\}$ satisfies the conditon of the absolute regularity,

$$(1.1) \qquad \beta(n)=E[\sup_{A\in\mathcal{F}_n^\infty}|P(A|\mathcal{F}_{-\infty}^0)-P(A)|]\downarrow 0\,,$$

as $n\to\infty$.

Suppose $EX_1=0$ and $E|X_1|^{2+\delta}<\infty$ for some $\delta>0$. Under these assumptions, if

$$\sum_{i=1}^\infty(\beta(i))^{\delta/2+\delta}<\infty\,,$$

then

$$(1.2) \qquad \sigma^2\equiv EX_1^2+2\sum_{i=1}^\infty E(X_iX_{i+1})<\infty\,.$$

(See e.g. [1]). Throughout this paper we assume $\sigma>0$.

Let $S_k=\sum_{i=1}^k X_i$, $S_0=0$ and $\{B(t), t\geqq 0\}$ be the standard Brownian motion. Let $f(s, x)\in C(\boldsymbol{R}^2)$ be a function such that f and its partial derivative of order one are of slow growth in x, i.e. f satisfies the inequality of the form,

$$(1.3) \qquad |Df(s, x)|\leqq K_1(1+|x|^a)\,,$$

where D denotes either the identity operator or first derivative, and a and K_1 are some positive constants independent of x and s. Moreover assume that the probability distribution

$$F(\lambda)=P\left(\int_0^1 f(t, B(t))dt\leqq\lambda\right)$$

satisfies a first order Lipshitz condition, i.e. there exists an absolute positive constant K_2 such that

(1.4) $$|F(t)-F(s)|\leqq K_2|t-s|,$$

uniformly in t and s.

Under these assumptions, Yoshihara [6] proved that if $E|X_1|^{4+\delta}<\infty$ for some $\delta>0$ and $\beta(n)=O(e^{-\gamma n})$ for some $\gamma>0$, then

(1.5) $$\Delta_n\equiv\sup_\lambda\left|P\left(\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}, \frac{S_k}{\sigma\sqrt{n}}\right)<\lambda\right)-F(\lambda)\right|=O(n^{-1/5}(\log n)^{1+a'}),$$

where $a'>a>0$.

The purpose of this paper is to estimate the left hand side of (1.5) for an absolutely regular sequence of random variables with absolute moments of all orders, by the method in the proofs given in [3] and [6], and to show the following theorem.

Theorem 1. *Let $\{X_i, i\in \boldsymbol{Z}\}$ be a sequence of strictly stationary random variables. Suppose that the sequence $\{X_i\}$ satisfies*

(1.6) $$\beta(n)=O(e^{-\gamma n})$$

for some $\gamma>0$. Suppose also that $EX_1=0$ and for any $\lambda\in\boldsymbol{R}^1$

(1.7) $$P(|X_1|\geqq\lambda)=O(\exp\{-\alpha\lambda^\delta\})$$

for some $\alpha>0$ and $\delta>0$. Moreover assume without loss of generality that $\sigma\equiv 1$. Let $f(s, t)\in C^1(\boldsymbol{R}^2)$ be a function satisfying the conditions (1.3) *and* (1.4). *We then have*

$$\Delta_n=O(n^{-1/4}(\log n)^\eta),$$

where $\eta=\eta(\alpha, \delta)>1+a$ is some positive constant depending only on α and δ.

2. Preliminary lemmas.

In this section we give five lemmas. The basic idea of the proof of Theorem 1 is to use the following appoximation theorem due to Yoshihara [5].

Lemma 1. *Let $\{X_i\}$ be an absolutely regular sequence of random variables such that $EX_1=0$ and $E|X_1|^{2+\delta}<\infty$ for some $\delta>0$. Let $g(t_1, \cdots, t_k)$ be any Borel function on $\boldsymbol{R}^k$ with $|g(t_1, \cdots, g_k)|\leqq K_g$ for some positive constant K_g. Then there exists a sequence of independent random variables $\{Y_i\}$ such that each Y_i has the same distribution as that of X_i and*

$$|E\{g(X_{i_1}, X_{i_2}, \cdots, X_{i_k})\}-E\{g(Y_{i_1}, Y_{i_2}, \cdots, Y_{i_k})\}|\leqq 2K_g k\beta(d)$$

for any $1\leqq i_1<i_2<\cdots<i_k$, where

$$d=\max_{1\leq j\leq k-1}(i_{j+1}-i_j).$$

The following lemma is also due to Yoshihara [7].

Lemma 2. *Let* $\{X_i\}$ *be a strictly stationary and absolutely regular sequence of random variables with zero mean and* $E|X_1|^s<\infty$ *for some* $s\geqq 3$. *If the assumption* (1.6) *holds and* $\sigma>0$ *in* (1.2), *then*

$$\sup_{t>((s-2)\log n)^{1/2}}\{t^s(\log n)^{-1}P(|S_n|\geqq t(EX_1^2)^{1/2}(\log n)^{1/2}n^{1/2})\}\leqq K_s n^{-(s-2)/2},$$

where K_s *is a positive constant depending only on* s.

We now define several sequences of random variables associated with $\{X_i, i\in \boldsymbol{Z}\}$ in the following way. Let $M=[n^{1/2}]+1$ and define the sequences $I_j=\{(j-1)[n^{1/2}]+1, (j-1)[n^{1/2}]+2, \cdots, j[n^{1/2}]\}$ for $j=1, \cdots, M-1$ and $I_M=\{(M-1)[n^{1/2}]+1, (M-1)[n^{1/2}]+2, \cdots, n\}$ and write

$$y_j=\sum_{i\in I_j} n^{-1/2}X_i,\ j=1, \cdots, M.$$

Furthermore letting $U_j=\{j[n^{1/2}]-[\theta(\log n)]+1, j[n^{1/2}]-[\theta(\log n)]+2, \cdots, j[n^{1/2}]\}$, define the random variables

$$v_j=\sum_{i\in U_j} n^{-1/2}X_i,\ j=1, \cdots, M-1$$

and $v_M=0$, where θ is some positive constant. Write $\xi_j=y_j-v_i$ for $j=1, \cdots, M$.

Using Theorem 1 in Yokoyama [3] we have the following lemma.

Lemma 3. *As* $n\to\infty$,

(i) $Ey_j^2=n^{-1/2}+O(n^{-1})$, $j=1, \cdots, M-1$,

(ii) $Ey_M^2=(n-(M-1)[n^{1/2}])/n+O(n^{-1})$

and

(iii) $Ev_j^2=(\theta(\log n))/n+O(n^{-1})$, $j=1, \cdots, M-1$.

Moreover for each positive integer m,

(iv) $E|y_j|^{2m}\leqq K_\beta K_m E|X_1|^{2m}n^{-m/2}$, $j=1, \cdots, M$,

and

(v) $E|v_j|^{2m}\leqq K_\beta K_m E|X_1|^{2m}(\theta(\log n))^m n^{-m}$, $j=1, \cdots, M-1$,

where $K_m=(2m)!(2m-1)$ *and* K_β *is a positive constant depending only on the coefficient* $\beta(\cdot)$.

Take M points a_k, $k=1, \cdots, M$ on $[0, 1]$ such that $a_k=k[n^{1/2}]/n$ for $k=1, \cdots, M-1$ and $a_M=1$. Let $\{X_n(t), 0\leqq t\leqq 1\}$, $\{\tilde{X}_n(t), 0\leqq t\leqq 1\}$ and $\{\hat{X}_n(t), 0\leqq t\leqq 1\}$ be continuous polygonal lines defined by

$$X_n(t)=\begin{cases} n^{1/2}tX_1, & \text{for } t\in\left[0, \dfrac{1}{n}\right], \\ n^{-1/2}S_k+(nt-k)n^{-1/2}X_{k+1}, & \text{for } t\in\left(\dfrac{k}{n}, \dfrac{k+1}{n}\right], \quad \text{for } k=1, \cdots, n-1, \end{cases}$$

$$\tilde{X}_n(t)=\begin{cases} \dfrac{t}{a_1}y_1, & \text{for } t\in[0, a_1], \\ \sum_{i=1}^{k} y_i+\dfrac{t-a_k}{a_{k+1}-a_k}y_{k+1}, & \text{for } t\in(a_k, a_{k+1}], \quad k=1, \cdots, M-1 \end{cases}$$

and

$$\hat{X}_n(t)=\begin{cases} \dfrac{t}{a_1}\xi_1, & \text{for } t\in[0, a_1], \\ \sum_{i=1}^{k} \xi_i+\dfrac{t-a_k}{a_{k+1}-a_k}\xi_{k+1}, & \text{for } t\in(a_k, a_{k+1}], \quad k=1, \cdots, M-1, \end{cases}$$

respectively. With these definitions, we have

Lemma 4. *Let* $\varepsilon_n=n^{-1/4}(\log n)^{\eta}$. *As* $n\to\infty$,

$$P(\sup_{0\leq t\leq 1}|f(t, X_n(t))-f(t, \tilde{X}_n(t))|\geqq\varepsilon_n)=o(\varepsilon_n).$$

Proof. Since, from the condition (1.3),

$$|f(t, x)-f(t, x')|\leqq K_1(1+\max(|x|^a, |x'|^a))|x-x'|, \tag{2.1}$$

we have for some positive b with $b>a$ and $b<\eta-1$,

$$\begin{aligned} &P(\sup_{0\leq t\leq 1}|f(t, X_n(t))-f(t, \tilde{X}_n(t))|\geqq\varepsilon_n) \\ &\leqq P(\sup_{0\leq t\leq 1}K_1|X_n(t)-\tilde{X}_n(t)|(1+\max(|X_n(t)|^a, |\tilde{X}_n(t)|^a))\geqq\varepsilon_n) \\ &\leqq P(\sup_{0\leq t\leq 1}K_1|X_n(t)-\tilde{X}_n(t)|(1+\max(|X_n(t)|^a, |\tilde{X}_n(t)|^a))\geqq\varepsilon_n, \\ &\qquad \sup_{0\leq t\leq 1}|X_n(t)|^a<K_1^{-1}(\log n)^b) \\ &\quad +P(\sup_{0\leq t\leq 1}|X_n(t)|^a\geqq K_1^{-1}(\log n)^b) \\ &\leqq P(\sup_{0\leq t\leq 1}|X_n(t)-\tilde{X}_n(t)|\geqq\varepsilon_n(\log n)^{-b}) \\ &\quad +P\left(\max_{1\leq k\leq n}\left|n^{-1/2}\sum_{i=1}^{k}X_i\right|\geqq K_1^{-1}(\log n)^{b/a}\right) \\ &\equiv A_1+A_2, \quad \text{say.} \end{aligned} \tag{2.2}$$

By the definitions of $X_n(t)$ and $\tilde{X}_n(t)$, we have

$$A_1=\sum_{i=1}^{M}P(\sup_{a_{i-1}\leq t\leq a_i}|X_n(t)-\tilde{X}_n(t)|\geqq\varepsilon_n(\log n)^{-b}) \tag{2.3}$$

$$\leqq MP\Big(\max_{1\leq k\leq [n^{1/2}]}\Big|\sum_{i=1}^{k} n^{-1/2}X_i\Big|\geqq \varepsilon_n(\log n)^{-b}\Big)$$

$$=MP\Big(\max_{1\leq k\leq [n^{1/2}]}\Big|\sum_{i=1}^{k} n^{-1/4}X_i\Big|\geqq (\log n)^{\eta-b}\Big).$$

Recall that $\eta-b>1$. Using Lemma 2 with $s=10$, we obtain that

$$P\Big(\Big|\sum_{i=1}^{k} n^{-1/4}X_i\Big|\geqq (\log n)^{\eta-b}\Big)=o(n^{-2})$$

uniformly in k with $1\leqq k\leqq M$. Thus we have from (2.3) that

$$A_1\leqq M\sum_{k=1}^{[n^{1/2}]}P\Big(\Big|\sum_{i=1}^{k} n^{-1/4}X_i\Big|\geqq (\log n)^{\eta-b}\Big)=o(\varepsilon_n). \tag{2.4}$$

Furthermore it is easily proved that $A_2=o(\varepsilon_n)$, too. Hence the lemma is shown by (2.3) and (2.4).

Lemma 5. *As $n\to\infty$,*

$$P(\sup_{0\leq t\leq 1}|f(t, \tilde{X}_n(t))-f(t, \hat{X}_n(t))|\geqq \varepsilon_n)=o(\varepsilon_n).$$

Proof. By the inequality (2.1) we have

$$P(\sup_{0\leq t\leq 1}|f(t, \tilde{X}_n(t)-f(t, \hat{X}_n(t))|\geqq \varepsilon_n) \tag{2.5}$$

$$\leqq P(\sup_{0\leq t\leq 1}K_1|\tilde{X}_n(t)-\hat{X}_n(t)|(1+\max(|\tilde{X}_n(t)|^a, |\hat{X}_n(t)|^a))\geqq \varepsilon_n)$$

$$\leqq P(\sup_{0\leq t\leq 1}|\tilde{X}_n(t)-\hat{X}_n(t)|\geqq \varepsilon_n(\log n)^{-b})$$

$$+P(\sup_{0\leq t\leq 1}\max(|\tilde{X}_n(t)|^a, |\hat{x}_n(t)|^a)\geqq K_1^{-1}(\log n)^b)$$

$$\equiv B_1+B_2, \quad \text{say.}$$

Since $\eta-b>1$, we apply Lemma 2 to B_1 with $s=4$ to see that

$$B_1=P\Big(\max_{1\leq k\leq M-1}\Big|\sum_{i=1}^{k} v_i\Big|\geqq \varepsilon_n(\log n)^{-b}\Big) \tag{2.6}$$

$$\leqq \sum_{k=1}^{M-1}P\Big(\Big|\sum_{i=1}^{k} v_i\Big|\geqq \theta^{-1/2}(Ev_i^2)^{1/2}(\log n)^{\eta-b-1/2}n^{1/4}\Big)$$

$$=o(n^{-1/2})$$

$$=o(\varepsilon_n)$$

We next estimate B_2 as follows. From the definitions of $\tilde{X}_n(t)$ and $\hat{X}_n(t)$ we have

$$B_2\leqq P(\sup_{0\leq t\leq 1}|\tilde{X}_n(t)|^a\geqq K_1^{-1}(\log n)^b)+P(\sup_{0\leq t\leq 1}|\hat{X}_n(t)|^a\geqq K_1^{-1}(\log n)^b) \tag{2.7}$$

$$=P\Big(\max_{1\leq k\leq M}\Big|\sum_{i=1}^{k} y_i\Big|\geqq K_1^{-1/a}(\log n)^{b/a}\Big)+P\Big(\max_{1\leq k\leq M}\Big|\sum_{i=1}^{k}\xi_i\Big|\geqq K_1^{-1/a}(\log n)^{b/a}\Big)$$

$$=B_{21}+B_{22}\,,\quad \text{say.}$$

Since $b/a>1$, we also apply Lemma 2 to B_{21} and B_{22} with $s=4$ and we obtain that

$$B_{21}\leqq \sum_{k=1}^{M} P\Big(\Big|\sum_{i=1}^{k} y_i\Big|\geqq K_1^{-1/a}(Ey_1^2)^{1/2}(\log n)^{b/a}n^{1/4}\Big) \tag{2.8}$$

$$=o(n^{-1/2})$$

$$=o(\varepsilon_n)$$

and

$$B_{22}=o(\varepsilon_n)\,. \tag{2.9}$$

Thus we obtain the lemma from (2.5)-(2.9).

3. Skorokhod embedding.

Let $\mathcal{M}_p^q$ be the σ-field generated by the random variables $\{\xi_p, \xi_{p+1}, \cdots, \xi_q\}$ for all p and q with $1\leqq p\leqq q\leqq M$. From the condition (1.1) we have

$$\max_{1\leq k\leq M-1} E\{\sup_{A\in\mathcal{M}_{k+1}^M}|P(A\,|\,\mathcal{M}_1^k)-P(A)|\}=O(n^{-\theta\gamma})\,.$$

Thus, from Lemma 1, there exists a sequence of independent random variables $\{Y_i,\ i=1, \cdots, M\}$ such that each Y_i has the same distribution as that of ξ_i and

$$\Big|P\Big(\max_{1\leq k\leq M}\Big|\sum_{i=1}^{k}\xi_i\Big|\geqq t\Big)-P\Big(\max_{1\leq k\leq M}\Big|\sum_{i=1}^{k}Y_i\Big|\geqq t\Big)\Big|=O(n^{-\theta\gamma+1/2}) \tag{3.1}$$

for any $t\geqq 0$.

On the other hand, applying the Skorokhod embedding theorem (see e.g. [2]) to $\{Y_i\}$, we can construct a Brownian motion $\{B(t),\ t\geqq 0\}$ and a sequence of independent and positive random variables $\{T_i,\ i=1, \cdots, M\}$ such that the joint distributions of $\Big\{B(T_1),\ B(T_1+T_2)-B(T_1), \cdots, B\Big(\sum_{i=1}^{M}T_i\Big)-B\Big(\sum_{i=1}^{M-1}T_i\Big)\Big\}$ are the same as those of $\{Y_1, Y_2, \cdots, Y_M\}$, and moreover

$$ET_i=EY_i^2 \quad\text{and}\quad E|T_i|^m\leqq 2m!\,E|Y_i|^{2m}\,,\qquad i=1, \cdots, M, \tag{3.2}$$

for each positive integer m.

Using this argument, we have the following lemma.

Lemma 6. *As* $n\to\infty$,

$$P\Big(\sum_{k=1}^{M}(a_k-a_{k-1})f\Big(a_k,\ B\Big(\sum_{i=1}^{k}T_i\Big)\Big)\leqq\lambda-2\varepsilon_n\Big)-o(\varepsilon_n)$$

$$\leqq P\Big(\int_0^1 f(t, \hat{X}_n(t))dt \leqq \lambda\Big)$$

$$\leqq P\Big(\sum_{k=1}^{M}(a_k - a_{k-1})f\Big(a_k, B\Big(\sum_{i=1}^{k} T_i\Big)\Big) \leqq \lambda + 2\varepsilon_n\Big) + o(\varepsilon_n).$$

The detail of the proof is just the same as that of Lemma 3.3 in Yoshihara [6] and is omitted.

4. Proof of Theorem 1.

To prove the theorem, it is enough to show that

$$P\Big(\Big|\sum_{i=1}^{M}(a_k - a_{k-1})f\Big(a_k, B\Big(\sum_{i=1}^{k} T_i\Big)\Big) - \int_0^1 f(t, B(t))dt\Big| \geqq \varepsilon_n\Big) = o(\varepsilon_n). \tag{4.1}$$

This is seen in the following way. Once (4.1) holds, we have

$$P\Big(\int_0^1 f(t, B(t))dt \leqq \lambda - \varepsilon_n\Big) - o(\varepsilon_n)$$

$$\leqq P\Big(\sum_{i=1}^{M}(a_k - a_{k-1})f\Big(a_k, B\Big(\sum_{i=1}^{k} T_i\Big)\Big) \leqq \lambda\Big)$$

$$\leqq P\Big(\int_0^1 f(t, B(t))dt \leqq \lambda + \varepsilon_n\Big) + o(\varepsilon_n),$$

it holds in view of (1.4) that

$$\Big|P\Big(\sum_{k=1}^{M}(a_k - a_{k-1})f\Big(a_k, B\Big(\sum_{i=1}^{k} T_i\Big)\Big) \leqq \lambda\Big) - P\Big(\int_0^1 f(t, B(t))dt \leqq \lambda\Big)\Big|$$

$$\leqq P\Big(\lambda - \varepsilon_n < \int_0^1 f(t, B(t))dt < \lambda + \varepsilon_n\Big) + o(\varepsilon_n)$$

$$\leqq 2K_2\varepsilon_n + o(\varepsilon_n).$$

From this we can prove the theorem from Lemmas 4-6.

Now we shall show (4.1) by the adaptation of Sawyer [3]. Define $f^A(s, x) \in C^1(\boldsymbol{R}^2)$ for any $A > 0$ to be such that

$$f^A(s, x) = f(s, x) \qquad \text{for} \quad |x| \leqq A,$$

and

$$Df^A(s, k) \leqq 2K_1(1 + A^\alpha) \qquad \text{for} \quad |x| > A,$$

uniformly in s and x, where D is the operator in (1.3). We then have that

$$P\Big(\Big|\sum_{k=1}^{M}(a_k - a_{k-1})f\Big(a_k, B\Big(\sum_{i=1}^{k} T_i\Big)\Big) - \int_0^1 f(t, B(t))dt\Big| \geqq \varepsilon_n\Big) \tag{4.2}$$

$$\leqq P\Big(\Big|\sum_{k=1}^{M}(a_k - a_{k-1})f^A\Big(a_k, B\Big(\sum_{i=1}^{k} T_i\Big)\Big) - \int_0^1 f^A(t, B(t))dt\Big| \geqq \varepsilon_n\Big)$$

$$+P\Big(\sup_{0\leq t\leq 2}|B(t)|\geqq A\Big)+P\Big(\sum_{i=1}^{M}T_i>2\Big)$$

$$\equiv C_1+C_2+C_3\,,\quad \text{say.}$$

From the assumption (1.7), we have that as $m\to\infty$,

$$E|X_1|^{2m}\leqq 4mK_1\int_0^1 x^{2m-1}\exp(-\alpha x^{\delta})dx\leqq \frac{4mK_1}{\alpha^{(2m/\delta)-1}}\Gamma\Big(\frac{2m}{\delta}\Big)\leqq\Big(\frac{2m}{\delta}\Big)^{2m/\delta}.$$

Thus, from Lemma 2 and (3.2), there exists a positive constant K_3 depending only on δ such that

$$ET_i=E(y_i-v_i)^2=a_i-a_{i-1}+O(n^{-1}\log n) \tag{4.3}$$

and

$$ET_i^m\leqq 2m!\,E|y_i-v_i|^{2m}\leqq (K_3m)^{K_3m}n^{-m/2}, \tag{4.4}$$

for each $i=1,\cdots,M$. Therefore it follows from Tchebyshev's inequality and Burkholder's inequality that

$$\begin{aligned} C_3&=P\Big(\sum_{i=1}^{M}(T_i-ET_i)>2-\sum_{i=1}^{M}ET_i\Big) \\ &=P\Big(\sum_{i=1}^{M}(T_i-ET_i)>1+o(1)\Big) \\ &\leqq 2E\Big|\sum_{i=1}^{M}(T_i-ET_i)\Big|^2 \\ &\leqq K_4\sum_{i=1}^{M}E|y_i-v_i|^4 \\ &\leqq K_5n^{-1/2} \\ &=o(\varepsilon_n)\,, \end{aligned} \tag{4.5}$$

where K_4 and K_5 are absolute positive constants. Furthermore, setting $A=\log n$, we also obtain that

$$C_2\leqq 2P(|B(1)|\geqq\sqrt{2}\,A)\leqq\sqrt{2}\,A^{-1}\exp(-A^2)=o(\varepsilon_n)\,. \tag{4.6}$$

Finally we shall estimate C_1 as follows. According to the proof in [3], we see easily that

$$\Big|\sum_{k=1}^{M}(a_k-a_{k-1})f^A\Big(a_k,\,B\Big(\sum_{i=1}^{k}T_i\Big)\Big)-\int_0^1 f(t,\,B(t))dt\Big|$$

$$=\sum_{k=0}^{M-1}\int_{\zeta_k}^{\zeta_{k+1}}\frac{\partial}{\partial s}f^A\Big(a_k+\bar{\rho}_{kn}(s),\,B\Big(\sum_{i=1}^{k}T_i\Big)+\rho_{kn}(s)\Big)(s-a_k)ds$$

$$+\sum_{k=0}^{M-1}\int_{\zeta_k}^{\zeta_{k+1}}\frac{\partial}{\partial x}f^A\left(a_k+\bar{\rho}_{kn}(s),\ B\left(\sum_{i=1}^{k}T_i\right)+\rho_{kn}(s)\right)\left(B(s)-B\left(\sum_{i=1}^{k}T_i\right)\right)ds$$

$$+\sum_{k=0}^{M-1}f^A\left(a_k,\ B\left(\sum_{i=1}^{k}T_i\right)\right)(T_{k+1}-ET_{k+1})+f^A(0,\ 0)a_1$$

$$-f^A\left(1,\ B\left(\sum_{i=1}^{M}T_i\right)\right)(a_M-a_{M-1})+f^A\left(a_{M-1},\ B\left(\sum_{i=1}^{M}T_i\right)\right)(a_M-a_{M-1})$$

$$-\sum_{k=0}^{M-1}f^A\left(a_k,\ B\left(\sum_{i=1}^{k}T_i\right)\right)([\theta\log n]/n+O(n^{-1}))$$

$$-\int_1^{\zeta_M}f^A(t,\ B(t))dt$$

$$\equiv\sum_{i=1}^{8}G_i,\quad \text{say.}$$

where $|\bar{\rho}_{kn}(s)|\leqq 1$, $|\rho_{kn}(s)|\leqq 1$ and

$$\zeta_k=\sum_{i=1}^{k}T_i \qquad \text{for} \quad k=1,\cdots,M \quad \text{and} \quad \zeta_0=0.$$

By Tchebyshev's inequality, we have that

$$C_1=P\left(\left|\sum_{i=1}^{8}G_i\right|\geqq\varepsilon_n\right)\leqq\sum_{i=1}^{8}P(|G_i|\geqq\varepsilon_n/8) \tag{4.7}$$

$$\leqq\sum_{i=1}^{8}(\varepsilon_n/8)^{-m}E|G_i|^m.$$

Now we have from the definition of $f^A(\cdot,\ \cdot)$ that

$$E|G_3|^m\leqq(2K_1)^m(\log n)^{ma}E\left|\sum_{k=1}^{M}(T_k-ET_k)\right|^m.$$

Thus, using Burkholder's inequality and the inequality (4.4), we have that

$$(\varepsilon_n/8)^{-m}E|G_3|^m\leqq(K_1/4)^m(K_3m)^{K_3m}(\log n)^{(a-\eta)m}$$

$$\leqq(K_6m)^{K_6m}(\log n)^{(a-\eta)m},$$

where K_6 is a positive constant depending only on δ. Therefore, if we set $m=[(\log n)/K_6]$ then

$$(\varepsilon_n/8)^{-m}E|G_3|^m\leqq n^{(1+(a-\eta)/K_6)\log\log n}=o(\varepsilon_n),$$

for some $\eta>a+K_6$.

Now we can also estimate other $E|G_i|^m$ in (4.7) to have

$$(\varepsilon_n/8)^{-m}E|G_i|^m=o(\varepsilon_n)$$

for each i. We therefore have that $C_1=o(\varepsilon_n)$ and finish the proof of Theorem 1 from (4.2), (4.5) and (4.6).

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