YOKOHAMA MATHEMATICAL
JOURNAL VOL. 31, 1983

ASYMPTOTIC ANALYSIS OF STATISTICAL ESTIMATORS FOR SHIFT PARAMETERS BASED ON OBSERVATIONS SATISFYING SOME ABSOLUTE REGULARITY CONDITIONS

By

KEN-ICHI YOSHIHARA

(Received November 6, 1982)

1. Introduction.

Let $\{X_t\}$ be a sequence of random variables. For each positive integer n, let $f(x_1^n : \theta)=f((x_1, \cdots, x_n) : \theta)$ be the density, with respect to a σ-finite measure ν^n, of $X_1^n=(X_1, \cdots, X_n)$ at the point $x_1^n=(x_1, \cdots, x_n)$ (of the appropriate space) when θ is the value of the (unknown to the statistician) parameter.

In this paper, we study the limiting properties of certain estimators for shift parameters constructed on observations $\{\xi_n\}$ satisfying some absolute regularity conditions. Especially, we consider the behavior of the following estimators:

(I) a maximum likelihood estimator $t_n=t_n(\xi_1, \cdots, \xi_n)$, i.e., t_n is defined by

$$f((\xi_1-t_n, \cdots, \xi_n-t_n))=\max_{t\in\Theta} f((\xi_1-t, \cdots, \xi_n-t)), \tag{1.1}$$

(II) a Bayesian estimator t_n with respect to the loss function $W(x)$ and a priori density $\pi(t)$, i.e.,

$$\int W(t_n-y)p_n(y)dy=\min_d \int W(d-y)p_n(y)dy \tag{1.2}$$

where

$$p_n(y)=\frac{f((\xi_1-y, \cdots, \xi_n-y))\pi(y)}{\int f((\xi_1-z, \cdots, \xi_n-z))\pi(z)dz} \tag{1.3}$$

is the a posteriori density for the parameter t determined according to Bayes' formula for the a priori density $\pi(t)$, and

(III) Pitman's estimator $\tilde{\theta}_n$ defined by

(1.4) $$\tilde{\theta}_n=\frac{\int y f((\xi_1-y, \cdots, \xi_n-y))dy}{\int f((\xi_1-y, \cdots, \xi_n-y))dy}.$$

2. Assumptions.

Let Θ be an open set in the real line R^1. Suppose that a family of probability spaces $\{(\Omega, \mathcal{F}, P_t), t\in\Theta\}$ is given. Let $\{\xi_n, -\infty<n<\infty\}$ be a strictly stationary, absolutely regular sequence of random variables, i.e., $\{\xi_n\}$ satisfies an absolute regularity (a.r.) condition

(2.1) $$\beta_n=\beta(n)=\sup_{t\in\Theta}\beta(n, t)\downarrow 0$$

as $n\to\infty$. Here, for each $t\in\Theta$

(2.2) $$\beta(n, t)=E_t\{\sup_{A\in\mathcal{F}_n^\infty}|P_t(A|F_{-\infty}^0)-P_t(A)|\}$$

$E_t(.)$ denotes the expectation with respect to $P_t(.)$, and $\mathcal{F}_a^b$ is the smallest σ-algebra generated by $\xi_a^b=(\xi_a, \cdots, \xi_b)$.

We assume that corresponding to $\{\xi_n\}$ there exists a conditional probability density (with respect to Lebesgue measure) $f(x_p^n|x_1^{p-1}, t)$ $(t\in\Theta,\ n\geqq p\geqq 1)$ where $f(x_1^n|x_1^0, t)=f(x_1^p, t)$ denotes the probability density function.

Throughout this paper, we assume that for each $t\in\Theta$ and for all $n\geqq 1$

(2.3) $$f(x_n|x_1^{n-1}, t)=f_n(x_n-t|(x_1-t, \cdots, x_{n-1}-t))$$

and

(2.4) $$f(x_1^n, t)=f((x_1-t, \cdots, x_n-t)).$$

Let us formulate the restrictions to be imposed on the function $f_n(.|.)$ which are used in (2.3) and (2.4).

A_1. The parameter set Θ is an open interval (bounded or unbounded) of R^1.

A_2. For any n the functions $f_n(x_n|x_1^{n-1})$ are defined for all x_1^n and $\mathcal{B}^n$-measurable where $\mathcal{B}^n$ is the smallest σ-algebra of all Borel subsets of R^n. The functions $f_n(x|x_1^{n-1})$ are absolutely continuous in $x\in R^1$.

We put

(2.5) $$f_n'(u|x_1^{n-1})=\frac{\partial}{\partial u}f_n(u|x_1^{n-1})$$

and for all t

(2.6) $$f'(x_n|x_1^{n-1}, t)=f_n'(x_n-t|(x_1-t, \cdots, x_{n-1}-t))$$

A_3. For all n $(\geqq 1)$ and almost all x_1^{n-1}

(2.7) $$\int f'_n(x \mid x_1^{n-1})dx = \frac{\partial}{\partial x}\int f_n(x_n \mid x_1^{n-1})dx_n = 0.$$

A_4. There exists a positive number δ_1 such that

(2.8) $$\sup_n \int |x|^{\delta_1} f_n(x \mid x_1^{n-1})dx < \infty$$

for almost all x_1^{n-1}.

A_5. There exists a number δ_2 (>0) such that

(2.9) $$\sup_{n\geq 1} \int \left| \frac{f'_n(x_n \mid x_1^{n-1})}{f_n(x_n \mid x_1^{n-1})} \right|^{2+\delta_2} f(x_1^n)dx_1^n < \infty$$

Here, the integrand is assumed to vanish whenever $f_n(x \mid x_1^{n-1}) = 0$.

A_6. For some $K_1 > 0$, $\varepsilon > 0$ and all $|t - t_0| < \varepsilon$

(2.10) $$\sup_{k\geq 1} \int \left| \frac{f'(x_k \mid x_1^{k-1}, t)}{f(x_k \mid x_1^{k-1}, t)} - \frac{f'(x_k \mid x_1^{k-1}, t_0)}{f(x_k \mid x_1^{k-1}, t_0)} \right|^2 f(x_1^k, t_0)dx_1^k \leqq K_1 |t - t_0|^2.$$

A_7. As $n \to \infty$ $\beta(n) = O(e^{-\lambda_1 n})$ for some $\lambda_1 > 0$.

A_8. As $m \to \infty$

(2.11) $$\gamma(m) = \sup_{n\geq m} \left[\int \left| \frac{f'_n(x_n \mid x_1^{n-1})}{f(x_n \mid x_1^{n-1})} - \frac{f'_m(x_n \mid x_{n-m+1}^{n-1})}{f_m(x_n \mid x_{n-m+1}^{n-1})} \right|^{2+\delta_2} f(x_1^n)dx_1^n \right]^{1/(2+\delta_2)}$$
$$= O(e^{-\lambda_2 m})$$

for some $\lambda_2 > 0$ where δ_2 is the one used in A_5.

Put

(2.12) $$I = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[\int \left| \frac{f'_i(x_i \mid x_1^{i-1})}{f_i(x_i \mid x_1^{i-1})} \right|^2 f(x_1^i)dx_1^i \right.$$
$$\left. + 2 \sum_{j=i+1}^{\infty} \int \frac{f'_i(x_i \mid x_1^{i-1})}{f_i(x_i \mid x_1^{i-1})} \cdot \frac{f'_j(x_j \mid x_1^{j-1})}{f_j(x_j \mid x_1^{j-1})} f(x_1^j)dx_1^j \right]$$

(The existence of the limit in the right-hand side of (2.11) is verified below (Lemma 4.1)).

A_9. I is positive.

A_{10}. The function $\pi(t)$ is continuous and positive at the point t_0, the "true" value of the parameter t and satisfies the condition

(2.13) $$|\pi(y)| < K_2\{1 + |y|^{\delta_3}\}.$$

In the following sections, we denote all generic constants by the letter K and the integral part of s by $[s]$. $\bar{D}$ denotes the complement of a set D.

3. Results.

Let $C_0=C_0(-\infty, \infty)$ be the space of functions which are continuous on $(-\infty, \infty)$ and for which $\lim_{|x|\to\infty} f(x)=0$. Let

$$Z_n(y)=\frac{f(\xi_1^p,\ t_0+yn^{-1/2})}{f(\xi_1^p,\ t_0)}. \tag{3.1}$$

Theorem 1. *Suppose that* A_1-A_9 *hold. Then the distributions in the space* C_0 *generated by the process* $Z_n(y)$ *converge as* $n\to\infty$ *to a distribution in* C_0 *generated by the process*

$$Z(y)=\exp\left\{y\sqrt{I}\,\tilde{\xi}-\frac{y^2}{2}\right\} \tag{3.2}$$

where $\tilde{\xi}$ *is the standardized normal random variable, and* I *is the one defined by* (2.11).

Theorem 2. *Suppose that* A_1-A_{10} *hold. Then the following statements are true:*

(i) *The maximum likelihood estimator* $\hat{t}_n$ *and Bayes' estimator* $t_n^{(a)}$ *for the loss function* $|x|^a$, $a\geqq 1$, *and a priori density* $\pi(t)$ *are equivalent to each other in the sense that for all* p (>0)

$$E_{t_0}(\sqrt{n}\,|t_n^{(a)}-\hat{t}_n|)^p\to 0 \quad (n\to\infty). \tag{3.3}$$

(ii) *Each of these estimators is asymptotically normal with parameters* $(t_0,\ 2/nI)$ *and for all* p (>0)

$$E_{t_0}|(\hat{t}_n-t_0)\sqrt{nI}\,|^p\to\frac{2^p\Gamma((p+1)/2)}{\sqrt{\pi}}, \tag{3.4}$$

the same sort of relation also holding for the estimator $t_n^{(a)}$.

(iii) *The difference*

$$\Delta_n(y)=\frac{1}{\sqrt{n}}p_n\left(t_0+\frac{y}{n}\right)-\left(\frac{I}{2\pi}\right)^{1/2}\exp\left\{-I\left[\hat{t}_n-\left(t_0+\frac{y}{\sqrt{n}}\right)\right]\right\}^2 \tag{3.5}$$

also converges in probability to zero as $n\to\infty$.

4. Auxiliary results.

We remark first that the following statement holds.

Remark 4.1. Propositions 3.1-3.4 in [5] remain true if the moment condition $E|f'(\xi_n|\xi_1^{n-1},\ t)/f(\xi_n|\xi_1^{n-1},\ t)|^r$ $(r=4+\delta)$ is replaced by the moment condition of order $r=2+\delta$ $(\delta>0)$, since the latter condition is needed in those proofs.

The next lemma is a special case of Propositions 3.1 and 3.2 in [5].

Lemma 4.1. *If* A_1-A_8 *hold, then the limit in the right-hand side of* (2.11) *and is finite.*

Now, for arbitrary positive integers m and n, let

$$X_n=\frac{1}{\sqrt{n}}\cdot\frac{f'(\xi_1^n)}{f(\xi_1^n)}=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{f_j'(\xi_j|\xi_1^{j-1})}{f_j(\xi_j|\xi_1^{j-1})} \tag{4.1}$$

and

$$X_{m,n}^*=\frac{1}{\sqrt{n}}\cdot\frac{f'(\xi_{m+1}^{m+n}|\xi_1^m)}{f(\xi_{m+1}^{m+n}|\xi_1^m)}. \tag{4.2}$$

Theorem 3. *Suppose that* A_1-A_8 *hold. Then* $EX_n=0$, $EX_{m,n}^*=0$ $(m, n=1, 2, \cdots)$ *and* $\lim_{n\to\infty} EX_n^2=I$. *Moreover, if* $m=o(n^{1/2})$, *then*

$$\lim_{n\to\infty} EX_{m,n}^*=I. \tag{4.3}$$

In addition, if A_9 *holds, then* $\{X_n\}$ *converges weakly to a normal random variable* X *with parameters* $(0, I)$.

Proof. By A_3 and Remark 4.1 the first-half is easily obtained.

To prove the latter-half, put $p=[n^{3/4}]$, $q=[n^{1/5}]$ and $k=[n/(p+2q)]$. Let

$$u(i)=\frac{f_i'(\xi_i|\xi_1^{i-1})}{f_i(\xi_i|\xi_1^{i-1})}\quad (1\leqq i\leqq n),$$

$$v_j(i)=\frac{f_q'(\xi_i|\xi_{i-q+1}^{i-1})}{f_q(\xi_i|\xi_{i-q+1}^{i-1})}\quad (q\leqq i\leqq n),$$

$$\eta_j=\sum_{i=1}^{p} v_q(j(p+2q)+2q+i)\quad (0\leqq j\leqq k-1),$$

$$\zeta_j=\sum_{i=1}^{2q} v_q(j(p+2q)+i)\quad (1\leqq j\leqq k-1),$$

$$\zeta_k=\sum_{i=1}^{n-k(p+2q)} v_q(k(p+2q)+i).$$

Then the following statements are easily proved by A_3, A_7, A_8 and (4.3):
(i) $\eta_0, \cdots, \eta_{k-1}$ are identically distributed random variables. For $j(0\leqq j\leqq k-1)\eta_i \in F_{j(p+2q)+q+1}^{(j+1)(p+2q)}$ and the η_j satisfy the a.r. condition with $\beta_1(j)=\beta(jq)$, $E\eta_j=0(0\leqq j\leqq k-1)$ and

$$E\eta_0^2=pI(1+o(1))\quad \text{as } n\to\infty \tag{4.4}$$

(ii) $\zeta_0, \cdots, \zeta_{k-1}$ are identically distributed random variables with $E\zeta_0=0$ and $E\zeta_0^2\leqq Kq^2$.

Further, $E\zeta_k=0$ and

$$E\zeta_k^2 \leqq \begin{cases} Kkq^2, & \text{if } n-k(p+2q)=O(q\log q), \\ O(p), & \text{otherwise.} \end{cases}$$

Hence we have

$$\frac{1}{n}E\left|\sum_{j=1}^{k}\zeta_j\right|^2 \leqq \frac{2}{n}E\left|\sum_{j=1}^{k-1}\zeta_j\right|^2 + E|\zeta_k|^2$$

$$\leqq \frac{2}{n}\{k^2E\zeta_j^2+O(p)\} \leqq \frac{K}{n}\{k^2q^2+p\} \to 0$$

as $n\to\infty$.

(iii) $$\frac{1}{n}E\left|\sum_{i=2q+1}^{n}\{u(i)-v_q(i)\}\right|^2 \leqq n\beta(q)\to 0 \quad (n\to\infty).$$

(iv) $$\frac{1}{n}E\left|\sum_{i=1}^{2q}u(i)\right|^2 \leqq K\frac{q^2}{n}\to 0 \quad (n\to\infty).$$

From (ii)-(iv) $\left\{\frac{1}{\sqrt{nI}}\sum_{j=1}^{k}\zeta_j\right\}$, $\left\{\frac{1}{\sqrt{nI}}\sum_{i=2q+1}^{n}(u(i)-v_q(i))\right\}$, and $\left\{\frac{1}{\sqrt{nI}}\sum_{i=1}^{2q}u(i)\right\}$, respectively, converge to zero in probability. Hence, to prove the latter-half it is enough to show that

(4.5) $$\frac{1}{\sqrt{n}}\sum_{j=0}^{k}\eta_j \xrightarrow{D} X.$$

But, (4.5) is proved by the method used in the proof of Theorem 6 in Yoshihara [5] using (i). So, we have the desired conclusion.

Corollary. *Suppose that* A_1-A_9 *hold. Then the* X_n^2 *are uniformly integrable.*

Proof. By Theorem 3 $\{X_n^2\}$ converges weakly to X^2, and $\lim_{n\to\infty} EX_n^2=EX^2$. Hence, by Theorem 5.4 in Billingsley [1, p. 32] the conclusion follows.

5. Proofs.

In the following, we put $p=p(n)=[n^\alpha](1/2<\alpha<1)$ and $\varepsilon=\varepsilon_n=bn^{-1/2}$ $(b>0)$. Let

(5.1) $$a_m(s)=b_m(s)\sqrt{f(x_1^p, s)}=\frac{f'(x_1^m, s)}{f(x_1^m, s)}=\sum_{j=1}^{m}\frac{f_j'(x_j|x_1^{j-1}, s)}{f_j(x_j|x_1^{j-1}, s)}.$$

For any $\tau>0$, let

(5.2) $$A_{n\tau}=\left\{x_1^p : \left|\log\frac{f(x_1^p, t+\varepsilon)}{f(x_1^p, t)}\right|>\tau\right\}$$

and

(5.3) $$B_{nL}(s)=\left\{x_1^p : \frac{1}{p}a_p^2(s)<L\right\}.$$

Lemma 5.1. *Asumme that* A_1-A_9 *hold. Then for any* $t \in \Theta$

$$\lim_{n\to\infty} \frac{1}{p\varepsilon}\int_t^{t+\varepsilon}\int_{A_{n\tau}} b_p^2(s)dx_1^p ds = 0. \tag{5.4}$$

Proof. First, we note that from (5.2) and the proof of Lemma 4.2 in Yoshihara [5]

$$\max_{|s-t|<\varepsilon}\int_{A_{n\tau}} f(x_1^p, s)dx_1^p = O(n^{-1}p) \tag{5.5}$$

as $n\to\infty$.

Next, from (2.3) it follows that for all s $(t<s<t+\varepsilon)$

$$\int_{\overline{B_{nL}(s)}} \frac{1}{p} a_p^2(s) f(x_1^p, s)dx_1^p = \int_{\overline{B_{nL}(0)}} \frac{1}{p} b_p^2(0)dx_1^p. \tag{5.6}$$

Hence, by (5.6) we obtain that

$$\begin{aligned} &\frac{1}{p\varepsilon}\int_t^{t+\varepsilon}\int_{A_{n\tau}} b_p^2(s)dx_1^p ds \\ &\leqq \frac{1}{p\varepsilon}\int_t^{t+\varepsilon}\left\{\int_{A_{n\tau}\cap B_{nL}(s)} + \int_{\overline{B_{nL}(s)}}\right\} a_p^2(s) f(x_1^p, s)dx_1^p ds \\ &\leqq \frac{L}{p\varepsilon}\int_t^{t+\varepsilon}\int_{A_{n\tau}} f(x_1^p, s)dx_1^p ds + \int_{\overline{B_{nL}(0)}} \frac{1}{p} a_p^2(0) f(x_1^p)dx_1^p \\ &= J_{1n}(L) + J_{2n}(L), \qquad \text{(say)}. \end{aligned} \tag{5.7}$$

Since by (5.5) $\lim_{n\to\infty} J_{1n}(L)=0$ holds for all L, so to prove (5.4) it suffices to show that

$$\lim_{L\to\infty}\lim_{n\to\infty} J_{2n}(L) = 0. \tag{5.8}$$

But, as the $\frac{1}{m}a_m^2(0)$ are uniformly integrable by Corollary to Theorem 3, so we have (5.8) and the proof is completed.

Lemma 5.2. *For any* $t \in \Theta$

$$\lim_{n\to\infty} \frac{1}{p\varepsilon^2}\int |g_p(t, \varepsilon)|^2 dx_1^p = \frac{1}{4} I \tag{5.9}$$

where

$$g_p(t, \varepsilon) = \sqrt{f(x_1^p, t+\varepsilon)} - \sqrt{f(x_1^p, t)}. \tag{5.10}$$

Proof. By Jensen's inequality and (5.1)

$$\frac{1}{p\varepsilon^2}\int |g_p(t, \varepsilon)|^2 dx_1^p = \frac{1}{p\varepsilon^2}\int |g_p(0, \varepsilon)|^2 dx_1^p \tag{5.11}$$

$$=\frac{1}{4p\varepsilon^2}\int\left\{\int_0^\varepsilon b_p(s)ds\right\}^2 dx_1^p$$

$$\leqq\frac{1}{4p\varepsilon^2}\int_0^\varepsilon\int b_p^2(s)dx_1^p ds=\frac{1}{4p}\int b_p^2(0)dx_1^p .$$

Hence, by Theorem 3

(5.12) $$\lim_{n\to\infty}\frac{1}{p\varepsilon^2}\int|g_p(t,\ \varepsilon)|^2dx_1^p\leqq\frac{1}{4}I .$$

Now, we prove that

(5.13) $$\lim_{n\to\infty}\frac{1}{p\varepsilon^2}\int|g_p(t,\ \varepsilon)|^2dx_1^p\geqq\frac{1}{4}I .$$

We note first that

(5.14) $$\frac{1}{p\varepsilon^2}\int|g_p(t,\ \varepsilon)|^2dx_1^p$$

$$=\frac{1}{4p\varepsilon^2}\int\left\{\int_0^\varepsilon b_p(s)ds\right\}^2 dx_1^p=\int_0^\varepsilon\int_0^\varepsilon\int b_p(s)b_p(t)dx_1^p ds dt$$

$$=\frac{1}{4p\varepsilon^2}\int_0^\varepsilon\int_0^\varepsilon\int b_p(t-s)b_p(0)dx_1^p ds dt .$$

For the moment, let t and s be fixed and put $l=t-s$. Since by A_6

(5.15) $$\int|a_p(s)-a_p(0)|^2f(x_1^p,\ 0)dx_1^p\leqq Kp^2s^2,$$

so by Schwarz's inequality we obtain that

(5.16) $$\int b_p(l)b_p(0)dx_1^p=\int a_p(l)a_p(0)\sqrt{f(x_1^p,\ l)}\sqrt{f(x_1^p,\ 0)}dx_1^p$$

$$\geqq\int|a_p(0)|^2\sqrt{f(x_1^p,\ l)}\sqrt{f(x_1^p,\ 0)}dx_1^p$$

$$-\int|a_p(0)|\,|a_p(l)-a_p(0)|\sqrt{f(x_1^p,\ l)}\sqrt{f(x_1^p,\ 0)}dx_1^p$$

$$\geqq\left\{\int|a_p(0)|^2f(x_1^p,\ 0)dx_1^p-\int c_p(l)dx_1^p\right\}$$

$$-\left\{\int|a_p(0)|^2f(x_1^p,\ 0)dx_1^p\int|a_p(l)-a_p(0)|^2f(x_1^p,\ l)dx_1^p\right\}^{1/2}$$

$$\geqq\int b_p^2(0)dx_1^p-\int c_p(l)dx_1^p-Klp\left\{\int b_p^2(0)dx_1^p\right\}^{1/2}$$

where

(5.17) $$c_p(l)=|a_p(0)|^2|g_p(l,\ 0)|\sqrt{f(x_1^p,\ 0)} .$$

For any $M>0$, let

$$D_M=\{x_1^p : |g_p(l, 0)|<M\sqrt{f(x_1^p, 0)}\}.$$

Then

$$\frac{1}{p}\int_{D_M} c_p(l)dx_1^p=\left\{\int_{D_M\cap B_{nL}(0)}+\int_{D_M\cap\overline{B_{nL}(0)}}\right\}\frac{1}{p}c_p(l)dx_1^p \tag{5.18}$$

$$\leqq L\int\sqrt{f(x_1^p, 0)}\,|g_p(l, 0)|\,dx_1^p+M\int_{\overline{B_{nL}(0)}}\frac{1}{p}b_p^2(0)dx_1^p$$

$$\leqq L\int|g_p(0, l)|^2dx_1^p+M\int_{\overline{B_{nL}(0)}}\frac{1}{p}b_p^2(0)dx_1^p.$$

So, by (5.10)

$$\frac{1}{p}\int_{D_M} c_p(l)dx_1^p\leqq KLpl^2+M\int_{\overline{B_{nL}(0)}}\frac{1}{p}b_p^2(0)dx_1^p \tag{5.19}$$

and by Corollary to Theorem 3 the second term in the right-hand side of the above inequality tends to zero uniformly in p as $L\to\infty$. On the other hand

$$\int_{\bar{D}_M} c_p(l)dx_1^p\leqq\frac{1}{M}\int_{\bar{D}_M}|a_p(0)|^2|g_p(0, l)|^2dx_1^p \tag{5.20}$$

$$\leqq\frac{2}{M}\int|a_p(0)|^2\{f(x_1^p, l)+f(x_1^p, 0)\}\,dx_1^p$$

$$\leqq\frac{2}{M}\left[\int b_p^2(0)dx_1^p+2\left\{\int b_p^2(l)dx_1^p+\int|a_p(l)-a_p(0)|^2f(x_1^p, l)dx_1^p\right\}\right]$$

$$\leqq\frac{2}{M}\left[3\int b_p^2(0)dx_1^p+Kp^2l^2\right].$$

Hence, by (5.18)-(5.20) and Theorem 3 we have that

$$\lim_{M\to\infty}\lim_{L\to\infty}\lim_{n\to\infty}\frac{1}{p\varepsilon^2}\int_0^\varepsilon\int_0^\varepsilon\int c_p(t-s)dx_1^pdsdt=0. \tag{5.21}$$

It is obvious from Theorem 3 that

$$\lim_{n\to\infty}\frac{1}{p\varepsilon^2}\int_0^\varepsilon\int_0^\varepsilon|t-s|\,p\left\{\int b_p^2(0)dx_1^p\right\}^{1/2}dsdt=0 \tag{5.22}$$

Hence, (5.13) follows from (5.14), (5.16), (5.21) and (5.22), and the proof is completed.

The following lemma assures the validity of Condition IV in Yoshihara [5].

Lemma 5.3. *If* A_4 *holds, then for any* t_1 *and* t_2

$$\sup_{n\geq 1}|t_2-t_1|^{\delta_1/2}\int\sqrt{f(x_n|x_1^{n-1}, t_1)}\sqrt{f(x_n|x_1^{n-1}, t_2)}f(x_1^{n-1})dx_1^n<\infty \tag{5.23}$$

where δ_1 *is the one defined in* A_4.

Proof. Without loss of generality, we may assume that $t_1=0$. From the inequality $\left|\frac{1}{2}t_2\right|^{\delta_1/2} \leqq |x-t_2|^{\delta_1}+|x|^{\delta_1}$ and A_4 we have

$$\left(\frac{1}{2}t_2\right)^{\delta_1/2}\int\sqrt{f(x_n|x_1^{n-1}, 0)f(x_n|x_1^{n-1}, t_2)}f(x_1^{n-1})dx_1^n$$

$$\leqq\int\sqrt{f(x_n|x_1^{n-1}, 0)|x_n|^{2\delta_1}}\sqrt{f(x_n|x_1^{n-1}, t_2)}f(x_1^{n-1})dx_1^n$$

$$+\int\sqrt{f(x_n|x_1^{n-1}, 0)}\sqrt{f(x_n|x_1^{n-1}, t_2)|x_n-t_2|^{2\delta_1}}f(x_1^{n-1})dx_1^n$$

$$\leqq 2\iint|x_n|^{2\delta_1}f(x_n|x_1^{n-1}, 0)f(x_1^{n-1})dx_n dx_1^{n-1}\leqq K,$$

which implies (5.23). Hence, the proof is completed.

Proofs of Theorems 1 and 2. If we use Lemmas 5.1 and 5.2 instead of Lemmas 4.2 and 4.4 in Yoshihara [5], then noting Lemma 5.3 the proofs are obtained by the completely same methods as those corresponding theorems in Yoshihara [5] and so are omitted.

References

[1] Billingsley, P. (1968) : *Convergence of probability measures.* Wiley, New York.

[2] Ibragimov, I. A. and Khas'minskii, R. Z. (1972) : *Asymptotic behavior of statistical estimators in the smooth case I.* Study of the likelihood ratio. Theory Probab. Appl. **17** 445-462.

[3] Ibragimov, I. A. and Khas'minskii, R. Z. (1973) : *Asymptotic behavior of some statistical estimators* II. *Limit theorems for the a posteriori density and Bayes'estimators.* Theory Probab. Appl. **18**, 79-91.

[4] Yoshihara, K. (1978) : *Probability inequalities for sums of absolutely regular processes and their applications.* Z. Wahrscheinlichkeitstheorie verw. Gebiete **43** 319-329.

[5] Yoshihara, K. (1982) : *Asymptotic behavior behavior of statistical estimators constructed on absolutely regular sequences.* Yokohama Math. J. **30** 121-139.

Yokohama National University
Department of Mathematics
Faculty of Engineering
156, Tokiwadai, Hodogaya,
Yokohama, Japan