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SOME THEOREMS OF PHRAGMÉN-LINDELÖF TYPE FOR THE HEAT OPERATOR ON A CERTAIN MANIFOLD

By

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1. Introduction.

A. Friedman [5] has generalized the classical theorems of Phragmén-Lindelöf type for the parabolic equations. In this paper we discuss some theorems of Phragmén-Lindelöf type for the heat equation on manifolds with "nice" properties and their applications to the solutions of the heat equation.

In § 2, we suppose that M is a complete C^{∞}-Riemannian manifold with Ricci curvature bounded from below. Then we have a similar theorem for the heat operator as in Theorem 7 in [5] and may apply it to prove the uniqueness of the solutions with Cauchy data which are not necessarily bounded. Our result is an extension of the theorem in [3] which gives the uniqueness of bounded solutions.

In § 3, we suppose that M is a "model" with Ricci curvature bounded from below. Then we have a similar theorem for the heat equation as in Theorem 8 in [5] and that the solution with Cauchy data vanishing at infinity vanishes uniformly as time tends to infinity.

In § 4, we apply theorems in § 3 to differential forms.

2. The uniqueness of the solutions with Cauchy data.

Let L be a linear, locally uniformly parabolic operator on a C^{∞}-Riemannian manifold M of dimension n. In local coordinates L may be written as

$$Lu := \sum_{i,j=1}^{n} a^{ij}(x, t)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(x, t)\frac{\partial u}{\partial x^i} - \frac{\partial u}{\partial t}, \tag{2.1}$$

and for a neighborhood of each point of $M\times \boldsymbol{R}$ ($\boldsymbol{R}$: the reals) there exist two positive constants C and Λ so that at every point of this neighborhood the following inequalities hold:

$$\left| \sum_{i=1}^{n} b^i(x, t)\xi_i \right| \leqq C\left(\sum_{i=1}^{n} \xi_i{}^2 \right)^{1/2} \tag{2.2}$$

$$(2.3)\qquad \Lambda^{-1}\left(\sum_{i=1}^{n}\xi_i^2\right)\leqq\sum_{i,j=1}^{n}a^{ij}(x,t)\xi_i\xi_j\leqq\Lambda\left(\sum_{i=1}^{n}\xi_i^2\right)$$

for every choice of real constants $\xi_1, \xi_2, \cdots, \xi_n$.

Definition 2.1 ([1], [3]). Given a parabolic operator (2.1) L satisfying (2.2) and (2.3), a continuous function u on a domain $\Omega\subset M\times\boldsymbol{R}$ and a function ϕ on Ω with no restriction whatever, we define that

$$Lu(\text{weakly})\leqq\phi$$

on any subset $B\subset\Omega$ if, for every $(x_0, t_0)\in B$ and every $\varepsilon>0$, there exists a neighborhood $V:=V(\varepsilon, x_0, t_0)\subset\Omega$ of (x_0, t_0) and a function u_{ε,x_0,t_0} on V, of class C^2 in the manifold variables and of class C^1 in t such that the difference $u-u_{\varepsilon,x_0,t_0}$ attains its maximum at (x_0, t_0), i.e.

$$(2.4)\qquad u(x,t)-u_{\varepsilon,x_0,t_0}(x,t)\leqq u(x_0,t_0)-u_{\varepsilon,x_0,t_0}(x_0,t_0)$$

for $(x, t)\in V$, and

$$(2.5)\qquad Lu_{\varepsilon,x_0,t_0}\leqq\phi+\varepsilon \quad \text{at } (x_0, t_0).$$

Remark 2.2. If u is sufficiently smooth, then $Lu(\text{weakly})\leqq\phi$ coincides with $Lu\leqq\phi$ ([3]).

Hereafter, we suppose that M is a complete C^∞-Riemannian manifold, and $r(x)$ is the geodesic distance from some fixed point x_0 to x. Let $\Omega\subset M\times(0,\infty)$ be an unbounded domain, and we define some subsets of Ω in the following:

$$T_R:=\Omega\cap\{(x,t)\in M\times\boldsymbol{R}\,|\,r(x)+|t|=R\}$$

$$C_R:=\Omega\cap\{(x,t)\in M\times\boldsymbol{R}\,|\,r(x)+|t|<R\}$$

and

$$D_R:=\Omega\cap\{(x,t)\in M\times\boldsymbol{R}\,|\,r(x)+|t|>R\}.$$

We may partition M into two complementary subsets $C(x_0)$ and $M(x_0)$, i.e. $M=C(x_0)\cup M(x_0)$, where $C(x_0)$ is the cut locus of x_0 and $M(x_0)=M\backslash C(x_0)$. It is well known that $M(x_0)$ is an open subset in M and the function $r(x)$ is of class C^∞ in $M(x_0)\backslash\{x_0\}$.

Lemma 2.3 ([1], [3]). *Let ϕ be a nondecreasing function of class C^2 defined on the half-line $(0,\infty)$. If the Ricci curvature of M is bounded from below, then the function $f:=\phi(r)$ satisfies the inequality*

$$\Delta f\leqq\phi''(r)+\left(\frac{n-1}{r}+C\right)\phi'(r)$$

on $M(x_0)\backslash\{x_0\}$, where the constant C depends only on the lower bound of the Ricci curvature and Δ denotes the Laplacian on M.

Hereafter, we suppose that L is the heat operator, $L=\Delta-\partial/\partial t$, on $M\times\boldsymbol{R}$.

Theorem 2.4. *Let M be a complete, C^∞-Riemannian manifold with Ricci curvature bounded from below and Ω an unbounded domain in $M\times(0,\infty)$. Let L be the heat operator $\Delta-\partial/\partial t$. If $u(x,t)\geqq 0$ on $\partial\Omega$, Lu(weakly)$\leqq 0$ on Ω and*

$$\lim_{R\to\infty}\frac{1}{R}\inf_{(x,t)\in T_R}u(x,t)=0, \tag{2.6}$$

then $u(x,t)\geqq 0$ in Ω.

Proof. Choose a nondecreasing function ϕ of class C^2 on $(0,\infty)$ such that $\phi(s)=0$ for $s\in(0,\delta/2)$ and $\phi(s)=s$ for $s\geqq\delta$, where $\delta>0$ is so small that the geodesic ball with center at x_0 and of radius δ is contained in a normal coordinate neighborhood at x_0. Set $\rho(x):=\phi(r(x))$. Then $\rho: M\to\boldsymbol{R}$ is continuous and ρ restricted to $M(x_0)$ is of class C^2. We set

$$V_R(x,t):=\frac{\rho(x)+\tilde{K}t+\delta}{R},\quad(\tilde{K}>1,\text{ a constant})$$

for $(x,t)\in\bar{\Omega}$. Clearly we have the following:

$$V_R(x,t)\geqq 0\quad\text{if }(x,t)\in\partial\Omega,\ r(x)+|t|\leqq R. \tag{2.7}$$

$$V_R(x,t)\geqq 1\quad\text{if }(x,t)\in\Omega,\ r(x)+|t|=R. \tag{2.8}$$

$$LV_R(\text{weakly})\leqq 0\quad\text{in }C_R\text{ if }\tilde{K}\text{ is sufficiently large.} \tag{2.9}$$

In fact, for $(x,t)\in C_R$ with $x\in M(x_0)$,

$$LV_R=\frac{1}{R}(\Delta\rho-\tilde{K}),$$

and, by Lemma 2.3, there exists a constant N, depending only on ϕ and the lower bound of Ricci curvature, such that $\Delta\rho\leqq N$. Then, we have

$$LV_R\leqq 0\quad\text{for a large constant }\tilde{K}.$$

Next, let $(x,t)\in C_R$ with $x\in C(x_0)$. Choosing one geodesic γ of the shortest length $r(x)$ joining x_0 with x, let x_ε be a point on γ at a distance $\varepsilon>0$ from x_0 arbitrarily small. Then, replacing x_0 by x_ε, we have that $x\in M(x_\varepsilon)$ because the part of γ between x_ε and x is the unique geodesic of length $r(x)-\varepsilon$ joining x_ε with x. We set

$$V_{\varepsilon,x,t}:=V_{\varepsilon,x}\times(0,\infty)\cap C_R$$

$$r_{\varepsilon,x}(y):=\text{the distance from }x_\varepsilon\text{ to }y$$

$$\rho_{\varepsilon,x}(y):=\phi(r_{\varepsilon,x}(y))$$

for $y\in V_{\varepsilon,x}$, where $V_{\varepsilon,x}\subset M(x_\varepsilon)$ is a neighborhood of x, and

$$V_{R,\varepsilon,x,t}(y, s):=\frac{\rho_{\varepsilon,x}(y)+\tilde{K}s+\delta}{R}$$

for $(y, s)\in V_{\varepsilon,x,t}$. Then, we have

$$LV_{R,\varepsilon,x,t}\leqq 0 \quad \text{at } (x, t).$$

The triangle inequality implies that

$$r-r_{\varepsilon,x}\leqq\varepsilon\,,$$

the equality holds at x. Choosing δ sufficiently small, we have

$$\rho(y)-\rho_{\varepsilon,x}(y)\leqq\varepsilon \quad \text{for } y\in V_{\varepsilon,x}\,,$$

the equality holds at x. Then we have

$$V_R(y, s)-V_{R,\varepsilon,x,t}(y, s)\leqq V_R(x, t)-V_{R,\varepsilon,x,t}(x, t)$$

for $(y, s)\in V_{\varepsilon,x,t}$. This completes the proof of (2.9).

Now, we consider the function

$$\tilde{u}(x, t):=u(x, t)-\{\min(0, \inf_{(x,t)\in T_R} u(x, t))\}V_R(x, t)\,.$$

Then $\tilde{u}(x, t)$ is non-negative on ∂C_R by (2.7) and (2.8), and $L\tilde{u}$(weakly)$\leqq 0$ in C_R by (2.9). Using the weak maximum principle ([3], [6]), we have $\tilde{u}(x, t)\geqq 0$ in C_R. Letting $R\to\infty$ and using (2.6) we have $u(x, t)\geqq 0$. q. e. d.

Corollary 2.5. *Let M be a complete, C^∞-Riemannian manifold with Ricci curvature bounded from below. Suppose that $u: M\times[0, \infty)\to\boldsymbol{R}$ is a solution of the heat equation with Cauchy data u_0, i.e.*

$$Lu=0 \quad \text{on } M\times(0, \infty),$$

and

$$u(x, 0)=u_0(x) \quad \text{on } M.$$

If

$$\lim_{R\to\infty}\frac{1}{R}\sup_{(x,t)\in T_R}|u(x, t)|=0\,, \tag{2.10}$$

then u is the unique solution with Cauchy data u_0.

Proof. Suppose that $u_0=0$ on M. Using Theorem 2.4 for $\Omega=M\times(0, \infty)$, we have $u\geqq 0$ on $M\times(0, \infty)$. And $-u$ also satisfies the condition of Theorem 2.4, then we have $u\leqq 0$ on $M\times(0, \infty)$. Therefore, we have $u=0$ on $M\times[0, \infty)$. This implies the uniqueness. q. e. d.

Remark 2.6. J. Dodziuk [3] has proved the uniqueness of bounded solutions of the heat equation with Cauchy data on a complete C^∞-Riemannian manifold with Ricci curvature bounded from below. Of course, the condition (2.10) is satisfied for bounded solutions.

3. The property of the solutions with Cauchy data.

A Riemannian manifold M of dimension n is a model if the following conditions hold:

(i) The exponential map of the tangent space $T_{x_0}(M)$ at some fixed point x_0 onto M is a diffeomorphism.
(ii) For every $r>0$, the mean curvature H of the geodesic sphere $S_r(x_0)$ satisfies the inequality

$$H \leqq \frac{K}{(n-1)r} \quad (K<1, \text{ a constant})$$

with respect to the outer normal.

Remark 3.1. Our terminology differs slightly from that of [2] and [4]. If M is a model in the sense of [4] with negative semi-definite Ricci curvature, then the condition (ii) holds.

Lemma 3.2. *Let M be a model and $\Omega \subset M \times (0, \infty)$ an unbounded domain. Then, for any $R_0>0$, there exists a function $\omega(x, t)$ with the following properties defined in D_{R_0}:*

(3.1) $$\omega(x, t) \geqq 0 \quad \textit{if } (x, t) \in \partial D_{R_0}.$$

(3.2) $$\omega(x, t) \geqq 1 \quad \textit{if } (x, t) \in \Omega, \ r(x)+|t|=R_0.$$

(3.3) $$L\omega \leqq 0 \quad \textit{in } D_{R_0}.$$

(3.4) $$\omega(x, t) \to 0 \quad \textit{uniformly in } D_{R_0} \textit{ as } r(x)+|t| \to \infty.$$

Proof. We consider the function

$$\omega(x, t) := \frac{C}{(t+1)^{\varepsilon}} \exp\left(\frac{-Gr(x)^2}{t+1}\right) \quad (C, G, \varepsilon: \text{positive constants}).$$

For (x, t) with $x \in M \setminus \{x_0\}$,

$$L\omega = \omega\left\{\frac{r^2}{(t+1)^2}(4G^2-G) + \frac{1}{t+1}(\varepsilon - 2Gr\Delta r - 2G)\right\}.$$

Note that $\Delta r = -(n-1)H$ ([11]). Then we have

$$L\omega \leqq \omega\left\{\frac{r^2}{(t+1)^2}(4G^2-G) + \frac{1}{t+1}(\varepsilon + 2GK - 2G)\right\}.$$

If

(3.5) $$4G^2 - G \leqq 0 \quad \text{and} \quad \varepsilon + 2GK - 2G \leqq 0,$$

then $L\omega \leqq 0$ for (x, t) with $x \in M \setminus \{x_0\}$. Clearly we may choose G and ε so that

(3.5) holds and we also have $L\omega \leqq 0$ for (x_0, t). Therefore (3.3) holds. Then (3.1) and (3.4) also hold, and we may choose C so that (3.2) holds. q.e.d.

Theorem 3.3. *Let M be a model with Ricci curvature bounded from below and $\Omega \subset M \times (0, \infty)$ an unbounded domain. If $Lu=0$ in Ω,*

$$\lim_{R\to\infty} \frac{1}{R} \sup_{(x,t)\in T_R} |u(x, t)| = 0 \tag{3.6}$$

and $u(x, t) \to 0$ uniformly on Ω as $r(x)+|t| \to \infty$, then $u(x, t) \to 0$ uniformly on Ω as $r(x)+|t| \to \infty$.

Proof. Given $\varepsilon > 0$, there exists $R_0 > 0$ such that $|u(x, t)| < \varepsilon$ for $(x, t) \in \partial\Omega$, $r(x)+|t| \geqq R_0$. We consider the function

$$v(x, t) := u(x, t) + \{\max_{r(x)+|t|=R_0} |u(x, t)|\}\omega(x, t) + \varepsilon,$$

where $\omega(x, t)$ is the function in Lemma 3.2. Then we have $v(x, t) \geqq 0$ for $(x, t) \in \partial D_{R_0}$, and $Lv \leqq 0$ in D_{R_0}. Theorem 2.4 implies that $v(x, t) \geqq 0$ in D_{R_0}. Choosing R_1 such that

$$\{\max_{r(x)+|t|=R_0} |u(x, t)|\}\omega(x, t) < \varepsilon$$

in D_{R_1}, we have

$$0 \leqq v(x, t) < u(x, t) + 2\varepsilon$$

in D_{R_1}, and so, $u(x, t) > -2\varepsilon$ in D_{R_1}. Similarly we have $u(x, t) < 2\varepsilon$ in D_{R_1}. Therefore, theorem is proved. q.e.d.

Remark 3.4. If Ricci curvature of $M \geqq -C$ (C: a positive constant), then

$$H \geqq -\frac{1}{r} - \left(\frac{C}{n-1}\right)^{1/2}.$$

Therefore H satisfies the inequality

$$-\frac{1}{r} - \left(\frac{C}{n-1}\right)^{1/2} \leqq H \leqq \frac{K}{(n-1)r} \quad (K<1).$$

Corollary 3.5. *Let M be a model with Ricci curvature bounded from below. If u_0 is a continuous function on M which vanishes at infinity and u is the solution of the heat equation with initial data u_0 and satisfies*

$$\lim_{R\to\infty} \frac{1}{R} \sup_{(x,t)\in T_R} |u(x, t)| = 0,$$

then $u(x, t) \to 0$ uniformly on M as $t \to \infty$.

Proof. Use Theorem 3.3 for $\Omega = M \times (0, \infty)$. Then, we have $u(x, t) \to 0$ uniformly on M as $t \to \infty$. q.e.d.

Remark 3.6. If M is a complete C^∞-Riemannian manifold with Ricci curvature bounded from below and u_0 is a continuous function on M which vanishes at infinity, then the bounded solution of the heat equation with Cauchy data u_0 vanishes at infinity for $t>0$ ([3], [7]).

4. An application to p-forms.

Let M be a C^∞-Riemannian manifold of dimension n. We denote ∇ and ∇^* the Levi-Civita connection and its dual respectively.

The Weitszenböck formula is well known:

$$\Delta\alpha = -\nabla^*\nabla\alpha - F_p\alpha \tag{4.1}$$

for a C^∞-p-form α on M depending on t, where F_p is an algebraic operator depending only on the curvature of M. We denote $|\alpha|$ the norm of α.

Theorem 4.1. *Let M be a model with Ricci curvature bounded from below. Suppose that $F_p \geqq 0$ at every point of M. If α is the solution of the heat equation on p-forms with Cauchy data vanishing at infinity and satisfies*

$$\lim_{R\to\infty}\frac{1}{R}\sup_{(x,t)\in T_E}|\alpha(x, t)|^2 = 0,$$

then $\alpha(x, t)\to 0$ uniformly on M as $t\to\infty$.

Proof. By the proof of Theorem 5.2 in [3], we have

$$L|\alpha|^2 \geqq 0.$$

We consider the function

$$v(x, t) := -|\alpha(x, t)|^2 + \{\max_{r(x)+|t|=R_0}|\alpha(x, t)|^2\}\omega(x, t) + \varepsilon$$

where $\omega(x, t)$ is the function in Lemma 3.2. Then, as in § 3, we have that $v(x, t) \geqq 0$ in D_{R_0}. Therefore, we have $|\alpha(x, t)|^2 < 2\varepsilon$ in D_{R_1}. q.e.d.

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